Teaching Notes

Extending runs of composite numbers

It is a familiar observation that the gaps between successive prime numbers can be arbitrarily large; for example $n! + j$ for $2 \leqslant j \leqslant n$ gives a run of $n - 1$ consecutive composite numbers. In fact, it gives a run of consecutive composites for $2 \leqslant j \leqslant p - 1$ where p is the smallest prime greater than n, but this will not be a longer run if $p = n + 1$. However, for $n \geqslant 4$, $n! + j$ is composite for $j = 0$ and for $2 \leqslant |j| \leqslant n$, so we would have a run of $2n + 1$ consecutive composites if we could deal with $j = \pm 1$ as well. For some values of n we may be lucky: for example, $5! - 1 = 119$ and $5! + 1 = 121$ are both composite, but $6! - 1 = 719$ is prime.

In this short Note, we suggest three ways of tweaking the construction to guarantee a run of $2n + 1$ consecutive composites: there is scope for many variations which make it an attractive topic for classroom investigations. For clarity, we write $X = n!$ below.

1. Consider $X^3 + j$ for $|j| \leqslant n$ with $n \geqslant 2$.
   Then $X^3 + j$ is divisible by X for $j = 0$ and by j for $2 \leqslant |j| \leqslant n$.
   For $j = \pm 1$, the algebraic factorisations $X^3 \pm 1 = (X \pm 1)(X^2 \mp X + 1)$ show that $X^3 \pm 1$ are also composite for $n \geqslant 3$.

2. Consider $4X^4 + j$ for $|j| \leqslant n$ with $n \geqslant 2$.
   Then $4X^4 + j$ is divisible by X for $j = 0$ and by j for $2 \leqslant |j| \leqslant n$.
   For $j = \pm 1$, the algebraic factorisations $4X^4 - 1 = (2X^2 - 1)(2X^2 + 1)$ and $4X^4 + 1 = (2X^2 - 2X + 1)(2X^2 + 2X + 1)$ show that $4X^4 \pm 1$ are also composite for $n \geqslant 2$.

3. Are there quadratic examples?
   Let a, b, c be any Pythagorean triple with $a^2 + b^2 = c^2$ where, necessarily, ab is even.
   Consider the quadratic expression $\frac{1}{2}abX^2 + cX + j$ for $|j| \leqslant n$. It is divisible by X for $j = 0$ and by j for $2 \leqslant |j| \leqslant n$. For $j = \pm 1$, the discriminant is $c^2 \mp 2ab = (a \mp b)^2$ so the pair of quadratics $\frac{1}{2}abX^2 + cX \pm 1$ both factorise.
   When $j = 1$, $\frac{1}{2}abX^2 + cX + 1 = (AX + 1)(BX + 1)$ where $A = \frac{1}{2}(c + a - b)$ and $B = \frac{1}{2}(b + c - a)$ are positive (by the triangle inequality) and integers (by a parity argument on $a^2 + b^2 = c^2$). These factors give rise to numerical factors greater than 1 for $n \geqslant 1$.
   When $j = -1$, $\frac{1}{2}abX^2 + cX - 1 = (CX + 1)(DX - 1)$ where $C = \frac{1}{2}(a + b - c)$ and $D = \frac{1}{2}(a + b + c)$. As above, $CX + 1$ gives rise to a numerical factor greater than 1 for $n \geqslant 1$, as does $DX - 1$ since $D \geqslant 6$, the semi-perimeter of the 3,4,5 triangle.

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Finally, we note that $n!$ can be replaced by the product $p_1p_2\ldots p_n$ where $p_1< p_2 < \ldots < p_n$ denote the primes in ascending order of size. In each of the three examples above, this produces a run of consecutive composites for $|j| \leqslant p_{n+1} - 1$.

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Two quick observations about rough inclined planes

1. In Figure 1, two masses m_1 and m_2 are connected by a light rod and projected up (or down) a rough plane inclined at angle θ to the horizontal: θ could be zero!

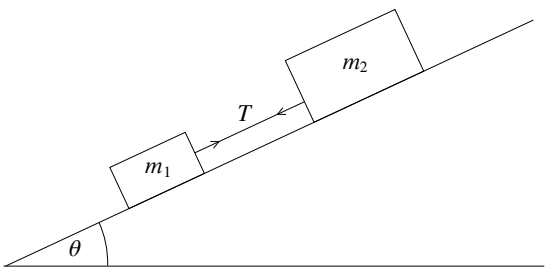


FIGURE 1

If a is the acceleration up the slope and T the force in the connecting rod, then resolving forces parallel to the slope shows that the equations of motion of the individual masses are

$$m_1a = T - m_1g \sin\theta - F_1 \tag{1}$$

and

$$m_2a = -T - m_2g \sin\theta - F_2 \tag{2}$$

where F_1, F_2 are the friction forces on m_1, m_2 respectively. Eliminating a between (1) and (2) then gives

$$m_2\left(T - m_1g \sin\theta - F_1\right) = m_1\left(-T - m_2g \sin\theta - F_2\right)$$

so that $T = \dfrac{m_2F_1 - m_1F_2}{m_1 + m_2}$. (If the masses are moving down the slope, the sign of the terms is reversed.)

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