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COMPLEX DYNAMICS IN LUCAS' TREE ASSET PRICING MODEL WITH DYNAMIC SELF-CONTROL PREFERENCES

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This paper studies the global equilibrium dynamics implied by a Lucas' tree asset pricing model where the representative agent is subject to temptation in consumption choices, and displays dynamic self-control preferences, as defined by Gul and Pesendorfer [(2004) *Econometrica* 72, 119–158.]. It shows that endogenous cycles of period 2 and higher, as well as chaotic dynamics exist provided temptation utility is sufficiently important (with respect to standard commitment utility) and sufficiently convex. For parameterizations leading to complex deterministic dynamics, a stochastic version of the model admits rational expectations equilibria displaying excess volatility with respect to the underlying fundamentals.

Keywords: Asset Pricing, Temptation, Self-Control, Endogenous Cycles, Chaotic Dynamics, Sunspot Equilibrium

1. INTRODUCTION

It is well known that, for any realistic parameterization, representative agent consumption-based asset pricing models with standard constant relative risk aversion (CRRA) preferences—for example, the Lucas's tree asset pricing model—fail to capture several empirical features of stock price data, among which is the fact that stock prices appear to be much more volatile and persistent than the underlying fundamentals. One reason behind the failure is that, in such framework, under rational expectations, the pricing equation for the risky stock (linking current to future stock prices) is a first-order stochastic *linear* difference equation whose unique stable solution features either a constant or a stationary price–dividend ratio.

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While several attempts to generate excess stock price volatility without abandoning the great analytical tractability of the Lucas' tree representative agent asset pricing framework have, to different extents, involved some departure from the rational expectations hypothesis, this paper explores a different route. It provides a detailed *global equilibrium* analysis of a CRRA Lucas's tree asset pricing model where the representative agent is characterized by temptation and dynamic self-control (henceforth, DSC) preferences, as formalized by Gul and Pesendorfer (2001, 2004). Under DSC preferences, in every period, the agent faces a temptation to liquidate his entire financial wealth for the purpose of immediate consumption. Resisting temptation involves cognitive effort (or self-control) and hence some disutility. Optimal behavior therefore trades off the temptation for immediate satisfaction with long-run optimal consumption smoothing.[1] Gul and Pesendorfer (2004) prove the existence of a recursive representation for dynamic optimization problems with DSC preferences, showing how the stock's pricing kernel becomes a function of future stock prices. While they discuss about the quantitative implication of DSC preferences for the equity premium puzzle in a Lucas' tree environment, they do not characterize the global equilibrium dynamics generated by the *nonlinear* asset pricing equation.[2] This is the main objective of this paper.

I show that under DSC preferences the stochastic discount factor (or pricing kernel) of the risky stock is a function of financial wealth. As a result, the equilibrium stock price can display time variation, even if fundamentals are constant. More specifically, I identify parametric conditions for the existence of both endogenous periodic cycles—that is, equilibria where the stock price switches deterministically between a finite set of values—and chaotic dynamics—that is, highly volatile aperiodic (but still deterministic) equilibrium trajectories. I show that the fundamental steady-state equilibrium is the unique feasible equilibrium when temptation utility is weakly concave. On the contrary, when temptation utility is strictly convex (and sufficiently so), endogenous cycles of periodicity 2 and higher arise as temptation becomes more important. Once the parameter indexing the degree of temptation passes a certain threshold, the stock price dynamics become chaotic.[3]

I also show that, for parameterizations leading to complex deterministic dynamics, it is possible to construct stochastic rational expectations equilibria (REE) where stock prices fluctuate because of exogenous shocks to either nonfundamentals (sunspots) or fundamentals (dividends). In the latter case, the underlying presence of deterministic cycles acts like an endogenous amplification mechanism and therefore leads to excess stock price volatility.

As discussed by Gul and Pesendorfer (2004), DSC preferences are consistent with the existing evidence on preference reversal in intertemporal consumption choices, as documented by Frederick et al. (2002). Ameriks et al. (2007), Mullainathan and Banerjee (2010), Mani et al. (2013), and Bernheim et al. (2015) provide empirical support for the presence of temptation and self-control issues in consumption behavior. Recent works by Toussaert (2018, 2019) elicit the

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presence of temptation and self-control in human behavior via laboratory and field experiments. Huang et al. (2015) estimate by generalized method of moments the consumption Euler equation derived from solving an infinite-horizon representative agent model with DSC preference. Their analysis provides empirical support for DSC preferences, and explores some implications for capital accumulation and the welfare cost of business cycle fluctuations. Earlier applications of these preferences for the study of notable macroeconomic puzzles are DeJong and Ripoll (2007) and Krusell et al. (2002) for the equity premium, and Krusell et al. (2010) for optimal taxation.[4]

The rest of the paper is organized as follows. Section 2 lays out the model and derives the key equilibrium equations. Section 3 fully characterizes the global equilibrium dynamics and derives analytical (as well as numerical) conditions for the existence of complex dynamics. Section 4 studies a stochastic version of the model. Section 5 concludes.

2. THE MODEL

Consider a Lucas' tree economy where a continuum of identical infinitively lived agents trade shares of the tree (henceforth, the "stock") in a competitive market. To start with, assume there is no intrinsic uncertainty in the economy: the stock generates constant dividends d per period.[5] The representative agent buys stock shares $a_t \geq 0$ at the market price p_t per share.[6] He uses the financial wealth generated by the stock to finance consumption of a single perishable good and purchases of new shares. His period budget constraint is

$$c_t + p_t a_t = (p_t + d)\, a_{t-1}. \tag{1}$$

The agent has DSC preferences, as formalized by Gul and Pesendorfer (2004). DSC preferences imply that, in each period, the agent faces the temptation to liquidate all his financial wealth for current consumption purposes, and that in order to resist this temptation he has to exert a self-control cognitive effort. Following Gul and Pesendorfer (2004), the agent's intertemporal utility maximization problem has the following recursive representation:

$$W_t = \max_{c_t,\, a_t} \{u(c_t) + v(c_t) + \beta E_t W_{t+1}\} - \max_{\tilde{c}_t,\, \tilde{a}_t} v(\tilde{c}_t), \tag{2}$$

where $\beta \in (0, 1)$, and u and v are both Von Neuman–Morgenstern utility functions. Although fundamentals are constant, the dynamic programming problem includes the (rational) expectations operator E_t to allow for the possibility of sunspot-driven fluctuations (extrinsic uncertainty). While $u_t + \beta E_t W_{t+1}$ represents a standard *commitment* utility, v_t captures *temptation*. Letting $\tilde{c}_t$ and $\tilde{a}_t$ denote, respectively, the optimal levels of consumption and stock shares chosen by the agent in period t if he falls to temptation, the term $v(c_t) - \max_{\tilde{c}_t,\, \tilde{a}_t} v(\tilde{c}_t) \leq 0$ corresponds to the disutility due to self-control that he would suffer if choosing c_t instead of $\tilde{c}_t$.

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I assume that temptation utility v takes the following form: $v(c_t) = \lambda z(c_t)$, where z is strictly increasing in consumption, and $\lambda > 0$ indexes the strength of temptation in preferences. For $\lambda = 0$, there is no temptation, as in the benchmark Lucas-type asset pricing model. As I am going to show, the existence of complex dynamics depends on whether the function z is concave or convex with respect to consumption. In the following analysis, I will consider these two cases separately.[7]

Given the strict monotonicity of z, if falling to temptation, the agent would not save—hence $\tilde{a}_t = 0$—but simply consume all available wealth, namely, $\tilde{c}_t = (p_t + d)\, a_{t-1}$. The dynamic programming problem (2) can therefore be written as follows:

$$W_t = \max_{c_t,\, a_t} \{u(c_t) + \lambda z(c_t) + \beta E_t W_{t+1}\} - \lambda z((p_t + d)\, a_{t-1}) \tag{3}$$

subject to the budget constraint (1). After taking first-order conditions with respect to choice variables and using the Envelope Theorem, it is straightforward to obtain the following asset pricing condition:

$$p_t = E_t\left[M_{t,t+1}^{DSC}(p_{t+1} + d)\right], \tag{4}$$

where $M_{t,t+1}^{DSC}$ is the individual stochastic discount factor (henceforth, SDF)—or pricing kernel of the risky stock—defined as

$$M_{t,t+1}^{DSC} \equiv \beta \frac{u'(c_{t+1}) + \lambda z'(c_{t+1}) - \lambda z'((p_{t+1} + d)\, a_t)}{u'(c_t) + \lambda z'(c_t)}. \tag{5}$$

Making use of the market clearing conditions $c_t = d$ and $a_t = 1$ holding in every period (where, without loss of generality, I have assumed a constant supply of stock shares and normalized it to unity), $M_{t,t+1}^{DSC}$ reduces to

$$M_{t,t+1}^{DSC} = m(p_{t+1}) \equiv \beta \frac{u'(d) + \lambda z'(d) - \lambda z'(p_{t+1} + d)}{u'(d) + \lambda z'(d)}, \tag{6}$$

The SDF is therefore a continuous function m of the next period stock price p_{t+1}. Since $v' = \lambda z' > 0$, temptation makes the SDF lower with respect to the benchmark no-temptation CRRA case (where $\lambda = 0$ gives $v' = 0$), that is, $M_{t,t+1}^{DSC} < M_{t,t+1}^{CRRA} = \beta$. This implies that, for a given future stock price and given dividends, under temptation, the risky stock becomes less valuable with respect to the CRRA case as higher financial wealth increases the cost of self-control for the agent. Notice that the SDF can be either increasing or decreasing in p_{t+1} depending on whether temptation utility is concave or convex:

$$m'(p_{t+1}) \gtreqless 0 \qquad \text{for} \qquad z'' \lesseqgtr 0.$$

Since the sign of $m'(p_{t+1})$ will be crucial for the existence of endogenous cycles, the curvature of temptation utility v deserves some discussion. Consider the numerator of $M_{t,t+1}^{DSC}$ in (6) which represents tomorrow's (equilibrium) total benefit from saving an extra unit of consumption today. This benefit is made of two terms: $u'(d)$—that corresponds to the standard marginal utility of wealth

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appearing in a benchmark CRRA environment—and $\lambda z'(d) - \lambda z'(p_{t+1} + d)$—the derivative of the disutility due to self-control. When z is convex, the latter takes a negative sign: namely, higher savings today increase the future cost of self-control. It follows that, under convex temptation, the benefits from saving today are smaller with respect to the case of standard CRRA preferences. Now, let's suppose the agent experiences an unexpected increase in dividends d. On the one hand, he would like to smooth the increment across periods: hence, savings will increase. On the other hand, due to temptation, he feels the urge to consume all of it today. The latter effect can be tamed (and therefore lead to some savings) only through self-control. However, as just discussed, because of convexity in z, higher savings today will make self-control costlier tomorrow. As a result, the agent has an additional incentive to fall to temptation today, and further increase current consumption.

The opposite holds under a concavity. In particular, since, in this case, higher savings will lower the cost of future self-control, the agent prefers to exert more cognitive effort to resist to temptation. If the latter effect is strong enough, the consumption smoothing motive dominates, and the overall increase in current consumption will be rather small. Based on this, we may conclude that, under a convex temptation utility, the consumption smoothing channel is weaker with respect to what implied by standard CRRA preferences. Overall, agents appear to be more myopic as consumption tends to overrespond to unexpected changes in income. This appears more consistent with the empirical evidence on hand-to-mouth behavior in developed economies, as documented by the fact that, even in the USA, a large share of the population does not hold interest-bearing assets.

Letting $\alpha \equiv u'(d) + \lambda z'(d)$ and $w_t \equiv p_t + d$ denote cum-dividend beginning-of-period wealth, we can use the SDF equation (6) to write the asset pricing equation (4) as follows:

$$w_t = d + \frac{\beta}{\alpha} E_t \left\{ \left[\alpha - \lambda z'(w_{t+1}) \right] w_{t+1} \right\}. \tag{7}$$

Since the model does not feature intrinsic uncertainty, I start the analysis by studying nonexplosive perfect foresight equilibrium (henceforth, PFE) sequences $\{w_t\}_{t=0}^{\infty}$ solving the deterministic version of the nonlinear difference equation (7):

$$w_t = d + \frac{\beta}{\alpha} \left[\alpha - \lambda z'(w_{t+1}) \right] w_{t+1}. \tag{8}$$

Notice that, for $\lambda = 0$, equation (8) reduces to $w_t = d + \beta w_{t+1}$, which corresponds to a Lucas' tree asset pricing equation with standard CRRA utility and constant dividends. It is straightforward to show that, in this case, the unique nonexplosive solution is $w_t = w^{CRRA} \equiv \frac{d}{1-\beta}$ (or equivalently, $p_t = \frac{\beta d}{1-\beta}$). With constant fundamentals (i.e., constant dividends), equilibrium wealth and the stock price are constant as well. For $\lambda > 0$ instead, the right-hand side of (8) is highly nonlinear, and there could be time-varying PFE trajectories despite having constant dividends. The purpose of the analysis is to provide a full characterization of

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those trajectories. In particular, I will assess the possibility of periodic cycles and (topologically) chaotic dynamics.[8]

DEFINITION 1. ***Period-n Cycle*** *A value "q" is a point of a period-n cycle if it is a fixed point of the nth iterate of a mapping f(.) that is* $q = f^n(q)$, *but not a fixed point of an iterate of any lower order. If "q" is such, we call the sequence* $\{q, f(q), f^2(q), ..., f^{n-1}(q)\}$ *a period-n cycle.*

Topological Chaos *A mapping f (.) is topologically chaotic if there exists a set "S" of uncountable many initial points, belonging to its domain, such that no orbit that starts in "S" will converge to one another or to any existing periodic orbit.*

3. GLOBAL DETERMINISTIC DYNAMICS

This section provides full characterization of the global PFE dynamics implied by the nonlinear difference equation (8). To guarantee the existence of an interior steady-state equilibrium in a more general case, I introduce the following assumption.

ASSUMPTION 1. $u'(d) > 0$.

I start by considering the case where temptation utility is strictly concave, $z'' < 0$. As Proposition 2 shows, periodic cycle do not exist. The unique stationary PFE is the steady state, for which the stock price is lower than what implied by the benchmark Lucas' tree model without temptation.[9]

PROPOSITION 2. *Assume that* $z'' < 0$*—that is, temptation utility v is strictly concave. Moreover, assume that the function z satisfies the following properties: (i)* $\lim_{w \to +\infty} z' + wz'' = 0$ *and (ii)* $2z'' + wz''' < 0$ *for every* $w \geq d$.[10] *Then, the dynamic equation (8) possesses a unique steady-state equilibrium* $w^{ss} \in \left(d, \frac{d}{1-\beta}\right)$. *This is also the unique stationary PFE.*

Proof. See Appendix A. ■

A necessary (but not sufficient) condition for the existence of equilibrium cycles is non-monotonicity in the relationship between w_t and w_{t+1} described in (8), which, in turn, requires temptation utility to be strictly convex. For analytical tractability, throughout the rest of the analysis, I assume the following functional forms for u and v:

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \lambda \frac{c_t^{1+\chi}}{1+\chi}, \qquad v(c_t) = \lambda \frac{c_t^{1+\chi}}{1+\chi}. \tag{9}$$

for $\sigma \geq 0$, $\lambda \geq 0$, and $\chi > 0$.[11] This specification has the following features. First, it nests the CRRA utility specification used in benchmark asset-pricing models (for $\lambda = 0$), and the specification used in Section 6.4 in Gul and Pesendorfer (2004) (for $\sigma = 0$ and $\chi = 1$) to discuss about the equity premium implications

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of DSC preferences. Second, while $v(c_t)$ is strictly convex, total momentary utility $u(c_t) + v(c_t)$ is strictly increasing and strictly concave. As in Gul and Pesendorfer (2004), this guarantees that the problem is globally concave, which in turn rules out risk-loving behavior.[12] By simple calculus and algebra, under the utility specification (9), the dynamic equation (8) takes the following form:

$$w_t = F(w_{t+1}) \equiv 1 + \beta\left[1 - \lambda w_{t+1}^{\chi}\right] w_{t+1}, \tag{10}$$

where, to simplify the notation, I have set dividends equal to unity: $d = 1$. Notice that under the chosen specification, Assumption 1 reduces to $\lambda < 1$.

PROPOSITION 3. *Let* $\bar{w} \equiv \lambda^{-\frac{1}{\chi}}$. *Under the utility specification in (9) and the normalization* $d = 1$, *the model possesses a unique steady-state equilibrium* $w^{ss} \in (1, \bar{w})$. *Moreover,* w^{ss} *is a strictly decreasing function of* λ *and surely smaller than* w^{CRRA} *(no-temptation steady state).*

Proof. See Appendix A. ■

The next proposition shows that if temptation utility is sufficiently convex (large enough χ), then there exists a non-empty range for the temptation parameter λ for which the mapping F defined in (10) maps the interval $[1, \bar{w}]$ onto itself, that is the set is invariant under mapping F, or, globally stable in the sense that all trajectories generated by (10) remain in $[1, \bar{w}]$. Moreover, for that same range, F is single-peaked.

PROPOSITION 4. *Assume that* $\chi > \chi^*$ *(that is, temptation utility is sufficiently convex), where* $\chi^* > \beta^{-1}$ *is the unique solution to the equation* $1 + \chi = \left[\frac{1+\chi(1+\beta)}{1+\chi}\right]^{\chi}$.

Then, there exists a range (λ^*, λ^m), *such that, for any* $\lambda \in (\lambda^*, \lambda^m)$, *the mapping* F *defined in equation (10) has the following properties: (i)* $F(1) = 1 + \beta(1 - \lambda) > 1$ *and* $F(\bar{w}) = 1$; *(ii)* F *is single-peaked at* $w^* \equiv [\lambda(1 + \chi)]^{-\frac{1}{\chi}} \in (1, w^{ss})$, *with* $F'(w_{t+1}) \gtreqless 0$ *for* $w_{t+1} \lesseqgtr w^*$ *and* $F'(w^{ss}) < 0$; *(iii)* $F : [1, \bar{w}] \to [1, \bar{w}]$.

Proof. See Appendix A. ■

Proposition 4 shows that for a degree of temptation λ belonging to the interval (λ^*, λ^m), the nonlinear difference equation (10) has well-defined backward dynamics; that is, for any given $w_{t+1} \in [1, \bar{w}]$, there exists a unique $w_t \in [1, \bar{w}]$. On the contrary, the forward dynamics are not well-defined: since the mapping F is non-invertible over the entire domain $[1, \bar{w}]$, F^{-1} can be a correspondence.[13]

For the purpose of establishing the existence of periodic orbits, studying the backward equilibrium dynamics does not constitute a problem: an N-period cycle is such both going forward and backward. This is indeed the strategy followed in overlapping generations (OLG) and/or monetary models with multi-valued forward dynamics. This methodological equivalence is less obvious when one is interested in chaotic dynamics as sensitive dependence on initial conditions—a

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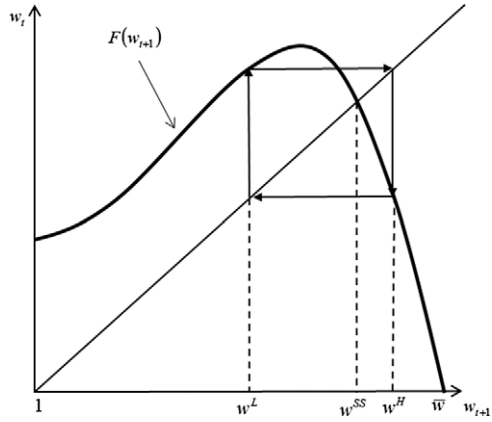


FIGURE 1. Two-period cycle with convex temptation.

key distinguishing component of chaos—goes in opposite direction in forward and backward dynamics. Luckily, Kennedy and Stockman (2008) provide an equivalence result. Using the theory of inverse limits, the authors are able to extend to multi-valued functions some key properties of chaos which were previously defined only for single-valued ones. By doing so, they show that, as long as the backward map is a continuous function on a compact metric space, then the backward map F is chaotic if and only if the associated (multi-valued) forward map F^{-1} is chaotic as well.

Based on these results, I focus the analysis on the backward map F. The single-peakedness and global stability proven by Proposition 4 are necessary but not sufficient conditions for the existence of periodic and chaotic dynamics. The next proposition establishes a sufficient condition for period-2 cycles to occur in this model.

PROPOSITION 5. *Recall the characterization of mapping F in Proposition 4. There exists a threshold* $\lambda^c \in (\lambda^*, \lambda^m)$ *such that if* $\lambda^c < \lambda < \lambda^m$ *the model exhibits a period-2 cycle.*

Proof. See Appendix A. ■

Figure 1 depicts the case of a period-2 cycle. This is defined as a pair $\{w^L, w^H\}$ for $w^L < w^{ss} < w^H$ such that $w^L = F\left(w^H\right)$ and $w^H = F\left(w^L\right)$, or, equivalently, $w^i = F^2\left(w^i\right)$ for $i = L, H$. This result implies that there exist a non-empty parametric range for the degree of temptation λ for which the economy switches deterministically between a high stock price value $p^H = w^H - 1$ and a low stock price value $p^L = w^L - 1$, thus leading to endogenous boom-bust cycles.

To construct a simple economic intuition for the existence of a period-2 cycle under convex temptation utility consider the pricing equation (4), which, under perfect foresight, reads

$$p = \beta\left[1 - \frac{v'\left(p'+1\right)}{u'\left(1\right) + v'\left(1\right)}\right]\left(p'+1\right), \tag{11}$$

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where I have denoted by p and p', respectively, the current and the future stock price. Suppose the economy is initially at the steady-state equilibrium $p^{ss} = w^{ss} - 1$, and that, suddenly, agents believe the stock price will be higher next period: $p' = p^H > p^{ss}$, where again $p^H = w^H - 1$. By the pricing equation (11), the new equilibrium price p is

$$p = \beta \left[1 - \frac{v'\left(p^H + 1\right)}{u'(1) + v'(1)} \right] \left(p^H + 1\right). \tag{12}$$

With v linear or concave, the right-hand side of (12) would be strictly increasing in the future price. This would make the current price p also higher than the steady-state price ($p > p^{ss}$), which would be inconsistent with a period-2 cycle. On the contrary, with a sufficiently convex temptation utility, the right-hand side of (12) becomes strictly decreasing in the future price p'. In this case, a high future price $p' = p^H$ can be consistent with a low current price $p = p^L < p^{ss}$, as well as a low future price can be consistent with a high current price.

The economic intuition for the result is related to our previous discussion on the implications of convex temptation in Section 2. A higher (expected) future stock price, $p' > p^{ss}$, increases (expected) future wealth, which in turn raises the cognitive cost of future self-control: it will be harder for the agent to resist to future temptation if he could liquidate a larger amount of wealth. This makes current savings (respectively, consumption) less (respectively, more) attractive. Hence, the demand for stock shares drops, pushing the current stock price below its steady-state value, $p < p^{ss}$.

Based on the analytical results of Propositions 4 and 5, I construct the orbit bifurcation diagram for the nonlinear difference equation (10) with respect to the degree of temptation λ.[14] For this purpose I set $\beta = 0.96$ and $\sigma = 1$. The panels in Figure 2 presents the results for different degrees of convexity in temptation, as indexed by χ. Stronger temptation can lead to periodic cycles as long as temptation utility is sufficiently convex. For instance, if $\chi = 1$ (i.e., quadratic temptation), the steady state is the unique feasible PFE, for any $\lambda \in [0, 1)$, while period-2 cycles exists for λ larger than, about, 0.3 (when $\chi = 2$) and 0.07 (when $\chi = 3$). As we further increase χ, two interesting results emerge. First, period-2 cycles start occurring at much lower levels of temptation: for instance, for λ around 0.01 if $\chi = 4$, 0.0025 if $\chi = 5$, and 0.0005 if $\chi = 6$. Second, the equilibrium dynamics display also higher-order cycles (it is easy to spot period-4 cycles in both panels d and e, and period-8 cycles in panels e and f) and aperiodic trajectories (the dense shaded areas in panels e and f).

To assess whether these aperiodic trajectories are indeed chaotic, I compute the Lyapunov exponent associated with the dynamic equation (10). Following Definition 1, to be called chaotic, a dynamic system should display sensitive dependence on initial conditions, in the sense that two obits starting from two arbitrarily close initial conditions should separate exponentially fast. Consider, for instance, the following two initial conditions: w_0 and $\tilde{w}_0 = w_0 + \varepsilon_0$, for ε_0 arbitrarily small. Let $|\varepsilon_n|$ be the distance between the two orbits generated by the

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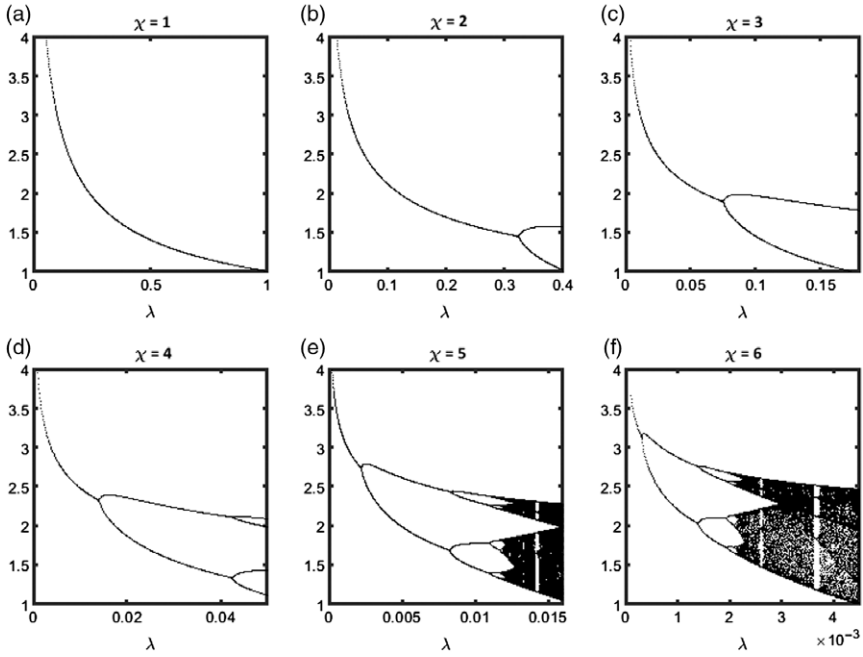


FIGURE 2. Orbit bifurcation diagrams: Convexity and strength of temptation utility.

difference equation (10) after n iterations. If $|\varepsilon_n| \approx |\varepsilon_0|\, e^{\gamma n}$, then γ is called the Lyapunov exponent. A positive γ is an index of chaotic behavior. Figure 3 plots the Lyapunov exponent for the case of $\chi = 5$ and $\chi = 6$, whose orbit bifurcation diagrams provided some visual evidence of chaos.[15] It clearly appears that a positive Lyapunov exponent (hence, chaos) starts occurring for λ between 0.01 and 0.015 when $\chi = 5$, and for λ larger than just 0.002 for $\chi = 6$.

4. STOCHASTIC DYNAMICS

How does the existence of cyclical/chaotic deterministic dynamics affect stock price fluctuations in presence of exogenous disturbances, for example, belief shocks and/or stochastic variation in dividends? I show that for parameterizations yielding deterministic cycles, a Lucas' tree model with temptation can generate (1) nonlinear sunspot-driven REE, that is stock price trajectories that fluctuate stochastically around the steady state despite constant fundamentals, as well as (2) linear REE whereby stock prices display excess volatility with respect to the underlying stochastic fundamentals. I study these two possibilities separately.

4.1. Sunspot Equilibria

In this section, I construct both stationary and nonstationary sunspot equilibria, along the lines of Azariadis and Guesnerie (1986). A sunspot equilibrium is an REE where endogenous variables depend on the realizations of an

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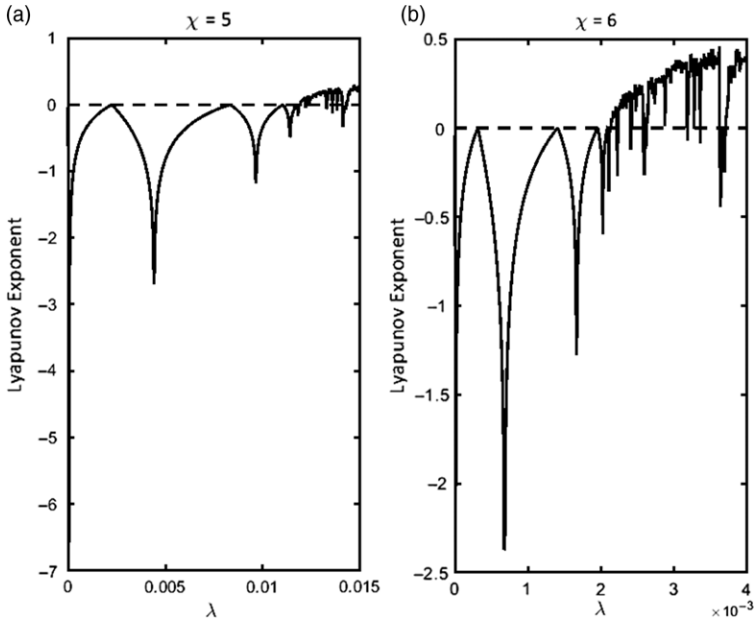


FIGURE 3. Lyapunov coefficient.

extrinsic (nonfundamental) sunspot shock. I start by studying a two-state *stationary* sunspot equilibrium (henceforth, 2-SSE). Consider a two-state sunspot $\zeta \in \{\zeta_1, \zeta_2\}$, with transition probabilities $\pi_{12} = prob\,(\zeta_{t+1} = \zeta_2 | \zeta_t = \zeta_1)$ and $\pi_{21} = prob\,(\zeta_{t+1} = \zeta_1 | \zeta_t = \zeta_2)$. In the context of my model, a 2-SSE is a time-invariant function $w\,(\zeta_t)$ such that $w_t = w_1 \equiv w\,(\zeta_1)$ if $\zeta_t = \zeta_1$ and $w_t = w_2 \equiv w\,(\zeta_2)$ if $\zeta_t = \zeta_2$. More specifically, a 2-SSE is a quadruple $\{w_1, w_2, \pi_{12}, \pi_{21}\}$ solving the following system:

$$w_1 = 1 + \beta \left[\pi_{12} \left(1 - \lambda w_2^{\chi}\right) w_2 + (1 - \pi_{12}) \left(1 - \lambda w_1^{\chi}\right) w_1\right], \tag{13}$$

$$w_2 = 1 + \beta \left[\pi_{21} \left(1 - \lambda w_1^{\chi}\right) w_1 + (1 - \pi_{21}) \left(1 - \lambda w_2^{\chi}\right) w_2\right]. \tag{14}$$

Notice that for $\pi_{12} = \pi_{21} = 1$, this system reduces to

$$w_1 = 1 + \beta \left[\left(1 - \lambda w_2^{\chi}\right) w_2\right] \qquad \text{and} \qquad w_2 = 1 + \beta \left[\left(1 - \lambda w_1^{\chi}\right) w_1\right]. \tag{15}$$

These last two equations define a period-2 cycle (i.e., a trivial 2-SSE), where w_t switches deterministically between the values w_1 and w_2. In this specific case, the conditions for the existence of a 2-SSE coincide with the conditions for the existence of a period-2 cycle, as stated in Proposition 5. It therefore follows that, given a parameterization of the model that satisfies the conditions in Proposition 5, by continuity of the system (13) and (14) in the probabilities π_{12} and π_{21}, there exist a pair w_1, w_2 solving (13) and (14) for some $\pi_{12}, \pi_{21} < 1$.

Figure 4 shows all quadruples $\{w_1, w_2, \pi_{12}, \pi_{21}\}$ solving the systems (13) and (14)—hence, the entire set of 2-SSE—for the case of $\chi = 5$ and $\lambda = 0.005$. We know from the analysis of Section 3 that, absent the sunspot shock, this

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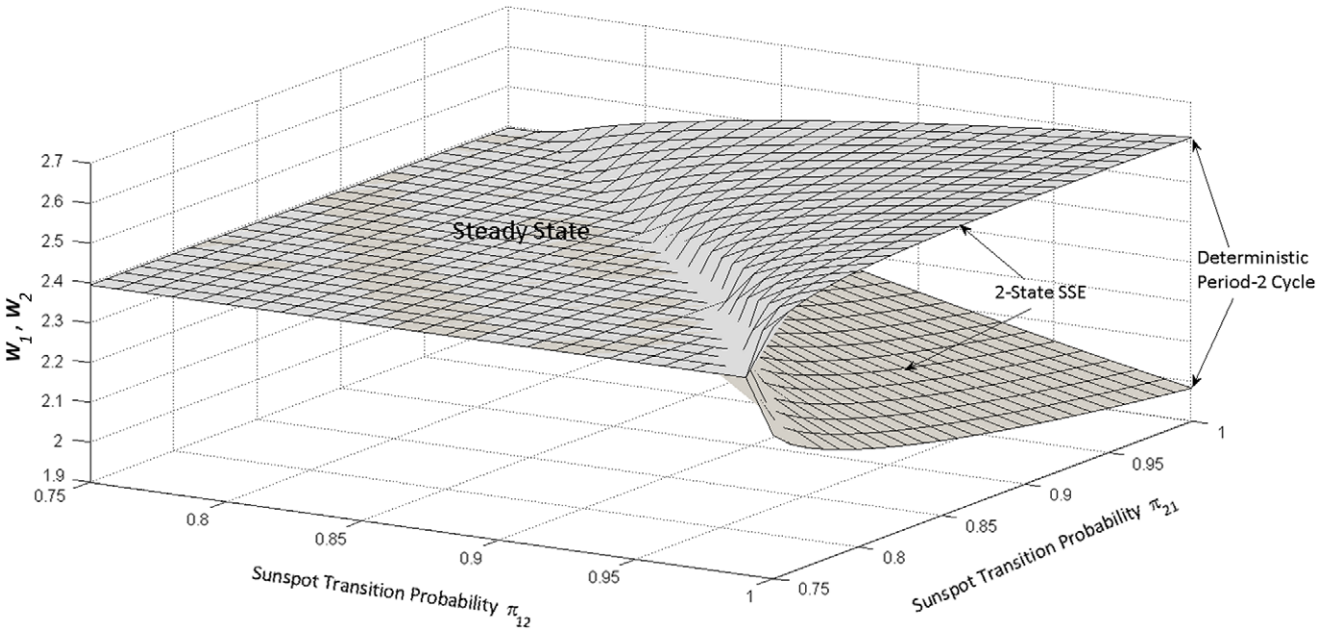


FIGURE 4. Two-state sunspot equilibria.

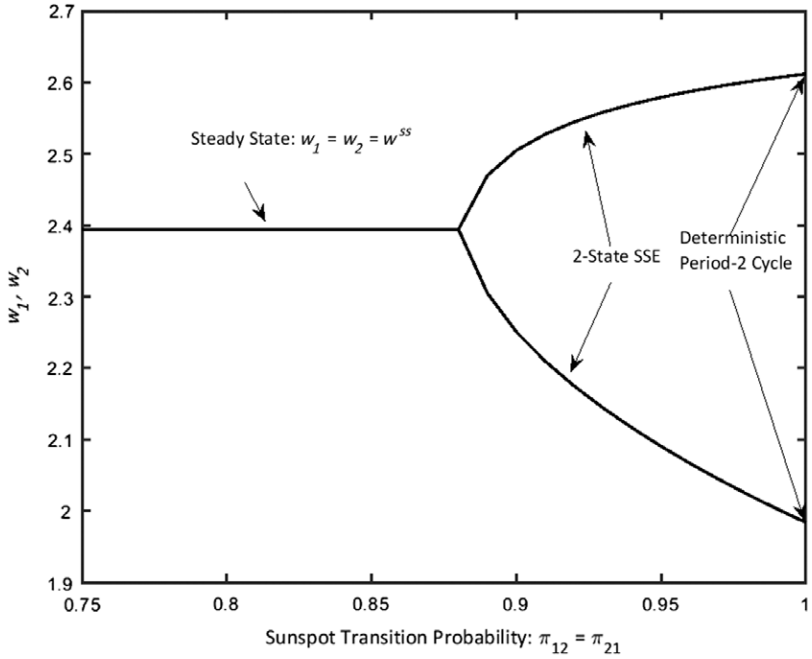


FIGURE 5. Symmetric two-state sunspot equilibria.

parameterization leads to a period-2 cycle (see panel e of Figure 2). For combinations $\{\pi_{12}, \pi_{21}\}$ for which the gray-colored surface is flat, we have that $w_1 = w_2 = w^{ss}$: the unique 2-SSE is the steady state (a trivial one). For combinations $\{\pi_{12}, \pi_{21}\}$ for which the surface splits into an upper and a lower part, there exists a 2-SEE given by a pair $w_2 > w_1$ (with w_2 belonging to the upper part, and w_1 belonging to the lower part). This occurs when the sunspot transition probabilities are sufficiently large. The figure also displays the deterministic period-2 cycle, which, as mentioned, occurs for $\pi_{12} = \pi_{21} = 1$. To get an even better sense of how an SSE looks like, I consider the case of a symmetric two-state sunspot shock: that is, I set $\pi_{12} = \pi_{21} = \pi$. Figure 5 shows that a symmetric two-state SSE exists for π larger than, about, 0.88. As π approaches unity from below, the SSE converges to the deterministic period-2 cycle.

Next, I construct a *nonstationary* sunspot equilibrium (NSSE). For this purpose I set $\chi = 5$ and $\lambda = 0.015$, a parameterization leading to higher-order cycles and chaos in the deterministic case (see panel e of Figure 2 again).[16] From the analysis in Section 3, we know that, under this parameterization, the map F defined in (10) is non-monotonic, hence it is not invertible over the entire admissible range $[1, \bar{w}]$. Letting $G \equiv F^{-1}$, we have in fact that, by the properties of F spelled in Proposition 4, G is single-valued and strictly decreasing for $w_t \in [1, F(1))$, a correspondence for $w_t \in [F(1), F(w^*)]$, and is not defined for $w_t \in (F(w^*), \bar{w}]$.

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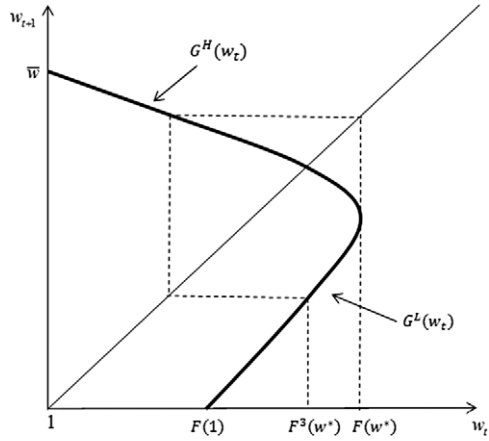


FIGURE 6. Forward map.

Figure 6 displays $w_{t+1} = G(w_t)$. Notice that for $w_t \in [F(1), F(w^*))$, G displays two branches: a strictly increasing lower branch, G^L, and a strictly decreasing upper branch G^H, which collide for $w_t = F(w^*)$, the point where G starts bending backward. From the figure, it also appears that, for any $w_t \in \left[F(1), F^3(w^*)\right)$, if $w_{t+1} = G^L(w_t)$, then $w_{t+2} = G^H(w_{t+1}) > F(w^*)$: the sequence falls outside the admissible range. Based on this observation, I define the interval $I \equiv \left[F^3(w^*), F(w^*)\right)$ and consider a two-state sunspot $\zeta \in \{\zeta_H, \zeta_L\}$, with arbitrary probabilities π and $1 - \pi$. I then construct the following NSSE:

$$w_0 \in I,$$

$$w_t = \begin{cases} G^H(w_{t-1}) \quad (\text{prob. } \pi) \text{ or } G^L(w_{t-1}) \quad (\text{prob. } 1-\pi), & \text{if } w_{t-1} \in I \\ G^H(w_{t-1}), & \text{if } w_{t-1} \notin I \end{cases} \quad \text{for } t \geq 1.$$

Figure 7 displays four alternative NSSE. Each panel assumes a different value for π (the probability of being on the higher branch of G), while each trajectory is generated starting from the same initial condition $w_0 \in I$. If the probability of being on the G^H branch is large ($\pi = 0.75$) or if the two branches are equally likely ($\pi = 0.5$), the NSSE converges to the unique steady-state equilibrium w^{ss} in finite time. This is because the steady state is determined by the intersection of the upper branch with the 45 degree line, and is stable in forward dynamics. However, once the lower branch G^L becomes more likely ($\pi = 0.25$ or $\pi = 0.05$), the NSSE displays much higher volatility and no tendency to convergence. This is because the branch G^L tends (at least for one iteration) to push the trajectory away from the steady state. When this becomes more frequent (lower π), convergence to the steady state is less likely, and the NSSE displays volatile cyclical fluctuations around it.

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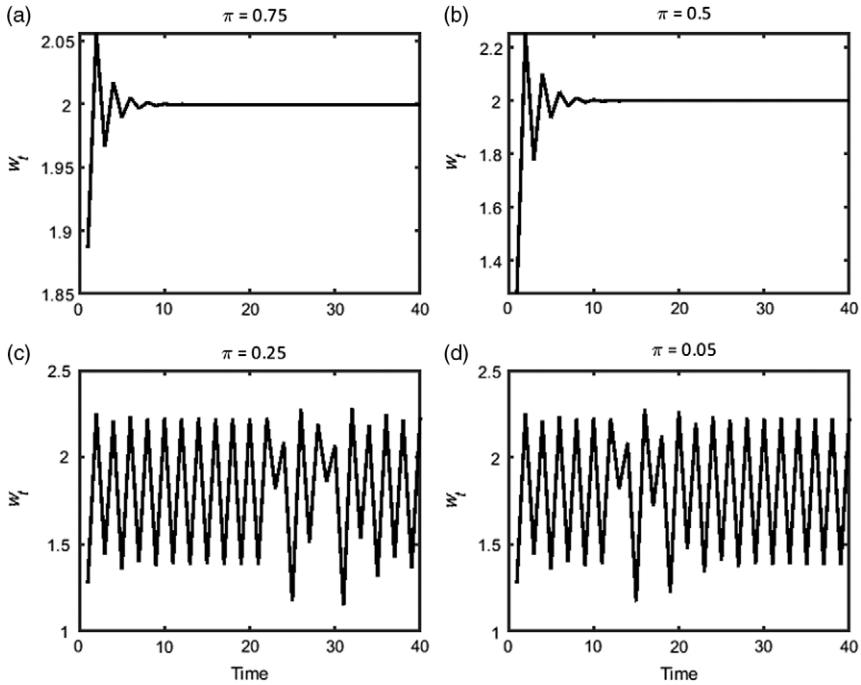


FIGURE 7. Nonstationary sunspot-driven trajectories.

4.2. Stochastic Dividends and Excess Volatility

Under the utility specification in (9), assuming stochastic variation in dividends, the model's dynamics are described by the following expectational difference equation:

$$w_t = d_t + \beta E_t \left[\left(\frac{d_{t+1}^{-\sigma} - \lambda w_{t+1}^{\chi}}{d_t^{-\sigma}} \right) w_{t+1} \right], \tag{16}$$

together with a process for d_t. Simple log-linearization around the unique nonstochastic steady state (where $w_t = w^{ss}$ and $d_t = 1$) gives

$$\hat{w}_t = aE_t\hat{w}_{t+1} + b\hat{d}_t, \tag{17}$$

for $\hat{w}_t \equiv \ln(w_t/w^{ss})$, $\hat{d}_t \equiv \ln d_t$, and

$$a \equiv \beta \left[1 - \lambda (1+\chi) (w^{ss})^{\chi} \right], \tag{18}$$

$$b \equiv \frac{1 + \sigma (w^{ss} - 1)}{w^{ss}} - \beta\sigma\rho > 0. \tag{19}$$

It is straightforward to show that, since $w^{ss} > 1$ (see Proposition 3), we have $b > 0$. Moreover, for $\sigma = 1$, the coefficient b equals $1 - \beta\rho \in (0, 1)$, and hence does not depend on the strength of temptation λ.

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Following standard practice, assume dividends, in deviation from their unconditional mean (normalized to unity, as in the deterministic case), are determined by the following first-order autoregressive, AR(1), stationary process:

$$\hat{d}_t = \rho \hat{d}_{t-1} + \hat{\varepsilon}_t, \qquad \hat{\varepsilon}_t \sim \text{iid}\left(0, \sigma_\varepsilon^2\right), \qquad \rho \in [0, 1). \tag{20}$$

PROPOSITION 6. *Assume temptation utility is sufficiently convex—namely, $\chi > \chi^* > \beta^{-1}$ (with χ^* defined in Proposition 4). Then, there exists a threshold $\lambda^c \in (0, 1)$ such that the linearized dynamic equation (17) satisfies the following properties:*

- *if $0 \leq \lambda < \lambda^c$, there exists a unique (locally determinate) REE, with minimum state variable (MSV) representation*

$$\hat{w}_t = \frac{b}{1 - a\rho} \hat{d}_t; \tag{21}$$

- *if $\lambda^c < \lambda < 1$, there exists a continuum of (locally indeterminate) REE, with the following autoregressive representation:*

$$\hat{w}_t = \frac{1}{a} \hat{w}_{t-1} - \frac{b}{a} \hat{d}_{t-1} + \hat{\eta}_t, \tag{22}$$

where the expectational error $\hat{\eta}_t \equiv \hat{w}_t - E_{t-1}\hat{w}_t$ is any martingale difference sequence for which $E_t \hat{\eta}_{t+1} = 0$.

Proof. See Appendix A. ■

Figure 8 displays the determinacy (white) and indeterminacy (gray) areas with respect to the strength of temptation, λ, and its convexity, χ, under the baseline calibration. The downward-sloping borderline separating the two areas corresponds to the threshold λ^c identified in Proposition 6. Clearly, the REE associated with the linearized model (17) is locally determinate when temptation is not too strong and/or not too convex. Within the determinacy region, the unique REE has a MSV representation as in (21), whereby wealth is linear in its fundamentals (dividends). As such, it inherits their persistence and volatility.

Under indeterminacy, things differ along two dimensions. First, as the only restriction on the expectational error is to have zero conditional mean, it can be conditioned on the realization of any mean-zero *iid* nonfundamental shock ζ_t (a sunspot), with finite variance σ_ζ^2. Since there are infinitely many of such processes, there exist infinitely many REE with a stationary AR(1) representation as in (22). Second, current wealth depends on its past realization, $\hat{w}_{t-1}$. This introduces an endogenous source of persistence and shock amplification, which is likely to lead to excess stock price volatility.

By close inspection of the results stated in Propositions 5 and 6, one can see that the lower bound threshold for the existence of 2-period cycles, λ^c, corresponds to the local indeterminacy boundary. As a result, in the presence of deterministic cycles, it is possible to construct linear REE displaying excess persistence and

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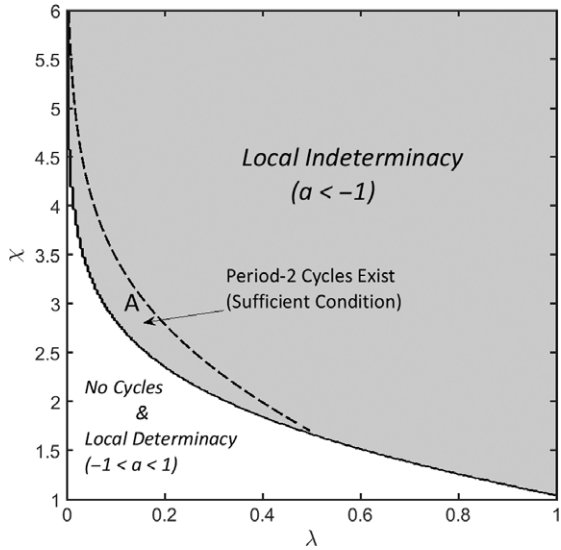


FIGURE 8. Cycles and equilibrium determinacy/indeterminacy.

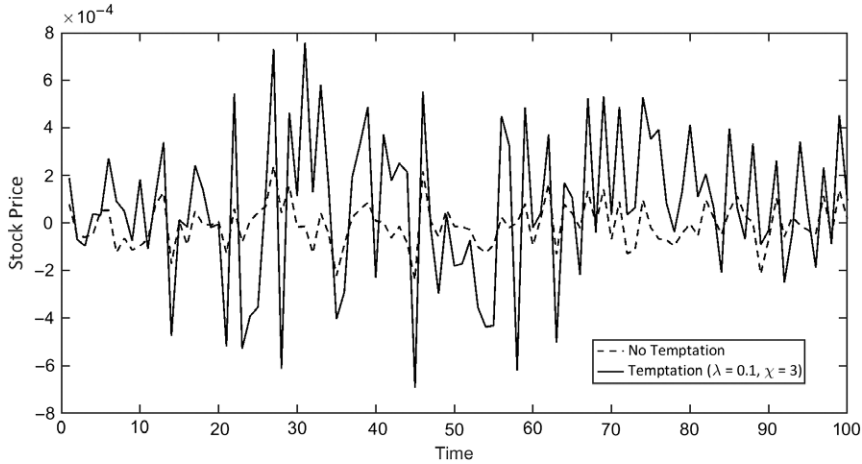


FIGURE 9. Simulated stock price dynamics.

volatility with respect to the underlying stochastic fundamentals. This occurs for parameterizations falling within the region labeled A in Figure 8.[17]

To get a better sense of the model's potential to generate excess volatility, I pursue the following simulation exercise. I draw 1000 observations for (log) dividends from the process (20), assuming $\rho = 0$ (hence, an *iid* process) and $\sigma_\varepsilon^2 = 0.01$. I feed the same series to the baseline model without temptation ($\lambda = 0$)—for which the unique linear REE is defined as in (21)—as well

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as to a version of the model featuring 2-period deterministic cycles ($\lambda = 0.1$ and $\chi = 3$, see panel c) in Figure 2)—for which there exists a continuum of REE with an AR(1) form as in (22).[18] Figure 9 displays the last 100 observations from the two simulations, where, in order to focus on the endogenous amplification of fundamental uncertainty, I have set the forecast error η_t equal to zero at all times (no sunspot-driven volatility).[19] The figure displays the (log) stock price defined as $\hat{p}_t = \frac{w^{ss}}{w^{ss}-1}\hat{w}_t - \frac{\hat{d}_t}{w^{ss}-1}$. Clearly, for given path of exogenous dividends, the presence of cyclical deterministic dynamics (possible under convex temptation) leads to excess volatility with respect to a no-temptation model where the unique deterministic equilibrium is the steady state.[20]

5. CONCLUSIONS

This paper studies the *global* PFE dynamics implied by a deterministic Lucas' tree asset pricing model where the representative agent has temptation and DSC preferences, as defined by Gul and Pesendorfer (2004). As shown, DSC preferences make the stock pricing kernel a function of future stock prices, which in turn introduces nonlinearities in the equilibrium stock price equation. I show this can lead to endogenous cycle of period 2 and higher, as well as chaotic dynamics. Hence, under these preferences, stock prices can display time variation even if the underlying fundamentals are constant. Once exogenous shocks (to either fundamentals or nonfundamentals) are introduced, for parameterizations yielding deterministic cycles, it is possible to construct REE displaying excess persistence and volatility.

NOTES

1. DSC preferences provide an interesting alternative to quasi-hyperbolic discounting models à-la Laibson (1997), where individual preferences are time-inconsistent.

2. DeJong and Ripoll (2007) also introduce DSC utility into a stochastic Lucas' tree model and study their consequences for the equity premium and stock price volatility. However, because of *local* log-linear approximation, their analysis overlooks the *global* nonlinear dynamics I am highlighting in this paper.

3. The introduction of DSC preferences is certainly not the only way to make the pricing kernel of a risky asset a function of financial wealth. Viable alternatives include prospect theory [Barberis et al. (2001)], Epstein-Zin preferences [Bansal and Yaron (2004)], and Max Weber's spirit of capitalism hypothesis [Bakshi and Chen (1996), Kamihigashi (2008), and Airaudo (2017)].

4. Recent contributions have studied the macroeconomic implications of DSC preferences for what concerns over-borrowing [Nakajima (2012)], social security [Kumru and Thanopoulos (2011)], and the Friedman rule [Hiraguchi (2018)].

5. I will consider the case of stochastic time variation in dividends in Section 4.

6. No short-selling is allowed.

7. The axioms in Gul and Pesendorfer (2004) hold for both the case of concave and convex temptation utilities.

8. See Devaney (2003) and Strogatz (1994) for an introduction to chaotic dynamical systems.

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9. The same result holds for the case of linear temptation.
10. These two conditions always hold for isoelastic (power) functions.
11. For $\chi \leq 0$, temptation utility would be weakly concave.
12. In Huang et al. (2015), temptation utility is assumed to take the following form: $v(c_t) = \lambda u(c_t)$. Since they consider economies with positive trend growth, this assumption guarantees the existence of a unique balanced-growth-path (BGP)—whereby the price–dividend ratio is constant—which, in turn, allows them to resort to log-linear techniques. This is not strictly necessary here since (a) I focus on a zero-growth economy, and (b) at least for what concerns the nonlinear analysis, I do not have to log-linearize the model around a stationary equilibrium. Moreover, as argued by Lettau and Van Nieurburgh (2008) and Favero et al. (2011), the standard assumption of stationarity in the (log) price–dividend ratio—hence, a long-run BGP—does not find strong support in the data.
13. This configuration is not new in the literature on economic dynamics. For instance, it often appears in works studying the global equilibrium dynamics of OLG models [Grandmont (1985) and Azariadis and Guesnerie (1986)], money-in-the-utility function models [Fukuda (1993)], or cash-in-advance models [Michener and Ravikumar (1998)].
14. The orbit bifurcation diagram displays all possible long-run (asymptotic) outcomes (steady state(s), periodic equilibria, chaotic attractors, etc..) of a dynamic system as a function of a specific bifurcation parameter. See Lorenz (1993) and Strogatz (1994) for a detailed description.
15. The Lyapunov coefficient has been computed using the E&F Chaos software by Diks et al. (2008). The plots have been done with Matlab.
16. While an NSSE exists also for $\lambda = 0.005$—that is, for the exact same parameterization used for the SSE case—I find that it always converges to the steady state in finite time. The case of $\lambda = 0.015$ delivers more interesting dynamics.
17. The upper dashed line delimiting region A corresponds to the threshold λ^m of Proposition 5. Beyond that, the existence of periodic cycles is not guaranteed.
18. This parameterization falls inside region A in Figure 8.
19. Obviously, allowing for sunspot shocks would make the simulated series under temptation even more volatile.
20. Although I present here only one simulation, this pattern holds true across different ones, as long as the parameterization of λ and χ remains inside region A. For the case displayed ($\sigma = 1$, $\rho = 0$), it is possible to show that $Var\left(\hat{p}_t\right) > Var(\hat{d}_t)$—where the latter corresponds to the variance of stock prices without temptation—provided $a \in (-\sqrt{2}, -1)$, which holds inside region A.

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APPENDIX A

APPENDIX A.1: PROOF OF PROPOSITION 2

First of all, notice that since the stock price cannot be negative, $p_t \geq 0$, we need to consider PFE sequences for which $w_t \geq d$. By equation (7), this requires that $\alpha = u'(d) + \lambda z'(d) \geq \lambda z'(w_{t+1})$. By Assumption 1 and the fact that z is strictly concave for any $w_{t+1} \geq d$, this inequality always holds. Let $F(w_{t+1})$ denote the right-hand side of (8). Simple calculus shows that $F'(w_{t+1}) > 0$ (by concavity of z), $F(d) > d$ (by Assumption 1), $\lim_{w_{t+1} \to +\infty} F'(w_{t+1}) = \beta$ (for $\lim_{w \to +\infty} z' + wz'' = 0$), and $F''(w_{t+1}) > 0$ (for $2z'' + wz''' < 0$).

It immediately follows that there exists a unique steady-state solution $w^{ss} > d$ to the equation $w = F(w)$. To show that indeed $w^{ss} < \frac{d}{1-\beta}$, the steady state implied by the benchmark Lucas' tree model without temptation, we simply need to show that $F\left(\frac{d}{1-\beta}\right) < \frac{d}{1-\beta}$. Using the definition of F, after simple algebra, this inequality is equivalent to $-\lambda z'\left(\frac{d}{1-\beta}\right) < 0$. The latter always holds since $z' > 0$.

From these results we can conclude that F is invertible. Hence, there exists a well-defined forward-mapping $w_{t+1} = G(w_t)$, for $G = F^{-1}$, with the following properties: (i) there exists a unique steady state $w^{ss} \in \left(d, \frac{d}{1-\beta}\right)$ such that $w^{ss} = G(w^{ss})$; (ii) $G'(w^{ss}) > 1$, that is, the steady state is dynamically unstable; (iii) $w_t = w^{ss}$ for every $t \geq 0$ is the unique stationary PFE since $w_t \to +\infty$ for any $w_0 > w^{ss}$, while w_t would eventually fall below d for any $w_0 < w^{ss}$.

APPENDIX A.2: PROOF OF PROPOSITION 3

Let $w_t = w_{t+1} = w$ in (10), and define $\bar{w} \equiv \lambda^{-\frac{1}{\chi}}$. As we are assuming $\lambda < 1$, we have that $\bar{w} > 1$. After a simple rearrangement of terms, the steady state is the solution to the following equation:

$$\frac{w-1}{w} = \beta(1 - \lambda w^{\chi}). \tag{A1}$$

Let $LHS(w)$ and $RHS(w)$ denote, respectively, the left- and the right-hand sides of (A1). First of all, recall that, by the nonnegativity of stock prices, we must have $w \geq 1$, which, by equation (A1), requires $w \leq \bar{w}$. Hence, feasibility requires $w \in [1, \bar{w}]$. Simple calculus shows that $LHS(1) = 0$ and $LHS'(w) > 0$, while $RHS(1) = \beta(1-\lambda) \in (0, \beta)$, $RHS(\bar{w}) = 0$, and $RHS'(w) < 0$. Hence, there exists a unique $w^{ss} \in (1, \bar{w})$ solving (A1). Moreover, it is straightforward to verify that $w^{ss} < w^{CRRA} \equiv (1-\beta)^{-1}$ (as the latter is larger than unity), and that $\frac{\partial w^{ss}}{\partial \lambda} < 0$.

APPENDIX A.3: PROOF OF PROPOSITION 4

The proof is build on the following Lemma.

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LEMMA 1. *Let* λ^*, λ^c, λ^m, *and* λ^f *be defined as follows:*

$$\lambda^* \equiv \frac{1}{1+\chi}\left[\frac{1+\chi\,(1-\beta)}{1+\chi}\right]^{\chi}, \qquad \lambda^f \equiv \frac{1}{(1+\chi)}, \tag{A2}$$

$$\lambda^m \equiv \left[1-\frac{\chi\beta}{1+\chi}\left(\frac{1}{1+\chi}\right)^{\frac{1}{\chi}}\right]^{\chi}, \quad \lambda^c \equiv \frac{1+\beta}{(1+\chi)\,\beta}\left[\frac{2+\chi\,(1-\beta)}{1+\chi}\right]^{\chi}. \tag{A3}$$

Assume that $\chi > \chi^*$. *Then, the following relationships hold: (a)* $\lambda^* < \lambda^c < 1$ *and (b)* $\lambda^* < \lambda^m < \lambda^f < 1$.

Proof. First, consider the following equation:

$$1+\chi = \left[\frac{1+\chi\,(1+\beta)}{1+\chi}\right]^{\chi}. \tag{A4}$$

Taking logs of both sides, (A4) becomes

$$\ln(1+\chi) = \chi \ln[1+\chi\,(1+\beta)] - \chi \ln(1+\chi). \tag{A5}$$

Let $LHS(\chi)$ and $RHS(\chi)$ denote, respectively, the left- and the right-hand sides of (A5). Simple calculus gives the following properties for $LHS(\chi)$: (i) $LHS'(\chi) > 0$; (ii) $LHS(0) = 0$ and $LHS'(0) = 1$; (iii) $LHS''(\chi) < 0$ with $\lim_{\chi\to\infty} LHS'(\chi) = 0$. Hence, $LHS(\chi)$ is a strictly increasing and strictly concave function. For what concerns $RHS(\chi)$, we instead have (i) $RHS'(\chi) > 0$; (ii) $RHS(0) = RHS'(0) = 0$; (iii) $RHS''(\chi) > 0$. Hence, $RHS(\chi)$ is a strictly increasing and strictly convex function. From these properties, it clearly follows that there exists a unique $\chi^* > 0$ solving (A5), and therefore (A4). Moreover, simple but tedious algebra shows that $\chi^* > \beta^{-1}$.

Assume that $\chi > \chi^*$. Consider the thresholds λ^* and λ^f defined in (A2). Given that $\chi > \chi^* > 0$, and since $\left[\frac{1+\chi(1-\beta)}{1+\chi}\right]^{\chi} \in (0,1)$, it immediately follows that $\lambda^* < \lambda^f < 1$. Next, consider the threshold λ^c defined in (A3). Simple algebra shows that, since $\chi > \beta^{-1}$, then $\lambda^* < \lambda^c < 1$. The last thing to show is that $\lambda^* < \lambda^m < \lambda^f$. Simple algebra shows that $\lambda^* < \lambda^m$, while the inequality $\lambda^m < \lambda^f$ can be written as $1+\chi < \left[\frac{1+\chi(1+\beta)}{1+\chi}\right]^{\chi}$. By the properties of the two sides of equation (A5) discussed above, this inequality always holds if $\chi > \chi^*$. ■

Assume temptation utility is sufficiently convex, that is $\chi > \chi^*$. Then, by the above Lemma, the following inequality holds: $\lambda^* < \lambda^m < \lambda^f < 1$. Assume that the temptation parameter λ belong to the interval (λ^*, λ^m). From equation (10) , we need $F(w_{t+1}) \equiv 1+\beta\left[1-\lambda w_{t+1}^{\chi}\right] w_{t+1} \geq 1$ for $w_t \geq 1$ (i.e., for the stock price p_t to be nonnegative). In turn, this requires $w_{t+1} \leq \bar{w} \equiv \lambda^{-\frac{1}{\chi}}$. Hence, for $\lambda \in (\lambda^*, \lambda^m)$, the set of feasible values for w_t is $[1, \bar{w}]$. By simple differentiation of $F(w_{t+1})$ with respect to w_{t+1} we obtain

$$F'(w_{t+1}) \gtreqless 0 \qquad \text{for} \qquad w_{t+1} \lesseqgtr w^* \equiv \left[\frac{1}{\lambda\,(1+\chi)}\right]^{\frac{1}{\chi}}.$$

Simple algebra shows that $w^* \in (1, \bar{w})$ given that $\lambda < \lambda^m < \lambda^f \equiv \frac{1}{1+\chi}$. By the properties of $LHS(w)$ and $RHS(w)$ defined in the proof of Proposition 3, we have that $w^* \gtreqless w^{ss}$ for $RHS(w^*) \lesseqgtr LHS(w^*)$, where w^{ss} is the unique interior steady state. After simple algebra this inequality reduces to the following condition:

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$$\lambda \lessgtr \lambda^* \equiv \frac{1}{1+\chi}\left[\frac{1+\chi\,(1-\beta)}{1+\chi}\right]^{\chi}.$$

Since we are restricting to the case of $\lambda > \lambda^*$, we have that $w^* < w^{ss}$, which also implies that $F'(w^{ss}) < 0$. Simple algebra shows that for $\lambda < \lambda^m \equiv \left[1 - \frac{\chi\beta}{1+\chi}\left(\frac{1}{1+\chi}\right)^{\frac{1}{\chi}}\right]^{\chi}$ we also have $F(w^*) < \bar{w}$. Since $F(w^*)$ is the peak of F, it follows that $F : [1, \bar{w}] \to [1, \bar{w}]$.

APPENDIX A.4: PROOF OF PROPOSITION 5

Recall the definition of λ^c in (A3), and assume that $\lambda^c \in (\lambda^*, \lambda^m)$. Then, define the auxiliary function $H(w) \equiv w - F^2(w)$. A period-2 cycle is a solutions to $H(w) = 0$ which is not the steady state w^{ss} (the latter is a trivial period-2 cycle since $F^2(w^{ss}) = w^{ss}$). By the properties of the mapping F in Proposition 4, we have that $H(1) = 1 - F[F(1)] < 0$ (this is because $F(1) \in (1, \bar{w})$ and then $F[F(1)] \in (1, \bar{w})$ as well) and $H(\bar{w}) = \bar{w} - F^2(\bar{w}) = \bar{w} - F(1) > 0$ (this is because $F(1) < F(w^*) < \bar{w}$). Hence, by continuity of the function $H(w)$ on the domain $[1, \bar{w}]$, a sufficient condition for the existence of a period-2 cycle is that $H'(w^{ss}) < 0$, or, equivalently, $F'(w^{ss}) < -1$. More explicitly, we need

$$\beta\left[1 - \lambda\,(1+\chi)\,(w^{ss})^{\chi}\right] < -1.$$

Simple rearrangement of terms gives

$$w^{ss} > \left[\frac{1+\beta}{\beta\lambda\,(1+\chi)}\right]^{\frac{1}{\chi}}. \tag{A6}$$

Let $\breve{w} \equiv \left[\frac{1+\beta}{\beta\lambda(1+\chi)}\right]^{\frac{1}{\chi}}$. From the proof of Proposition 3, recall that the steady state w^{ss} is the unique solution to the following equation:

$$\frac{w-1}{w} = \beta\,(1 - \lambda w^{\chi}). \tag{A7}$$

Let $LHS(w)$ and $RHS(w)$ denote the left- and the right-hand sides of (A7). It is straightforward to show the inequality (A6) holds if and only if $LHS(\breve{w}) < RHS(\breve{w})$. As simple algebra shows, the latter can be written as follows:

$$\lambda > \lambda^c \equiv \frac{1+\beta}{(1+\chi)\,\beta}\left[\frac{2+\chi\,(1-\beta)}{1+\chi}\right]^{\chi}.$$

APPENDIX A.5: PROOF OF PROPOSITION 6

Consider the dynamic equation (16):

$$\hat{w}_t = aE_t\hat{w}_{t+1} + b\hat{d}_t.$$

With $\hat{w}_t$ being a non-predetermined variable, the REE associated with the linearized model (17) is locally determinate (respectively, indeterminate) when the value of a falls inside (respectively, outside) the unit circle. As $\chi > 0$, we know that for sure $a < 1$. It remains to establish under which conditions $a \gtrless -1$. Since a corresponds to the slope of the deter-

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ministic nonlinear map F (as defined in (10)) evaluated at the steady state, from the proof of Proposition 5, we know that

$$a \gtreqless -1 \qquad \text{if and only if} \qquad \lambda \lesseqgtr \lambda^c \equiv \frac{1+\beta}{(1+\chi)\,\beta}\left[\frac{2+\chi\,(1-\beta)}{1+\chi}\right]^{\chi}.$$

Moreover, from Lemma 1, provided temptation utility is sufficiently convex, that is $\chi > \chi^* > \beta^{-1}$, we have that $\lambda^c < 1$.

Suppose $0 \leq \lambda < \lambda^c$, such that $a \in (-1, 1)$ and the REE is locally determinate. Then, by forward iteration on (17), after imposing the appropriate transversality condition, the unique nonexplosive solution is $\hat{w}_t = \frac{b}{1-a\rho}\hat{d}_t$. The latter is also the unique MSV representation of an REE.

Consider now $\lambda^c < \lambda < 1$, such that $a < -1$ and the REE is locally indeterminate. Letting $\hat{\eta}_{t+1} \equiv \hat{w}_{t+1} - E_t\hat{w}_{t+1}$ denote the expectational error, and lagging variables by one period, equation (17) can be written as $\hat{w}_t = \frac{1}{a}\hat{w}_{t-1} - \frac{b}{a}\hat{d}_{t-1} + \hat{\eta}_t$. Clearly, the latter (also known as the canonical form) is a stationary solution to (17) as long as $E_t\hat{\eta}_{t+1} = 0$. It therefore follows that there are infinitely many of such solutions as $\hat{\eta}_t$ is an arbitrary martingale difference sequence.

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