THE WEAK EXTENSION PRINCIPLE

ALESSANDRO VIGNATI AND DENIZ YILMAZ

Abstract. We prove a rigidity result for maps between Čech–Stone remainders under fairly mild forcing axioms.

§1. Introduction. This article focuses on Čech–Stone remainders of locally compact topological spaces, spaces of the form $X^* = \beta X \setminus X$, where βX is the Čech–Stone compactification of X, and maps between them.

A natural way to construct a continuous map between Čech–Stone remainders is to consider the restriction of the compactification of a continuous map between the underlying spaces (perhaps forgetting about compact sets). To be precise, if X and Y are locally compact noncompact spaces, and $V_X \subseteq X$ is an open set with compact closure[1], a proper continuous map $X \setminus V_X \to Y$ extends to a map $\beta(X \setminus V_X) \to \beta Y$ whose restriction to X^* gives a continuous map $X^* \to Y^*$. Does this construction recover all continuous maps between Čech–Stone remainders, at least without appealing to unnatural back-and-forth constructions? This question was first analyzed for the simplest space among all, $\mathbb{N}$ with the discrete topology, and it was given a negative answer: in [2], Dow constructed (in ZFC) a nontrivial copy of $\mathbb{N}^*$ inside $\mathbb{N}^*$ which is nowhere dense, and therefore cannot come from a map $\mathbb{N} \to \mathbb{N}$ as above (further considerations on nontrivial copies of $\mathbb{N}^*$ were recently made in [3, 4]). Conjecturally, this is all there is in ZFC, and under some reasonably weak forcing axioms, all maps between Čech–Stone remainders arise as a blend of two maps as above. This intuition was formalised by Farah, who in [6, Section 4] introduced the *weak Extension Principle*. This principle was stated originally for pairs of zero-dimensional spaces; the following adapts it to the class of locally compact noncompact second countable spaces.

Definition 1.1. Let X and Y be locally compact noncompact second countable topological spaces. We say that X and Y satisfy the weak Extension Principle, and write $\mathsf{wEP}(X, Y)$, if the following happens for all pairs of naturals $d, \ell \geq 1$.

For every continuous map $F\colon (X^*)^d \to (Y^*)^\ell$, there exists a partition into clopen sets $(X^*)^d = U_0 \cup U_1$, an open set with compact closure $V_X \subseteq X$ such that $F[U_0]$

Received August 16, 2024.

2020 *Mathematics Subject Classification.* Primary 03E35, Secondary 03E57, 03E65.

Key words and phrases. Čech–Stone remainders, homeomorphisms, forcing axioms.

[1]The need of V_X is justified, for example, by the fact that the spaces $\mathbb{R}^*$ and $((-\infty, 0] \cup [1, \infty))^*$ are homeomorphic, but no continuous function $\mathbb{R} \to (-\infty, 0] \cup [1, \infty)$ can induce a homeomorphism between them.

© The Author(s), 2025. Published by Cambridge University Press on behalf of The Association for Symbolic Logic.
0022-4812/00/0000-0000
DOI:10.1017/jsl.2025.10124

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

is nowhere dense in $(Y^*)^\ell$, and a continuous function $G\colon (\beta(X \setminus V_X))^d \to (\beta Y)^\ell$, which restricts to F on U_1.

By wEP, we denote the statement "wEP(X, Y) holds whenever X and Y are locally compact, noncompact second countable spaces".

The weak Extension Principle cannot be a consequence of ZFC, as witnessed by the existence of nontrivial autohomeomorphisms of $\mathbb{N}^*$ under the Continuum Hypothesis CH [20]. More in depth, by a classical result of Parovičenko [19], CH implies that if X is a zero-dimensional locally compact noncompact second countable space, then X^* is homeomorphic to $\mathbb{N}^*$. All of these nontrivial homeomorphisms witness the failure of wEP. Further, again under CH, all Čech–Stone remainders (of second countable spaces) are continuous images of $\mathbb{N}^*$, and all connected ones are continuous images of $[0, 1)^*$. A simple counting argument shows that most of these maps cannot arise from continuous maps from $\mathbb{N}$ (or $[0, 1)$) to the locally compact space of interest. In general, CH and back-and-forth constructions make it essentially impossible to classify maps between Čech–Stone remainders.

Is it then possible that wEP is consistent? Instances of wEP were analysed first for $X = Y = \mathbb{N}$, and more generally to study maps between finite powers of $\mathbb{N}^*$ in different models of set theory (see [5, 8, 15, 16, 22] and [23], or [9] for more related questions). Farah, while formally introducing the wEP, showed under some fairly mild forcing axioms (which are Todorcevič's OCA and Martin's Axioms $\mathsf{MA}_{\aleph_1}$), wEP(X, Y) holds for pairs of countable locally compact noncompact spaces. Further results were obtained in [13] and [24], focusing on just homeomorphisms instead of general continuous maps. For a lengthy and detailed treatment on how forcing axioms impact the structure of homomorphisms between quotient structures, see [12].

The main result of this article is that the weak Extension Principle holds unconditionally on the spaces of interest, again under the same forcing axioms assumptions.

THEOREM 1.2. *Assume* OCA *and* $\mathsf{MA}_{\aleph_1}$. *Then* wEP *holds.*

As a consequence, we show that the thesis of [7, Theorem 5.1] holds (under OCA and $\mathsf{MA}_{\aleph_1}$) for all pairs of locally compact noncompact second countable spaces: if κ and λ are cardinals with $\kappa < \lambda$ then there is no continuous surjection $(X^*)^\kappa \to (Y^*)^\lambda$. This result does not hold under CH, as under this assumption $\mathbb{N}^*$ continuously surjects onto every compact topological space of weight $\leq \mathfrak{c}$, and specifically onto every finite power of every Čech–Stone remainder of a locally compact second countable topological space, while $[0, 1)^*$ continuously surjects onto every finite power of every such connected Čech–Stone remainder.

The proof of our main result proceeds as follows: first, we use a result of Farah [7] on maps between powers of Čech–Stone remainders (more generally, $\beta\mathbb{N}$-spaces) to simplify the situation and show that it is enough to show wEP for maps, where $d = \ell = 1$ (Proposition 2.2). Secondly, we rely on Gel'fand's duality, and study *-homomorphisms between coronas of abelian C*-algebras. We build on a strong lifting theorem proved in [24] (see Proposition 2.3) which gives that, under forcing axioms, *-homomorphisms between abelian corona C*-algebras have well-behaved liftings on certain blocked subalgebras. The main content of our proof is to glue and

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

analyse such liftings. This approach is fairly different from the one of [6, Section 4], where Boolean algebras of clopen sets were used. Once the dimension increases, Stone duality is replaced by Gel'fand's one, as clopen sets do not recover completely the topology in this case.

For all background material on C*-algebras theory, we refer the reader to [1], or to [11] for the connections between C*-algebras and set theory. For notions in topology, we refer to the standard book of Munkres [18] and to [14] more specifically for Čech–Stone compactifications and remainders.

§2. Proof of the main result. This main section is fully dedicated to the proof of Theorem 1.2. As mentioned, our first step is to allow us to only consider maps between Čech–Stone remainders. This is due to a powerful result of Farah. Some context: If $d \geq 1$, a function between topological spaces $F\colon X^d \to Y$ is said to depend on at most one coordinate on $U \subseteq X^d$ if there exists $j \leq d$ and $H\colon X \to Y$ such that $F \restriction U = H \restriction \pi_j(U)$, $\pi_j\colon X^d \to X$ being the projection on the j-th coordinate. A rectangle in X^d is a set of the form $\prod_{i\leq d} A_i$, where $A_i \subseteq X$. The following is a reformulation of [7, Theorem 3][2].

THEOREM 2.1. *Let K be a compact space, Y be locally compact, noncompact, and second countable, $d \geq 1$, and let $F\colon K^d \to Y^*$ be a continuous map. Then K^d can be covered by finitely many clopen rectangles on each of which F depends on at most one coordinate.*

PROPOSITION 2.2. *Suppose the* wEP *holds for all continuous maps between Čech–Stone remainders. Then the* wEP *holds.*

PROOF. The statement of the proposition asserts that if we can prove the weakening of the wEP where we only consider maps between Čech–Stone remainders, i.e., $d = \ell = 1$, then the wEP holds.

At first, we show that we can assume $\ell = 1$. For this, assume the wEP holds for all maps $(X^*)^d \to Y^*$ (where X and Y satisfy the appropriate hypotheses, and $d \geq 1$ is arbitrary). Let $F\colon (X^*)^d \to (Y^*)^\ell$ be a continuous map, and, for $i \leq \ell$, compose F with the projection on the i-th coordinate maps. This gives continuous maps $F_i = \pi_i \circ F\colon (X^*)^d \to Y^*$. Applying the weak Extension principle to these maps, we get, for $i \leq \ell$, a clopen decomposition $(X^*)^d = U_0^i \cup U_1^i$ and an open set with compact closure V_X^i such that $F_i[U_0^i]$ is nowhere dense and there is a continuous $G_i\colon (\beta X \setminus V_X^i)^d \to \beta Y$ which restricts to F_i on U_1^i. The sets $U_0 = \bigcup_i U_0^i$, $U_1 = \bigcap_i U_1^i$, $V_X = \bigcup_i V_X^i$ and the map $G\colon (\beta X \setminus V_X^i)^d \to (\beta Y)^\ell$ sending x to $(G_1(x), \dots, G_\ell(x))$ witness that F satisfies the wEP.

Let us now suppose we have a continuous map of the form $F\colon (X^*)^d \to Y^*$, and apply Theorem 2.1. Then there exists a finite cover of $(X^*)^d$ by clopen rectangles $R_1, \dots, R_n$. Let $R_i = A_1^i \times \cdots \times A_d^i$. Since each R_i is clopen in the compact space $(X^*)^d$, we can assume each $A_j^i \subseteq X^*$ is clopen, hence we can find open sets $Z_{i,j} \subseteq X$ such that $A_j^i = Z_{i,j}^*$. For every i, let $j(i)$ be such that $F \restriction R_i$ depends on the $j(i)$-th coordinate. Hence we can find a continuous $H_i\colon A_{j(i)}^i = Z_{i,j(i)}^* \to Y^*$. Applying

[2]Theorem 3 in [7] works even if d is an infinite cardinal. Further, the codomain space Y needs to be a $\beta\mathbb{N}$-space, and not necessarily a remainder space.

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

the wEP to each H_i, for $i \leq n$, we get open sets with compact closure $V_i \subseteq Z_{i,j(i)}$, partitions into clopen sets $Z^*_{i,j(i)} = U^i_0 \cup U^i_1$ and continuous maps $G_i \colon \beta(Z_{i,j(i)} \setminus V_i) \to \beta Y$ lifting H_i on U^i_1. For $i \leq n$, define

$$\tilde{G}_i \colon \beta Z_{i,1} \times \cdots \times \beta(Z_{i,j(i)} \setminus V_i) \times \cdots \times \beta Z_{i,d} \to \beta Y$$

by mapping a tuple to the G_i image of its $j(i)$-th entry. Fix $V_X = \bigcup V_i$. Gluing all of these maps together gives a continuous lift on

$$U_1 = \bigcup_{i \leq n} (Z^*_{i,1} \times \cdots \times \underbrace{U^i_1}_{j(i)} \times \cdots \times Z^*_{i,d})$$

while the F-image of the complement of U_1 is a finite union of the nowhere dense sets $F[Z^*_{i,1} \times \cdots \times U^i_0 \times \cdots \times Z^*_{i,d}]$. This concludes the proof. ⊣

From now on, we only deal with continuous maps between Čech–Stone remainders (i.e., we assume that $d = \ell = 1$ in the definition of the wEP). We approach these via Gel'fand's duality, by looking at the C*-algebras $C(X^*)$ and $C(Y^*)$ and at *-homomorphisms between them. In this article, we only consider abelian C*-algebras, i.e., algebras of continuous functions on locally compact topological spaces where the topology is induced by the sup-norm and operations are defined pointwise. For an introduction and basic terminology, see [1, Chapter II.2].

Algebras of the form $C(X^*)$ are typical examples of corona C*-algebras ([1, Chapter II.7] or [11, Section 13]). The structure of *-homomorphisms between corona algebras and their behaviour under forcing axioms was extensively studied in [17, 24], after the use of forcing axioms to analyse massive quotients of C*-algebras was popularised by the seminal [10].

If X and Y are locally compact noncompact second countable topological spaces then Gelf'and's duality gives that $C(\beta X) \cong C_b(X)$, the latter being the algebra of bounded continuous functions on X, and that $C(X^*) \cong C_b(X)/C_0(X)$, where $C_0(X)$ is the algebra of continuous functions on X vanishing at infinity. We denote by π_X the canonical quotient map $\pi_X \colon C_b(X) \to C(X^*)$.

Since our spaces are second countable and locally compact, they are σ-compact (alternatively, the C*-algebras $C_0(X)$ and $C_0(Y)$ are σ-unital). The following notation is set up to match the hypotheses of [24, Theorem 4.3]. We let

- Y_n be an increasing sequence of open subsets of Y such that $Y = \bigcup Y_n$, each Y_n has compact closure, and $\overline{Y}_n \subseteq Y_{n+1}$ for all $n \in \mathbb{N}$. We assume each $Y_{n+1} \setminus \overline{Y}_n$ is not empty, and that $Y_0 = \emptyset$.
- $U^e_n = Y_{10n+7} \setminus \overline{Y}_{10n}$ and $U^o_n = Y_{10n+12} \setminus \overline{Y}_{10n+5}$.

The letters e and o are for even and odd partitions. Recall [1, Chapter II.3.4] that if A is a C*-algebra, a C*-subalgebra $B \subseteq A$ is hereditary if for all positive $a \leq b$, $b \in B$ implies that $a \in B$, where $a \in A$ is positive if $0 \leq a$ (in the abelian setting, a takes only values in $[0, \infty)$).

We have the following properties:

(1) for every $n \neq m$ and $i \in \{e, o\}$, $\overline{U}^i_n \cap \overline{U}^i_m = \emptyset$;
(2) for each $i \in \{e, o\}$, $\prod_n C_0(U^i_n)$ is a hereditary C*-subalgebra of $C(\beta Y)$ such that $(\prod_n C_0(U^i_n)) \cap C_0(Y) = \bigoplus_n C_0(U^i_n)$.

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

(3) $C_b(Y) = \prod_n C_0(U_n^e) + \prod_n C_0(U_n^o)$, and consequently

$$C(Y^*) = \prod_n C_0(U_n^e)/\bigoplus_n C_0(U_n^e) + \prod_n C_0(U_n^o)/\bigoplus_n C_0(U_n^o).$$

Note that the sum above is not a direct sum, as $U_n^e \cap U_n^o \neq \emptyset$ for all n.

If $i \in \{e, o\}$ and $f \in \prod_n C_0(U_n^i)$, since the sets U_n^i, for $n \in \mathbb{N}$, are pairwise disjoint, we can view f as a sequence (f_n), where $f_n \in C_0(U_n^i)$. If $S \subseteq \mathbb{N}$, we write f_S for the element of $\prod_n C_0(U_n^i)$ defined by

$$(f_S)_n = \begin{cases} f_n & \text{if } n \in S \\ 0 & \text{else.} \end{cases}$$

For fixed $S \subseteq \mathbb{N}$ and $i \in \{e, o\}$, the map $f \mapsto f_S$ gives a *-homomorphism $\prod_n C_0(U_n^i) \to \prod_{n\in S} C_0(U_n^i)$, dual to the inclusion $\bigcup_{n\in S} U_n^i \subseteq \bigcup_n U_n^i$.

Recall, a *-homomorphism between nonunital C*-algebras $\psi\colon A \to B$ is nondegenerate [1, Chapter II.7.3.8] if the ideal generated by the image of ψ is dense (in norm topology). This is equivalent to ask that approximate unit for A is sent to an approximate unit of B. In the abelian setting, Gel'fand's duality asserts that nondegenerate *-homomorphisms are dual to proper continuous maps. (Recall, a map between topological spaces is proper if inverse images of compact subsets are compact.)

The following is derived from [24]. For the reader's convenience, we shortly sketch its proof, referring heavily to the notation of [24].

PROPOSITION 2.3. *Assume* OCA *and* $\mathsf{MA}_{\aleph_1}$. *Suppose that* $\Phi\colon C(Y^*) \to C(X^*)$ *is a* **-homomorphism. Then there are a natural* $\bar{n}$, *open subsets of* X *with compact closure* V_n^e *and* V_n^o, *for* $n \geq \bar{n}$, *and nondegenerate* **-homomorphisms*

$$\psi_n^i\colon C_0(U_n^i) \to C_0(V_n^i) \text{ for } i \in \{e, o\}$$

such that:

(1) *For* $i \in \{e, o\}$ *and* $n \neq m$, $\overline{V_n^i} \cap \overline{V_m^i} = \emptyset$, *and, if* $n > \bar{n}$,

$$\overline{V_n^e} \subseteq V_{n-1}^o \cup V_n^e \cup V_n^o, \ \overline{V_n^o} \subseteq V_n^e \cup V_n^o \cup V_{n+1}^e.$$

(2) *Every compact subset of* X *intersects only finitely many of the sets* $\{V_n^i \mid n \geq \bar{n}, i \in \{e, o\}\}$.

(3) *For* $i \in \{e, o\}$,

$$\Psi^i = \sum_{n\geq\bar{n}} \psi_n^i\colon \prod_{n\geq\bar{n}} C_0(U_n^i) \to \prod_{n\geq\bar{n}} C_0(V_n^i)$$

is a **-homomorphism such that for every* $f = (f_n) \in \prod_{n\geq\bar{n}} C_0(U_n^i)$ *there is a nonmeager ideal* $\mathcal{I}_f \subseteq \mathcal{P}(\mathbb{N})$ *such that*

$$\pi_X(\Psi^i(f_S)) = \Phi(\pi_Y(f_S))$$

for all $S \in \mathcal{I}_f$, *and*

(4) *If* $\gamma_n^i\colon V_n^i \to U_n^i$ *is the proper continuous map dual to* ψ_n^i, *then* γ_n^e *and* γ_m^o *(equivalently, the* **-homomorphisms* ψ_n^e *and* ψ_m^o*) agree on their common domain.*

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

PROOF. Write X as an increasing union of nonempty open sets $X = \bigcup X_n$, where for each n, we have that $\overline{X_n} \subseteq X_{n+1}$, $X_{n+1} \setminus \overline{X_n}$ is nonempty, and each $\overline{X_n}$ is compact. We furthermore ask that if $K \subseteq X$ is compact then $K \subseteq X_n$ for some n. This can be done since X is σ-compact. Writing B for $C_0(X)$, we let $e_n^B \in C_0(X)$ be a positive contraction[3] such that $e_n^B[\overline{X_n}] = 1$ and $e_n^B[X \setminus X_{n+1}] = 0$, so that $e_{n+1}^B e_n^B = e_n^B$ for all n.

We are ready to construct the sets V_n^i. For convenience, we only give the details on how to construct the sets V_n^e; the 'odd' sets can be constructed in the same way after appropriately re-indexing.

Let e_n^A be an approximate identity of positive contractions for $A = C_0(Y)$ such that $e_n^A[\overline{Y}_n] = 1$ and $e_n^A(y) \neq 0$ if and only if $y \in Y_{n+1}$, and let $J_n = [10n - 1, 10n + 7]$, so that the algebra A_{J_n} given in [24, Notation 3.3] is precisely $C_0(U_n^e)$. Applying the results of Sections 3 and 4 of [24], and specifically Lemma 4.2 therein, we get mutually orthogonal[4] functions $\alpha_n \colon C_0(U_n^e) \to C_0(X)$ such that the function

$$\Gamma^e = \sum \alpha_n \colon \prod_n C_0(U_n^e) \to C_b(X)$$

has the property that for every $f = (f_n) \in \prod_n C_0(U_n^e)$ we have a nonmeager ideal $\mathcal{I}_f$ such that

$$\pi_X(\Gamma^i(f_S)) = \Phi(\pi_Y(f_S))$$

for all $S \in \mathcal{I}_f$. The mutually orthogonal α_ns come with two sequences of naturals $j_n < k_n$ such that the range of α_n is contained in $(e_{k_n}^B - e_{j_n}^B)C_0(X)(e_{k_n}^B - e_{j_n}^B)$ and $\lim_n j_n = \infty$. This is the key property in showing condition (2), as it gives that if $K \subseteq X$ is compact then the set

$$\{n \mid \exists f \in C_0(U_n^e) \exists x \in K \, (\alpha_n(f)(x) \neq 0)\}$$

is finite.

We will now slightly modify the maps α_n, which themselves are not *-homomorphisms. By Ulam stability for approximate maps between abelian C*-algebras (a result proved by Sěrml in [21], see also [24, Section 5] or, for details, [12, Section 5.4 and Theorem 5.18]) shows that we can find $\bar{n}$ and mutually orthogonal *-homomorphisms $\psi_n^e \colon C_0(U_n^e) \to (e_{k_n}^B - e_{j_n}^B)C_0(X)(e_{k_n}^B - e_{j_n}^B)$, for $n \geq \bar{n}$ such that $\Psi^i = \sum \psi_n^e$ still lifts elements of $\prod_n C_0(U_n^e)$ on nonmeager ideals. The sets V_n^e are now constructed from the maps ψ_n^e as in [24, Section 4.1][5], by setting

$$V_n^e = \bigcup_{f \in C_o(U_n^e)} \operatorname{supp}(\psi_n^e(f)).$$

Since $\lim_n j_n = \infty$ and $V_n^e \subseteq X_{k_n} \setminus X_{j_n}$, if a K is compact then K can intersect at most finitely many of the V_n^es. This shows condition (2). The other properties are

[3] A contraction in a C*-algebra is an element whose norm is ≤ 1.

[4] Meaning that if $n \neq m$ then for all $f \in C_0(U_n^e)$ and $f' \in C_0(U_m^e)$ we have $\alpha_n(f)\alpha_m(f') = 0$.

[5] Injectivity of the maps ψ_n^e is stated as an assumption there, but it is not used there.

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

verified exactly as in Section 4.1 in [24]. This concludes the construction of the sets V_n^e, and the proof of the proposition. ⊣

If f and g are elements of $C_b(X)$, we write $f =_{C_0(X)} g$ for $f - g \in C_0(X)$.

Proof of Theorem 1.2. By Proposition 2.2, it is sufficient to prove wEP for maps between Čech–Stone remainders (i.e., when $d = \ell = 1$). Fix then a continuous function $F\colon X^* \to Y^*$, and let $\Phi\colon C(Y^*) \to C(X^*)$ be the dual *-homomorphism. Let $\tilde{\Phi}\colon C_b(Y) \to C_b(X)$ be any set theoretic lift of Φ. All of our notation will be as in Proposition 2.3 and the discussion preceding it.

Let

$$V_1 = \bigcup_{n\geq \bar{n}} V_n^e \cup V_n^o \text{ and } W = \bigcup_{n\geq \bar{n}} U_n^e \cup U_n^o.$$

Claim 2.4. *V_1 is, modulo compact, clopen.*

Proof. We show that $\overline{V}_1 \setminus V_1$ is contained in the compact set $\overline{V}^e_{\bar{n}}$. Combining conditions (1) and (2) of Proposition 2.3 gives that

$$\overline{V_1} = \overline{\bigcup_{n\geq \bar{n}} V_n^e \cup V_n^o} = \bigcup_{n\geq \bar{n}} \overline{V_n^e} \cup \bigcup_{n\geq \bar{n}} \overline{V_n^o} \subseteq \overline{V_{\bar{n}}^e} \cup \bigcup_{n\geq \bar{n}} V_n^e \cup V_n^o,$$

where the equality $\overline{\bigcup_{n\geq \bar{n}} V_n^e \cup V_n^o} = \bigcup_{n\geq \bar{n}} \overline{V_n^e} \cup \bigcup_{n\geq \bar{n}} \overline{V_n^o}$ is given by the fact that compact subsets of X can only intersect finitely many of the sets $V_n^e \cup V_n^o$. This shows that $\overline{V}_1 \setminus V_1$ is contained in the compact set $\overline{V_{\bar{n}}^e}$. ⊣

(As a side note, a deep understanding of [24, Section 4.1] shows that if the spaces of interest can be written as a union of pairwise disjoint compact open sets, V_1 can be chosen to be clopen. We shall not need this fact.)

As the continuous proper maps γ_n^i agree on their common domains, the function

$$\Gamma = \bigcup_{n\geq \bar{n}} \gamma_n^e \cup \gamma_n^o\colon V_1 \to W$$

is a well-defined continuous function. Furthermore, Γ is proper: if $K \subseteq W$ is compact, then there exists $\bar{n}'$ such that $K \subseteq \bigcup_{\bar{n}\leq n\leq \bar{n}'} U_n^e \cup U_n^o$, hence $\Gamma^{-1}[K]$ is a closed subset of $\bigcup_{\bar{n}\leq n\leq \bar{n}'} V_n^e \cup V_n^o$. As the latter has compact closure, $\Gamma^{-1}[K]$ is compact. Since Γ is proper, it induces a *-homomorphism

$$\Psi\colon C_b(W) \to C_b(V_1)$$

mapping $C_0(W)$ to $C_0(V_1)$ and whose dual is $\beta\Gamma\colon \beta V_1 \to \beta W \subseteq \beta Y$.

We are ready to define the necessary sets and maps to prove the wEP: first, let $V_X \subseteq X$ be an open set with compact closure such that V_1 is clopen in $X \setminus V_X$, and set $V_0 = (X \setminus V_X) \setminus V_1$. Note that $\beta(X \setminus V_X) = \beta V_1 \sqcup \beta V_0$ and therefore $U_0 := V_0^*$ and $U_1 := V_1^*$ is a clopen partition of X^*, by compactness of $\overline{V_X}$. Pick a point in $y \in \beta Y$, let $G'\colon \beta V_0 \to \{y\}$ be the constant map, and let $G = \beta\Gamma \cup G'$.

We are now going to show that these objects satisfy wEP(X, Y), that is, that $G \restriction U_1 = F$ and that $F[U_0]$ is nowhere dense.

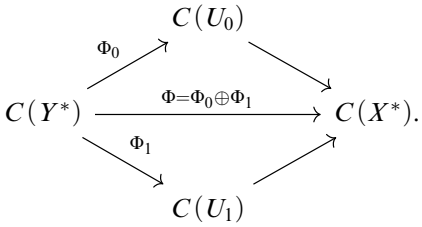
Claim 2.5. $G \restriction U_1 = F$.

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

PROOF. First, write Φ as $\Phi = \Phi_0 \oplus \Phi_1$, where

$$\Phi_i(f) = \chi_{U_i}\Phi(f),\ i \in \{0,1\},$$

$\chi_{U_i} \in C(X^*)$ being the characteristic function of U_i. We have the following diagram:



Both Φ_0 and Φ_1 are *-homomorphisms, as $C(X^*)$ is abelian. To show that $G \restriction U_1 = F$, it is enough to show that the *-homomorphism Ψ lifts Φ_1.

Since every function $f \in C_b(Y)$ can be written (modulo $C_0(Y)$) as a sum $f = f_e + f_o$ where, for $i \in \{e,o\}$, $f_i \in \prod_{n\geq\bar{n}} C_0(U_n^i)$, it is enough to show Ψ lifts Φ_1 on $\prod_{n\geq\bar{n}} C_0(U_n^i)$, again for $i \in \{e,o\}$. For convenience, let $i = e$, and fix mutually orthogonal positive contractions $g_n \in C_b(X)$ such that for all $n \geq \bar{n}$

$$g_n[V_n^e] = 1 \text{ and } \operatorname{supp}(g_n) \subseteq V_{n-1}^o \cup V_n^e \cup V_n^o.$$

Let $g_n = 0$ if $n < \bar{n}$. Such functions exist as distinct V_n^es have disjoint closures. For $S \subseteq \mathbb{N}$, let $g_S = \sum_{n\in S} g_n$. Note that, even if V_1 is not clopen, it is clopen modulo compact, and therefore the function χ_{V_1} is equal, modulo $C_0(X)$, to a continuous function.

SUBCLAIM 2.6. *For every $f \in \prod_{n\geq\bar{n}} C_0(U_n^e)$ and $S \subseteq \mathbb{N}$, we have that*

$$g_S\tilde{\Phi}(f) =_{C_0(X)} \chi_{V_1}\tilde{\Phi}(f_S).$$

PROOF. Fix $f \in \prod_{n\geq\bar{n}} C_0(U_n^e)$ and $S \subseteq \mathbb{N}$. We can suppose S is infinite, otherwise both g_S and $\tilde{\Phi}(f_S)$ belong to $C_0(X)$. We will show that for every $\varepsilon > 0$ there is a compact set K such that

$$\text{if } (x \in V_1 \setminus K \text{ and } |\tilde{\Phi}(f_S)(x)| > \varepsilon) \text{ then } x \in \bigcup_{n\in S} V_n^e. \tag{2.1}$$

This gives that $g_S\tilde{\Phi}(f_S) =_{C_0(X)} \chi_{V_1}\tilde{\Phi}(f_S)$. As S is arbitrary, the same reasoning applied to $\mathbb{N}\setminus S$, gives that $g_S\tilde{\Phi}(f_{\mathbb{N}\setminus S}) \in C_0(X)$. Altogether, we get that

$$\chi_{V_1}\tilde{\Phi}(f_S) =_{C_0(X)} g_S\tilde{\Phi}(f_S) =_{C_0(X)} g_S\tilde{\Phi}(f_S) + g_S\tilde{\Phi}(f_{\mathbb{N}\setminus S}) =_{C_0(X)} g_S\tilde{\Phi}(f),$$

which is the claim.

Suppose (2.1) fails for $\varepsilon > 0$. Then there is a sequence of points $x_k \in V_1 \setminus (\bigcup_{n\in S} V_n^e)$ such that $|\tilde{\Phi}(f_S)(x_k)| > \varepsilon$ and $x_k \to \infty$ as $k \to \infty$ (meaning for every compact $K \subseteq X$ we have that $K \cap \{x_k\}$ is finite). Passing to a subsequence, we can assume that there is $i \in \{e,o\}$ such that $\{x_k\} \subseteq \bigcup_n V_n^i$. Let n_k be such that $x_k \in V_{n_k}^i$. As $x_k \to \infty$ as $k \to \infty$, we can assume, by further shrinking the sequence $\{x_k\}$, that $\{n_k\}$ is strictly increasing and that $n_0 > \bar{n}$.

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

Since the homomorphisms ψ^i_n are nondegenerate, then for all $n > \bar{n}$ and $i \in \{e, o\}$,

$$\begin{aligned} V^i_n &= \bigcup \{\mathrm{supp}(\psi^i_n(h)) \mid h \in C_0(U^i_n)\} \\ &= \{x \in X \mid \exists h \in C_0(U^i_n), 0 \leq h \leq 1, \psi^i_n(h)(x) = 1\} \\ &= \{x \in X \mid \exists h \in C_0(U^i_n), 0 \leq h \leq 1, \Psi(h)(x) = 1\}. \end{aligned}$$

Hence, for each k, we can find positive contractions $h_{n_k} \in C_0(U^i_{n_k})$ such that

$$\psi^i_{n_k}(h_{n_k})(x_k) = \Psi(h_k)(x_k) = 1.$$

Since $x_k \notin \bigcup_{n\in S} V^e_n = \Gamma(\bigcup_{n\in S} U^e_n)$, we can further assume that $\mathrm{supp}(h_{n_k}) \cap \bigcup_{n\in S} U^e_n = \emptyset$.

Set $h_j = 0$ if $j \notin \{n_k \mid k \in \mathbb{N}\}$, and let

$$h = \sum h_k \in \prod_{n\geq \bar{n}} C_0(U^i_n).$$

Notice that for every $T \subseteq \mathbb{N}$ we have $\Psi(h_T)(x_k) = 1$ whenever k is such that $n_k \in T$ and $h_T f_S = 0$, since $\mathrm{supp}(f_S) \subseteq \bigcup_{n\in S} U^e_n$ which is disjoint from $\mathrm{supp}(h) \supseteq \mathrm{supp}(h_T)$.

Let $\mathcal{I}_h$ be the nonmeager ideal given to us from Proposition 2.3, and let T be an infinite set in $\mathcal{I}_h$ such that $T \subseteq \{n_k\}$, so that $\Psi(h_T) =_{C_0(X)} \tilde{\Phi}(h_T)$. Since $|\tilde{\Phi}(f_S)(x_k)| > \varepsilon$ then

$$\limsup_{k\to\infty} |\tilde{\Phi}(f)\tilde{\Phi}(h_T)(x_k)| = \limsup_{k\to\infty} |\tilde{\Phi}(f)\Psi(h_T)(x_k)| \geq \varepsilon.$$

Since $x_k \to \infty$ as $k \to \infty$, this contradicts that $\tilde{\Phi}(f)\tilde{\Phi}(h_T) \in C_0(X)$, as $\tilde{\Phi}$ lifts the *-homomorphism Φ and h_T and f_S are orthogonal. ⊣

By the subclaim, $g_{\mathbb{N}}\tilde{\Phi}(f)$ is a lift for $\Phi_1(f)$. Let

$$\mathcal{J}_f = \{S \subseteq \mathbb{N} \mid g_S(\Psi(f) - \tilde{\Phi}(f)) \in C_0(X)\}.$$

Since $g_S\Psi(f) = \Psi(f_S)$ for all $S \subseteq \mathbb{N}$, and $\Psi(f_S) =_{C_0(X)} \tilde{\Phi}(f_S)$ whenever $S \in \mathcal{I}_f$, then $\mathcal{I}_f \subseteq \mathcal{J}_f$, and the latter is therefore nonmeager and contains all finite sets. Since the association $S \mapsto g_S$ is continuous (when $C_b(X)$ is given the strict topology), $C_0(X)$ is Borel in $C_b(X)$, and $\Psi(f)$ and $\tilde{\Phi}(f)$ are fixed, $\mathcal{J}_f$ is a Borel ideal. Since the only Borel nonmeager ideal containing Fin is $\mathcal{P}(\mathbb{N})$, $\mathcal{J}_f = \mathcal{P}(\mathbb{N})$, hence $g_{\mathbb{N}}(\Psi(f) - \tilde{\Phi}(f)) \in C_0(X)$. Since $g_{\mathbb{N}}\Psi(f) = \Psi(f)$ and $g_{\mathbb{N}}\tilde{\Phi}(f)$ lifts $\Phi_1(f)$, we have that $\Psi(f)$ lifts $\Phi_1(f)$, which is our thesis. The claim is then proved. ⊣

Claim 2.7. *$F \restriction U_0$ has nowhere dense range.*

Proof. Since F is a continuous map between compact spaces, it is closed, hence so is $F[U_0]$. Suppose for a contradiction that there is a nonempty open $U \subseteq F[U_0] \subseteq Y^*$. Let $O \subseteq \beta Y$ be open such that $O \cap Y^* = U$, and let $O' = O \cap Y$. Since the sets $\{U^i_n \mid i \in \{e, o\}, n \in \mathbb{N}\}$ cover Y, we can find an infinite $S \subseteq \mathbb{N}$ and $i \in \{e, o\}$ such that $O' \cap U^i_n \neq \emptyset$ for all $n \in S$. For $n \in S$, let h_n be a positive contraction whose support is in U^e_n and such that there is $x \in O'$ with $h_n(x) = 1$. Let $h = \sum h_n$. Note that for every infinite $T \subseteq S$ we have that $\left\|\Phi(h_T)\chi_{U_0}\right\| = 1$.

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

Find $T \subseteq S$ infinite such that Ψ lifts Φ on h_T, and notice that $\chi_{U_0}\pi_X(\Psi(h_T)) = 0$. This is a contradiction. ⊣

Claims 2.5 and 2.7 together show that G, U_0, and U_1 satisfy the hypotheses of $\mathsf{wEP}(X, Y)$. This concludes the proof of the theorem.

The following is a consequence of the result of Farah.

COROLLARY 2.8. *Assume* OCA *and* $\mathsf{MA}_{\aleph_1}$. *Let* $\kappa < \lambda$ *be cardinals, and suppose that* X *and* Y *are locally compact noncompact second countable spaces. Then there is no continuous surjection* $(X^*)^\kappa \to (Y^*)^\lambda$.

PROOF. This is a consequence of Theorem 5.1 in [7], which asserts that the thesis holds if we have wEP. The result then follows from Theorem 1.2. ⊣

The wEP has consequences on the existence of surjections from Čech–Stone remainders to powers of them: while CH implies that $\mathbb{N}^*$ surjects onto every compact space of density $\aleph_1$, and therefore onto every power of remainder of every (reasonable small) space, the same cannot happen if wEP holds, where, for example, that $(\mathbb{N}^*)^d$ surjects onto $(\mathbb{N}^*)^\ell$ if and only if $d \geq \ell$ (see [6, Section 4.6]). In future work, we intend to explore these phenomena, along with extension principles in the noncommutative setting.

Acknowledgments. The authors would like to thank I. Farah for useful conversations, and the anonymous referee for helpful comments.

Funding. This research originated from D.Y.'s Master's thesis, written under the guidance of A.V. at Université Paris Cité. In this period, D.Y. was supported by an INSMI (Institut National des Sciences Mathématiques et de leurs Interactions) grant for master's students.

REFERENCES

[1] B. BLACKADAR, ***Operator Algebras***, Encyclopaedia of Mathematical Sciences, 122, Springer-Verlag, Berlin, 2006, Theory of C*-algebras and von Neumann algebras, Operator Algebras and Noncommutative Geometry, III.

[2] A. DOW, *A non-trivial copy of* $\beta\mathbb{N} \setminus \mathbb{N}$. ***Proceedings of the American Mathematical Society***, vol. 142 (2014), no. 8, pp. 2907–2913.

[3] ———, *Non-trivial copies of* $\mathbb{N}^*$. ***Topology and its Applications***, vol. 355 (2024), Article no. 109008, 18 pp.

[4] ———, *Autohomeomorphisms of pre-images of* $\mathbb{N}^*$. ***Topology and its Applications***, vol. 368 (2025), Article no. 109348, 14pp.

[5] A. DOW and K. P. HART, ω^* *has (almost) no continuous images*. ***Israel Journal of Mathematics***, vol. 109 (1999), pp. 29–39.

[6] I. FARAH, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*. ***Memoirs of the American Mathematical Society***, vol. 148 (2000), no. 702, xvi+177.

[7] ———, *Dimension phenomena associated with* $\beta\mathbb{N}$*-spaces*. ***Topology and its Applications***, vol. 125 (2002), pp. 279–297.

[8] ———, *Powers of* $\mathbb{N}^*$. ***Proceedings of the American Mathematical Society***, vol. 130 (2002), pp. 1243–1246.

[9] ———, *The fourth head of* $\beta\mathbb{N}$, ***Open Problems in Topology II*** (E. Pearl, editor), Elsevier, Amsterdam, 2007, pp. 135–142.

[10] ———, *All automorphisms of the Calkin algebra are inner*. ***Annals of Mathematics (2)***, vol. 173 (2011), pp. 619–661.

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press

[11] ———, ***Combinatorial Set Theory and C*-Algebras***, Springer Monographs in Mathematics, Springer, Cham (Switzerland), 2019.

[12] I. Farah, S. Ghasemi, A. Vaccaro, and A. Vignati, *Corona rigidity*, ***The Bulletin of Symbolic Logic***. Published online 2025: 1–88. doi: 10.1017/bsl.2025.10084.

[13] I. Farah and P. McKenney, *Homeomorphisms of Čech–Stone remainders: the zero-dimensional case*. ***Proceedings of the American Mathematical Society***, vol. 146 (2018), no. 5, pp. 2253–2262.

[14] L. Gillman and M. Jerison, ***Rings of Continuous Functions***, Graduate Texts in Mathematics, 43, Springer-Verlag, New York, 1976, Reprint of the 1960 edition.

[15] W. Just, *The space* $(\omega^*)^{n+1}$ *is not always a continuous image of* $(\omega^*)^n$. ***Fundamenta Mathematicae***, vol. 132 (1989), pp. 59–72.

[16] ———, *A weak version of AT from OCA*. ***MSRI Publications***, vol. 26 (1992), pp. 281–291.

[17] P. McKenney and A. Vignati, *Forcing axioms and coronas of* C^**-algebras*. ***Journal of Mathematical Logic***, vol. 21 (2021), no. 2, Article no. 2150006, 73 pp.

[18] J. R. Munkres, ***Topology***, second ed., Prentice Hall, Inc., Upper Saddle River, NJ, 2000.

[19] I. I. Parovičenko, *A universal bicompact of weight* $\aleph$. ***Soviet Mathematics Doklady***, vol. 4 (1963), pp. 592–592.

[20] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*. ***Duke Mathematics Journal***, vol. 23 (1956), pp. 409–419.

[21] P. Šemrl, *Nonlinear perturbations of homomorphisms on* $C(X)$. ***The Quarterly Journal of Mathematics Oxford Second Series (2)***, vol. 50 (1999), no. 197, pp. 87–109.

[22] E. K. van Douwen, *Prime Mappings, Number of Factors and Binary Operations*, ***Dissertationes Math. (Rozprawy Mat.)***, Polska Akademia Nauk. Instytut Matematyczny. Dissertationes Mathematicae. Rozprawy Matematyczne 199, 1981, p. 35.

[23] J. van Mill, *An introduction to* $\beta\omega$, ***Handbook of Set-Theoretic Topology*** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 503–567.

[24] A. Vignati, *Rigidity conjectures for continuous quotients*. ***Annales Scientifiques de l'École Normale Supérieure (4)***, vol. 55 (2022), no. 6, pp. 1687–1738.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU - PARIS RIVE GAUCHE (IMJ-PRG)
UNIVERSITÉ PARIS CITÉ
INSTITUT UNIVERSITAIRE DE FRANCE
BÂTIMENT SOPHIE GERMAIN
8 PLACE AURÉLIE NEMOURS
75013 PARIS
FRANCE
E-mail: vignati@imj-prg.fr
URL: http://www.automorph.net/avignati

INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE (IRIF),
UNIVERSITÉ PARIS CITÉ
BÂTIMENT SOPHIE GERMAIN
8 PLACE AURÉLIE NEMOURS
PARIS 75013
FRANCE
E-mail: deniz.yilmaz@irif.fr
URL: https://denizyilmaz.fr

https://doi.org/10.1017/jsl.2025.10124 Published online by Cambridge University Press