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Automatic continuity with application to C^*-algebras

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The fact proved by Cleveland [**4**], that the topology of any (non-complete) algebra norm on a C^*-algebra is stronger than the topology of the usual norm, is reencountered as a direct consequence of a theorem, which we prove in this note, stating that homomorphisms from certain non-complete normed (associative) algebras onto some semisimple Banach algebras are automatically continuous.

The statement of this theorem involves classical concepts in the theory of normed algebras. One of them, namely minimality of norm [**2**, **3**, **5**], is of geometric nature and actually we are interested only in its underlying topological content. We recall that a normed algebra $(A, \|\ \|)$ has the property of minimality of the norm if, whenever $|||\ |||$ is an algebra norm on A with $|||\ ||| \leqslant \|\ \|$, we have $|||\ ||| = \|\ \|$. Also we say that $(A, \|\ \|)$ has the property of minimality of norm topology if, whenever $|||\ |||$ is an algebra norm on A with $|||\ ||| \leqslant K\|\ \|$ for some non-negative number K, we have that $|||\ |||$ and $\|\ \|$ are equivalent norms.

PROPOSITION 1. *For* (*associative*) *normed algebras, minimality of the norm implies minimality of norm topology.*

Proof. Assume that the normed algebra $(A, \|\ \|)$ has the property of minimality of the norm and let $|||\ |||$ be an algebra norm on A with $|||\ ||| \leqslant K\|\ \|$ for a suitable non-negative number K. Then the closed unit ball of A for the norm $\|\ \|$ is a $|||\ |||$-bounded subsemigroup of A and therefore, by [**3**], theorem 4·1, there exists an algebra norm p on A which is equivalent to $|||\ |||$ and satisfies $p(a) \leqslant 1$ for every a in A with $\|a\| \leqslant 1$. Since $p \leqslant \|\ \|$ and $(A, \|\ \|)$ has the property of minimality of the norm, we have $p = \|\ \|$. Therefore $|||\ |||$ and $\|\ \|$ are equivalent norms on A.

For later reference we record the following folklore result.

LEMMA 1. *Every C^*-algebra has the property of minimality of the norm.*

Proof. For any algebra norm $|||\ |||$ on a C^*-algebra $(A, \|\ \|)$ we have $\|a\|^2 \leqslant |||a^*|||\ |||a|||$ for all a in A by [**7**], corollary 4·8·4. Therefore, if $|||\ ||| \leqslant \|\ \|$, we have

$$\|a\|^2 \leqslant \|a^*\|\ |||a||| = \|a\|\ |||a|||,$$

so $\|\ \| \leqslant |||\ |||$, and so $\|\ \| = |||\ |||$.

Another concept involved in the statement of our main result is the one of normed Q-algebra. Following the paper by Yood [**9**], in which this notion is widely developed, a normed algebra A is called a Q-algebra if the set of quasi-invertible elements of A is open in A. Call a subalgebra C of an (associative) algebra B a full subalgebra of B if C contains the quasi-inverses of its elements which are quasi-invertible in B. Then

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full subalgebras of Banach algebras are examples of normed Q-algebras. But they are more than examples, because it can be proved that a normed algebra is a Q-algebra (if and) only if it is a full subalgebra of its completion. For an element a in a normed algebra $(A, \|\ \|)$, $r_{\|\ \|}(a)$ (or simply $r(a)$, when confusion is not possible) will denote the 'geometric' spectral radius of a, namely

$$r_{\|\ \|}(a) := \lim \{\|a^n\|^{1/n}\}.$$

In what follows we will use without comment the fact that $(A, \|\ \|)$ is a Q-algebra if and only if for every a in A the 'geometric' spectral radius $r_{\|\ \|}(a)$ agrees with the 'algebraic' spectral radius of a (the sup of $|\lambda|$ for λ in the spectrum of a relative to A): see [9], lemma 2·1. As a first application of this result, a well-known fact on C^*-algebras can be reformulated as follows.

PROPOSITION 2. *If $|||\ |||$ is any algebra norm on a C^*-algebra A, then $(A, |||\ |||)$ is a normed Q-algebra.*

Proof. By [7], theorem 4·8·3·(iii) we have $r_{|||}(a) = r_{\|\ \|}(a)$ for all a in A. But $(A, \|\ \|)$ is a Banach algebra, so $r_{\|\ \|}(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}\,(A, a)\}$. Therefore $r_{|||}(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}\,(A, a)\}$ and $(A, |||\ |||)$ is a Q-algebra, as required.

The proof of our main result is strongly based on the following fundamental lemma by Aupetit. Although not explicitly stated, it is the cornerstone in the proof of [1], theorem 1. Aupetit's proof has been simplified by Ransford (see [6]). If X and Y are Banach spaces and F is a linear mapping from X into Y we denote by $S(F)$ (the separating subspace for F) the set of those y in Y for which there is a sequence $\{x_n\}$ in X such that $0 = \lim\{x_n\}$ and $y = \lim\{F(x_n)\}$.

LEMMA 2 ([1]). *If A and B are complex Banach algebras and F is a homomorphism from A into B, then $r(b) = 0$ for every b in $S(F) \cap F(A)$.*

Now we prove our main result.

THEOREM. *Let A be a complex normed Q-algebra and let B be a semisimple complex Banach algebra with minimality of norm topology. Then every homomorphism from A onto B is continuous.*

Proof. The proofs of many results on Banach algebras depend only on the fact that Banach algebras are normed Q-algebras, and this is the case of [3], proposition 25·10. Therefore, if π is any homomorphism from the normed Q-algebra A onto the semisimple algebra B, $\mathrm{Ker}\,(\pi)$ is a closed ideal of A. But the quotient of a normed Q-algebra by a closed ideal is also a normed Q-algebra, as can easily be deduced from the fact that a normed algebra is a Q-algebra if and only if the 'algebraic' spectral radius is less than the norm ([9], lemma 2·1). Therefore, if $\tilde{\pi}$ denotes the isomorphism from $A/\mathrm{Ker}\,(\pi)$ onto B induced by π and if for b in B we define $|b| := \|\tilde{\pi}^{-1}(b)\|$, then we have that $(B, |\ |)$ is a normed Q-algebra. Now the desired continuity for π is equivalent to the existence of a non-negative number K such that $\|b\| \leqslant K|b|$ for all b in B. To prove this, let $(C, |\ |)$ denote the completion of B under the norm $|\ |$ and let F be the mapping $x \mapsto x$ from $(B, \|\ \|)$ into $(C, |\ |)$. Since F is a homomorphism with dense range, the separating subspace $S(F)$ for F is a closed ideal of $(C, |\ |)$ (by [8], lemma 1·2). Moreover, by [8], lemma 1·3.(i), the mapping $x \mapsto x + S(F)$ from $(B, \|\ \|)$

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into $(C/S(F), |\ |)$ is continuous, i.e. there is a non-negative number M such that $|x+S(F)| \leqslant M\|x\|$ for all x in B. On the other hand, for x in $B \cap S(F)$ we have in view of Lemma 2 that $r_{||}(x) = 0$ and, since $(B, |\ |)$ is a Q-algebra, this implies that x is quasi-invertible in B. Now $B \cap S(F)$ is a quasi-invertible ideal of B, so that $B \cap S(F) \subset \mathrm{Rad}\,(B) = 0$, and thus, if for b in B we define $|||b||| := |b+S(F)|$, then $|||\ |||$ is an algebra norm on B satisfying $|||\ ||| \leqslant M\|\ \|$. Since B has the property of minimality of norm topology, $|||\ |||$ and $\|\ \|$ are equivalent norms on B, so there is a non-negative number K such that

$$\|b\| \leqslant K|||b||| = K|b+S(F)| \leqslant K|b|$$

for all b in B, and the proof is concluded.

COROLLARY ([**4**]). *The topology of any algebra norm on a C*-algebra is stronger than the topology of the usual norm.*

Proof. Let A be a C^*-algebra and let $|||\ |||$ be any algebra norm on A. Then, by Proposition 2, $(A, |||\ |||)$ is a normed Q-algebra and, by Lemma 1 and Proposition 1, $(A, \|\ \|)$ is a semisimple Banach algebra with minimality of norm topology. Therefore, by the above theorem, the mapping $x \mapsto x$ from $(A, |||\ |||)$ onto $(A, \|\ \|)$ is continuous.

Remark. The Corollary has the following suggestive reformulation: if A is a C^*-algebra, then every radially bounded absorbent absolutely convex subsemigroup of A is actually bounded (see [**3**], proposition 1·9).

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