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On the dimension of triangular self-affine sets

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Abstract. As a continuation of a recent work [Bárány *et al*, On the dimension of self-affine sets and measures with overlaps. *Proc. Amer. Math. Soc.* **144** (2016) 4427–4440] of the same authors, in this note we study the dimension theory of diagonally homogeneous triangular planar self-affine iterated function systems.

1. *Introduction*

1.1. *The theme of the paper.*

The dimension theory of self-affine measures and sets is so complicated that even on $\mathbb{R}^2$, in the diagonal case (when the linear part of the mappings from the iterated function system (IFS) consists entirely of diagonal matrices), it is not fully understood. The authors of this note have recently investigated this question [**6**]. Namely, consider the diagonal self-affine IFS on the plane

$$\mathcal{S}^{\text{diag}} := \left\{ S_i^{\text{diag}}(x,\, y) := D_i \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_i \\ v_i \end{pmatrix} \right\}_{i=1}^{N} \quad \text{where } D_i := \begin{pmatrix} c_i & 0 \\ 0 & b_i \end{pmatrix}. \tag{1}$$

The projection of the coordinate axis naturally generates a self-similar IFS on both coordinate axes. Assume that it is not the case that both of the similarity dimensions of these projected self-similar IFS are greater than one. Then the dimension theory of diagonal self-affine systems on $\mathbb{R}^2$ is settled in [**6**], at least for all but a very small set of parameters.

In this note we take a step forward and consider a triangular self-affine IFS. That is, we assume that the linear part of the mappings from the IFS consists entirely of triangular

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matrices (all of them are lower triangular, say). More precisely, let

$$\mathcal{S} := \left\{ S_i(x, y) := T_i \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_i \\ v_i \end{pmatrix} \right\}_{i=1}^{N} \quad \text{where } T_i := \begin{pmatrix} c_i & 0 \\ d_i & b_i \end{pmatrix}. \tag{2}$$

We say that $\mathcal{S}$ is diagonally homogeneous if there exist constants $b, c \in (0, 1)$ such that, for all i, $c_i = c$ and $b_i = b$. We mostly investigate the diagonally homogeneous case (see §§4 and 5). However, in the general case, we give a result in the case where the affinity dimension is smaller than one (see §3).

1.2. *History.* A self-affine iterated function system (IFS) is a finite list of contracting affine mappings on $\mathbb{R}^d$. If we choose a closed ball $B \subset \mathbb{R}^d$ centered at the origin with sufficiently high radius then this ball will be mapped into itself by all the mappings from the IFS. The ellipses obtained by applying the mappings of the IFS, in any particular order, n-times on this ball B are the n-cylinders. As n tends to infinity, the ratio of the lengths of the semi-axis goes to zero. This makes it possible that even if the n-cylinders are pairwise disjoint, in some cases, they are not effective covers of the attractor (which is the set that remains after an infinite number of iterations of the mappings from the IFS on the ball B above).

In 1988 Falconer introduced the notion of affinity dimension [**10**] for self-affine fractals. We obtain it if we replace the 'most economic cover' in the definition of Hausdorff dimension with the most natural cover associated with the n-cylinders. In some sense the affinity dimension is the most natural guess for the Hausdorff dimension of a self-affine attractor. Since the affinity dimension $\dim_{\text{aff}}(\mathcal{F})$ of a self-affine IFS

$$\mathcal{F} := \{ f_i(x) = A_i \cdot \mathbf{x} + \mathbf{t}_i \}_{i=1}^{N} \tag{3}$$

depends only on the linear parts, it remains the same if we vary the translation vectors in $\mathcal{F}$. In 1988 Falconer proved that for almost all translates of a self-affine IFS, the Hausdorff dimension of the attractor and affinity dimension of the IFS are equal, if all of the mappings from the IFS have strong enough contraction. This upper bound on the contractions was originally $1/3$, which was improved 10 years later by Solomyak [**26**] to $1/2$. Solomyak [**26**] also showed that the bound $1/2$ is optimal. For a survey of results before 2014, see [**23**]. There has very recently been some very intensive activity in this field, partially due to the use of Furstenberg–Kifer measure (usually referred to as Furstenberg measure). See [**2–5**, **7**, **14**, **20**].

1.3. *Affinity dimension in the triangular case on* $\mathbb{R}^2$. In the special case of the triangular self-affine IFS, Falconer and Miao [**8**, Corollary 2.6] showed that if all the matrices are (e.g. lower) triangular then the affinity dimension can be given explicitly by a formula which depends only on the diagonal elements of the matrices. This formula is rather simple when we are on the plane. Namely, assume that the self-affine IFS is given by formula (2) and let s_x and s_y be the similarity dimension of the self-similar IFS

$$\mathcal{H} := \{ h_i(x) := c_i x + u_i \}_{i=1}^{N} \quad \text{and} \quad \mathcal{V} := \{ \varphi_i(x) := b_i y + v_i \}_{i=1}^{N}, \tag{4}$$

respectively. Clearly, $\mathcal{H}$ is the horizontal and $\mathcal{V}$ is the vertical projection of the corresponding diagonal system given in the form (1). Let

$$\widehat{s_x} := \min\{s_x, 1\} \quad \text{and} \quad \widehat{s_y} := \min\{s_y, 1\}.$$

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We define d_x and d_y as the solutions of the equations

$$\sum_{i=1}^{N} c_i^{\widehat{s}_x} b_i^{d_x-\widehat{s}_x} = 1 \quad \text{and} \quad \sum_{i=1}^{N} b_i^{\widehat{s}_y} c_i^{d_y-\widehat{s}_y} = 1. \tag{5}$$

Then (cf. [**1**, Theorem 4.1]) the affinity dimension of $\mathcal{S}$ is

$$\dim_{\mathrm{aff}}(\mathcal{S}) = \max\{d_x, d_y\}. \tag{6}$$

We say that the x-direction (y-direction) dominates if $\dim_{\mathrm{aff}}(\mathcal{S}) = d_x$ ($\dim_{\mathrm{aff}}(\mathcal{S}) = d_y$). It follows from the definition of the Hausdorff dimension that the affinity dimension is always an upper bound for the Hausdorff dimension of the attractor (see [**10**]).

1.4. *Notation.* Throughout this paper, all self-affine IFSs on the plane are supposed to be of the form of (2). Without loss of generality we may assume that

$$S_i([0,1]^2) \subset [0,1]^2 \quad \text{for all } i = 1, \ldots, N.$$

As we mentioned above, the attractor of $\mathcal{S}$ is

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i}\in\{1,\ldots,N\}^n} S_{\mathbf{i}}([0,1]^2),$$

where $S_{\mathbf{i}} := S_{i_1} \circ \cdots \circ S_{i_n}$ for $\mathbf{i} = (i_1, \ldots, i_n)$. Let μ be the uniform distribution measure on the symbolic space $\Sigma := \{1, \ldots, N\}^{\mathbb{N}}$. That is, μ is the $(1/N, \ldots, 1/N)$-Bernoulli measure. We define $\Pi : \Sigma \to \Lambda_{\mathcal{S}}$ in the natural way:

$$\Pi(\mathbf{i}) := \lim_{n\to\infty} S_{\mathbf{i}|_n}(0), \quad \mathbf{i} \in \Sigma.$$

The push-forward measure of μ is $\nu := \Pi_*\mu$. The attractor of the self-similar IFS $\mathcal{H}$ introduced in (4) is denoted by $\Lambda_{\mathcal{H}}$. The natural projection generated by $\mathcal{H}$ is

$$\Pi_{\mathcal{H}}(\mathbf{i}) := \lim_{n\to\infty} h_{\mathbf{i}|_n}(0), \quad \mathbf{i} \in \Sigma.$$

Clearly, $\Pi_{\mathcal{H}} = \mathrm{proj}_x \circ \Pi$, where proj_x is the orthogonal projection to the x-axis. The measure on the x-axis generated by μ is

$$\nu_x := (\Pi_{\mathcal{H}})_*\mu = (\mathrm{proj}_x)_*\nu. \tag{7}$$

We now introduce the Furstenberg–Kifer measure. The projection Π_{AF} below will be used to construct a method to check that the transversality condition holds.

1.5. *Furstenberg–Kifer measure.* In this subsection we study the action of our system $\mathcal{S}$ on the projective line, in the case

$$c_i > b_i \quad \text{for all } i = 1, \ldots, N. \tag{8}$$

In particular, the x-direction dominates.

This action can be identified with the action of a simple IFS on the line. Consider the vertical line $\xi := \{(1, z) \in \mathbb{R}^2 : z \in \mathbb{R}\}$ on the plane. We can identify $(1, z) \in \xi$ with $\widetilde{z} \in \mathbb{R}$. With this identification we define the self-similar IFS $\mathcal{F}$ on ξ by

$$\mathcal{F} := \left\{ f_i(\widetilde{z}) := \frac{b_i}{c_i}\widetilde{z} + \frac{d_i}{c_i} \right\}_{i=1}^{N}. \tag{9}$$

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(Recall that in this subsection $c_i > b_i$.) It follows from (8) that all f_i are strict contractions. So we can define the natural projection $\Pi_{AF} : \Sigma \to \xi$ in the usual way:

$$\Pi_{AF}(\mathbf{i}) := \frac{d_{i_1}}{c_{i_1}} + \sum_{k=2}^{\infty} \frac{d_{i_k}}{c_{i_k}} \cdot \prod_{\ell=1}^{k-1} \frac{b_{i_\ell}}{c_{i_\ell}}. \tag{10}$$

The importance of Π_{AF} is as follows: the action of $\{T_i\}_{i=1}^N$ on the projective line is described by the maps $\widetilde{T}_i : \xi \to \xi$,

$$\widetilde{T}_i(\widetilde{z}) := \frac{1}{c_i} \cdot T_i \cdot \begin{pmatrix} 1 \\ z \end{pmatrix}, \tag{11}$$

where $\widetilde{z} \in \xi$ is $\widetilde{z} = (1, z)$. Then

$$\Pi_{AF}(\mathbf{i}) = \widetilde{T}_i(\Pi_{AF}(\sigma \mathbf{i})) \tag{12}$$

is the natural projection for $\{\widetilde{T}_i\}_{i=1}^N$.

As a similar construction first appeared in [**12**], we will call the projection under Π_{AF} of an ergodic measure η defined on Σ the *Furstenberg–Kifer measure* corresponding to η.

1.6. *The transversality condition.* Consider two cylinders $S_{\mathbf{i}|_n}[0, 1]^2$ and $S_{\mathbf{j}|_n}[0, 1]^2$, $i_1 \neq j_1$. They are parallelograms with two vertical sides. Their angle can be defined as the angle between their non-vertical sides. The following condition holds if any two cylinders $S_{\mathbf{i}|_n}[0, 1]^2$ and $S_{\mathbf{j}|_n}[0, 1]^2$, $i_1 \neq j_1$, either are disjoint or have an angle uniformly bounded away from zero.

Definition 1.1. We say that the diagonally homogeneous IFS $\mathcal{S}$ satisfies the transversality condition if is there exists a $K_3 > 0$ such that for every n and for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_1 \neq j_1$ we have

$$\mathrm{proj}_x(S_{\mathbf{i}|_n}[0, 1]^2 \cap S_{\mathbf{j}|_n}[0, 1]^2) < K_3 \cdot b^n. \tag{13}$$

Below we present a natural geometric interpretation of this condition which provides a method to check it. First let us define the IFS $\widetilde{\mathcal{S}}$ which acts on $\mathbb{R}^3$:

$$\widetilde{\mathcal{S}} := \{\widetilde{S}_i(x, y, z) := (S_i(x, y), \widetilde{T}_i(z))\}_{i=1}^N. \tag{14}$$

Now we recall two separation properties of IFSs. For an IFS $\{F_i\}$ we say that it satisfies the strong separation property (SSP) if its natural projection (in the case of $\widetilde{\mathcal{S}}$ this is given by $\mathbf{i} \mapsto (\Pi(\mathbf{i}), \Pi_{AF}(\mathbf{i}))$) is a bijection. This is equivalent to the existence of a non-empty open set V satisfying:

(a) $F_i(V) \subset V$ for all $i = 1, \ldots, N$; and
(b) $\overline{F_i(V)} \cap \overline{F_j(V)} = \emptyset$ for all $i \neq j$,

where $\overline{A}$ means the closure of the set A.

We can define the open set condition (OSC) in an analogous way: an IFS $\{F_i\}$ satisfies the OSC if there exists a non-empty open set V satisfying:

(a) $F_i(V) \subset V$ for all $i = 1, \ldots, N$; and
(b′) $F_i(V) \cap F_j(V) = \emptyset$ for all $i \neq j$.

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LEMMA 1.2.

(i) *If $\widetilde{\mathcal{S}}$ satisfies the SSP then the transversality condition holds for $\mathcal{S}$.*

(ii) *If the transversality condition holds for $\mathcal{S}$ and $S_i([0,1]^2) \subset (0,1)^2$ for all $i = 1, \dots, N$, then $\widetilde{\mathcal{S}}$ satisfies the SSP.*

Proof. First we prove part (i). We will work with long and thin parallelograms. We will call the *principal axis* of a parallelogram the direction of its long side.

When $\mathbf{i} \mapsto (\Pi(\mathbf{i}), \Pi_{AF}(\mathbf{i}))$ is a bijection, the usual compactness argument shows that there exists $\ell > 0$ such that for any two symbolic sequences $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_1 \neq j_1$ either

$$\operatorname{dist}(\Pi(\mathbf{i}), \Pi(\mathbf{j})) > \ell$$

or

$$\operatorname{dist}(\Pi_{AF}(\mathbf{i}), \Pi_{AF}(\mathbf{j})) > \ell.$$

Take some $N > 2(-\log \ell)/\min(-\log c, -\log(b/c))$. In the first case, for $n > N$ the parallelograms $S_{\mathbf{i}|_n}[0,1]^2$, $S_{\mathbf{j}|_n}[0,1]^2$ do not intersect at all. In the second case, they intersect but transversally (the angle between their principal axes is larger than $\ell/2$). In both cases (13) holds. This proves assertion (i).

Now we prove part (ii). Assume now that (13) holds. By the assumption, there exists $\ell > 0$ such that $\Lambda \in (\ell, 1-\ell)^2$. Fix some large n, to be defined later. Fix also some interval I such that $\widetilde{T}_i(I) \subset I$ for all i.

Consider two words $\mathbf{i}|_n, \mathbf{j}|_n$ with different first digits $i_1 \neq j_1$. Assume for the moment that the parallelograms $S_{\mathbf{i}|_n}[\ell/2, 1-\ell/2]^2$, $S_{\mathbf{j}|_n}[\ell/2, 1-\ell/2]^2$ intersect each other. Then the parallelogram $S_{\mathbf{i}|_n}[0,1]^2$ must intersect an internal point of the parallelogram $S_{\mathbf{j}|_n}[0,1]^2$, a point which is at a distance at least $\ell b^n/2$ from the vertical boundary and at a distance at least $\ell c^n/2$ from the horizontal (non-vertical) boundary of $S_{\mathbf{j}|_n}[0,1]^2$. Hence, if (13) holds then the angle between the principal axes of the parallelograms is at least $K_4 = \ell/2K_3$. In particular, if n was so large that $(b/c)^n|I| < K_4/2$ then

$$\widetilde{S}_{\mathbf{i}|_n}([\ell, 1-\ell]^2 \times I) \cap \widetilde{S}_{\mathbf{j}|_n}([\ell, 1-\ell]^2 \times I) = \emptyset. \tag{15}$$

On the other hand, when the parallelograms $S_{\mathbf{i}|_n}[\ell/2, 1-\ell/2]^2$, $S_{\mathbf{j}|_n}[\ell/2, 1-\ell/2]^2$ do not intersect each other then (15) also holds. Hence, it holds for all pairs of words $\mathbf{i}|_n, \mathbf{j}|_n$ with different first digits. This implies that $\widetilde{S}$ satisfies the SSP for the set

$$V = \bigcup_{\mathbf{k}|_{n-1} \in \{1,\dots,N\}^{n-1}} \widetilde{S}_{\mathbf{k}|_{n-1}}((\ell, 1-\ell)^2 \times I^o).$$

This completes the proof of assertion (ii). □

1.7. *Lyapunov exponents, Lyapunov dimension, projection entropy and exponential separation condition.*

Notation 1.3. Let $\Psi = \{\psi_i\}_{i=1}^N$ be a strictly contracting IFS on $\mathbb{R}^d$. Let $\Sigma = \{1, \dots, N\}^{\mathbb{N}}$ be the symbolic space, σ the left-shift operator on Σ, and the natural projection is $\Pi(i_1, i_2, \dots) = \lim_{n\to\infty} \psi_{i_1} \circ \cdots \circ \psi_{i_n}(\underline{0})$. Let $\mathbf{p} := (p_1, \dots, p_N)$ be a probability

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vector. This generates a Bernoulli measure on Σ, which is denoted by $\mathbb{P}_{\mathbf{p}} := \{p_1, \ldots, p_N\}^{\mathbb{N}}$. The corresponding self-similar measure is

$$\nu_{\Psi,\mathbf{p}} := \Pi_* \mathbb{P}_{\mathbf{p}}. \tag{16}$$

Finally, for every $n \geq 1$ we put

$$\Sigma_n := \{1, \ldots, N\}^n \quad \text{and} \quad \Sigma_* := \bigcup_{n=1}^{\infty} \Sigma_n. \tag{17}$$

1.7.1. *Lyapunov exponents and Lyapunov dimension.* The general definition of Lyapunov exponents can be found, for example, in [**27**]. However, in the special case of systems generated by lower triangular matrices like in (2) the vertical direction is preserved by the system. Hence, the Lyapunov exponents can be expressed by the diagonal elements only [**8**].

The Lyapunov dimension of a self-affine measure was introduced in [**16**, Definition 3]. It is always an upper bound on the Hausdorff dimension of the measure. In the special cases we consider in this paper, we can write down a simpler formula for the Lyapunov dimension of $\nu_{\Psi,\mathbf{p}}$.

Example 1.4.

(a) If Ψ is a self-similar IFS on the line then $\nu_{\Psi,\mathbf{p}}$ has one Lyapunov exponent which is defined as

$$\chi_{\Psi,\mathbf{p}} := -\sum_{i=1}^{N} p_i \cdot \log |\psi_i'(0)|. \tag{18}$$

(b) Assume that $\Psi = \mathcal{S}^{\text{diag}}$ is a diagonally self-affine IFS on the plane defined as in (1). Then its horizontal and vertical projections are self-similar IFSs $\mathcal{H}$ and $\mathcal{V}$ on the line of the form (4). In this case, $\nu_{\Psi,\mathbf{p}}$ has two (not necessarily distinct) Lyapunov exponents: $\chi_{\mathcal{H},\mathbf{p}}$ and $\chi_{\mathcal{V},\mathbf{p}}$. We say that the x-direction dominates if the following counter intuitive condition holds:

$$\chi_{\mathcal{H},\mathbf{p}} \leq \chi_{\mathcal{V},\mathbf{p}}. \tag{19}$$

Assuming, for example, that the x-direction dominates, the Lyapunov dimension can be expressed as

$$D(\nu_{\Psi,\mathbf{p}}) = \begin{cases} \dfrac{h(\mathbb{P}_{\mathbf{p}})}{\chi_{\mathcal{H},\mathbf{p}}} & \text{if } h(\mathbb{P}_{\mathbf{p}}) \leq \chi_{\mathcal{H},\mathbf{p}}, \\ \dfrac{h(\mathbb{P}_{\mathbf{p}}) + \chi_{\mathcal{V},\mathbf{p}} - \chi_{\mathcal{H},\mathbf{p}}}{\chi_{\mathcal{V},\mathbf{p}}} & \text{if } \chi_{\mathcal{H},\mathbf{p}} < h(\mathbb{P}_{\mathbf{p}}) \leq \chi_{\mathcal{H},\mathbf{p}} + \chi_{\mathcal{V},\mathbf{p}}, \\ 2\dfrac{h(\mathbb{P}_{\mathbf{p}})}{\chi_{\mathcal{H},\mathbf{p}} + \chi_{\mathcal{V},\mathbf{p}}} & \text{otherwise.} \end{cases} \tag{20}$$

(c) Equation (20) depends only on the entropy and the Lyapunov exponents, which depend on the diagonal elements only. In the general lower triangular case when

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$\Psi = \mathcal{S}$ given by formula (2), the non-diagonal element of the matrix $A_{i_0} \cdots A_{i_n}$ has the form

$$d_{i_0,\dots,i_n} = \sum_{k=0}^{n} d_{i_k} \Bigg(\prod_{j=k+1}^{n} c_{i_j} \Bigg) \Bigg(\prod_{j=0}^{k-1} b_{i_j} \Bigg).$$

Thus, by ergodicity, simple algebraic manipulation shows that

$$\liminf_{n\to\infty} \frac{-\log|d_{i_0,\dots,i_n}|}{n} \geq \chi_{\mathcal{H},\mathbf{p}} \quad \text{almost surely.}$$

Hence, the Lyapunov dimension depends only on the diagonal elements of the matrices, so equation (20) still holds.

1.7.2. *Projection entropy.* We recall the definition of projective entropy, introduced by Feng and Hu [**11**].

Here we use Notation 1.3. That is ,let $\Psi = \{\psi_i\}_{i=1}^{N}$ be a strictly contracting IFS on $\mathbb{R}^d$. Moreover, let $\mathfrak{m}$ be a σ-invariant ergodic measure on Σ. Let $\mathcal{P} = \{[1], \dots, [N]\}$ be the partition of Σ, where $[i] = \{\mathbf{i} \in \Sigma : i_1 = i\}$, and we write $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}^d$. Feng and Hu [**11**, Definition 2.1] defined the *projection entropy of* $\mathfrak{m}$ *under* Π *with respect to* Ψ:

$$h_\Pi(\mathfrak{m}) := H_\mathfrak{m}(\mathcal{P} \mid \sigma^{-1}\Pi^{-1}\mathcal{B}) - H_\mathfrak{m}(\mathcal{P} \mid \Pi^{-1}\mathcal{B}),$$

where $H_\mathfrak{m}(\xi \mid \eta) = -\int \log \mathfrak{m}_x^\eta(\xi(x))\, d\mathfrak{m}(x)$ denotes the usual conditional entropy of ξ given η, in which $\mathfrak{m}_x^\eta$ denotes the conditional measure of $\mathfrak{m}$ supported on $\eta(x)$ (the element of the partition η, which contains x).

1.7.3. *Hochman's exponential separation condition.* Hochman introduced the following Diophantine-type condition in [**13**]. Let $\Psi = \{\psi_i\}_{i=1}^{N}$ be a self-similar IFS on $\mathbb{R}$. Let $\Delta_n(\psi)$ be the minimum of $\Delta(\bar{\imath}, \bar{\jmath})$ for distinct $\bar{\imath}, \bar{\jmath} \in \Sigma_n$, where

$$\Delta(\bar{\imath}, \bar{\jmath}) = \begin{cases} \infty, & \psi_{\bar{\imath}}'(0) \neq \psi_{\bar{\jmath}}'(0), \\ |\psi_{\bar{\imath}}(0) - \psi_{\bar{\jmath}}(0)|, & \psi_{\bar{\imath}}'(0) = \psi_{\bar{\jmath}}'(0). \end{cases}$$

Condition. We say that the self-similar IFS ψ satisfies *Hochman's exponential separation condition* if there exist an $\varepsilon > 0$ and an $n_k \uparrow \infty$ such that

$$\Delta_{n_k} > \varepsilon^{n_k}. \tag{21}$$

We remark that in our earlier paper [**6**, p. 2] we stated this condition in an unnecessarily strict way. We required that (21) holds for all elements of the sequence $\{\Delta_n\}_{n=1}^{\infty}$, while it is enough that this inequality holds only on a subsequence as stated above. However, all the assertions of [**6**] remain valid under the weaker condition (21) since in [**6**] we never used the condition $\Delta_n > \varepsilon^n$ directly, we used only the conclusions of Hochman's theorems [**13**].

2. *Hochman and Feng–Hu theorems*

Hochman proved the following very important assertion in [**13**, Theorem 1.1].

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THEOREM 2.1. (Hochman) *Here we use Notation 1.3. Let $\Psi = \{\psi_i\}_{i=1}^N$ be a self-similar IFS on the real line. Assume that Ψ satisfies Hochman's exponential separation condition. Let $\mathbf{p} = (p_1, \ldots, p_N)$ be an arbitrary probability vector. Then*

$$\dim_H(\nu_{\Psi,\mathbf{p}}) = \min\left\{1, \frac{h(\mathbb{P}_\mathbf{p})}{\chi_{\Psi,\mathbf{p}}}\right\}, \tag{22}$$

where $h(\mathbb{P}_\mathbf{p}) = -\sum_{i=1}^N p_i \log p_i$.

The ratio on the right-hand side of (22) is the similarity dimension of $\nu_{\Psi,\mathbf{p}}$; that is,

$$\dim_S(\nu_{\Psi,\mathbf{p}}) := \frac{h(\mathbb{P}_\mathbf{p})}{\chi_{\Psi,\mathbf{p}}}. \tag{23}$$

We note that the similarity dimension is a special case of Lyapunov dimension, if the affinity transformations are similarity transformations.

For the next two theorems we refer to [**11**, Theorems 2.8 and 2.11].

THEOREM 2.2. (Feng and Hu) *Here we use Notation 1.3 and notation from §1.7.2. Let Ψ be a self-similar IFS on the line and $\mathbf{p}$ be a probability vector. Then*

$$\dim_H(\nu_{\Psi,\mathbf{p}}) = \frac{h_\Pi(\mathbb{P}_\mathbf{p})}{\chi_{\Psi,\mathbf{p}}}. \tag{24}$$

If we combine Theorems 2.1 and Theorem 2.2 we obtain the following result.

COROLLARY 2.3. *Let Ψ be a self-similar IFS on the line which satisfies Hochman's exponential separation condition. Then $h_\Pi(\mathbb{P}_\mathbf{p}) = \min\{h(\mathbb{P}_\mathbf{p}), \chi_{\Psi,\mathbf{p}}\}$.*

Now we consider the diagonally self-affine case on the plane.

THEOREM 2.4. (Feng and Hu) *Given the diagonally self-affine IFS as in (1), we assume that it satisfies the SSP. Its horizontal and vertical parts (see (4)) are denoted by $\mathcal{H}$ and $\mathcal{V}$.*

Fix a probability vector $\mathbf{p} = (p_1, \ldots, p_N)$. We consider the Lyapunov exponents $\chi_{\mathcal{H},\mathbf{p}}$ and $\chi_{\mathcal{V},\mathbf{p}}$ as in (18). Without loss of generality we assume that the x-direction dominates, *which means that*

$$0 < \chi_{\mathcal{H},\mathbf{p}} \le \chi_{\mathcal{V},\mathbf{p}}$$

holds. Then

$$\dim_{\mathrm{H}}(\nu_{\mathcal{S}^{\mathrm{diag}},\mathbf{p}}) = \frac{h(\mathbb{P}_\mathbf{p})}{\chi_{\mathcal{V},\mathbf{p}}} + \left(1 - \frac{\chi_{\mathcal{H},\mathbf{p}}}{\chi_{\mathcal{V},\mathbf{p}}}\right) \cdot \dim_{\mathrm{H}}(\nu_{\mathcal{H},\mathbf{p}}). \tag{25}$$

Observe that the Hausdorff dimension of the measure $\nu_{\mathcal{S}^{\mathrm{diag}},\mathbf{p}}$ is equal to its Lyapunov dimension if and only if $\dim_{\mathrm{H}}(\nu_{\mathcal{H},\mathbf{p}}) = 1$.

3. *The trivial case and further notation*

3.1. *The trivial case.*

LEMMA 3.1. *Assume that the x-direction dominates and $\sum_{i=1}^N |c_i| < 1$. Further, we assume that the exponential separation condition holds for the IFS $\mathcal{H}$ defined in (4). Then* $\dim_{\mathrm{H}} \Lambda_\mathcal{S} = s_x$.

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The proof is immediate since in this case $\dim_{\mathrm{aff}}(\mathcal{S}) = s_x$, which is always an upper bound. The lower estimate comes from Hochman's theorem: the dimension of the attractor of $\mathcal{H}$ is equal to s_x and this attractor is a projection of Λ. Apart from the trivial case, we can get results only if we assume that

$$c := c_i \quad \text{and} \quad b := b_i \quad \text{hold for all } i. \tag{26}$$

3.2. *Further notation.* We use the notation of §1.4. That is, from now on we always assume that the matrices T_i in (2) are of the form

$$T_i = \begin{pmatrix} c & 0 \\ d_i & b \end{pmatrix}. \tag{27}$$

Clearly, if $c > b$ then the x-direction dominates and if $b > c$ then the y-direction dominates.

In §1.4 we introduced the uniformly distributed Bernoulli measure μ on the symbolic space Σ, and the natural projection Π; the projection of μ to the attractor $\Lambda \subset \mathbb{R}^2$ was was denoted by ν and the projection of ν to the x-axis was denoted by ν_x.

The measures μ and ν can be disintegrated, according to the partitions

$$\xi_a^s := \{\mathbf{i} \in \Sigma : \Pi_{\mathcal{H}(\mathbf{i})} = a\} \quad \text{and} \quad \xi_a := \{(x, y) : x = a,\ y \in [0, 1]\},$$
$$\mu(A^s) = \int \alpha_a^\mu(A)\, d\mu_x(a), \quad \nu(A) = \int \alpha_a^\nu(A)\, d\nu_x(a), \tag{28}$$

for any sets $A^s \subset \Sigma$ and $A \subset [0, 1]^2$. That is, the probability measures α_a^μ and α_a^ν are supported by ξ_a^s and ξ_a, respectively.

Clearly,

$$\mu_x = (\mathrm{proj}_x)_* \nu = \nu_x. \tag{29}$$

For an arbitrary $a \in [0, 1]$ we write Π_a for the restriction of the natural projection Π to ξ_a^s. That is, $\Pi_a : \xi_a^s \cap \Sigma \to \xi_a \cap \Lambda_{\mathcal{S}}$. Then we have

$$(\Pi_a)_* \alpha_a^\mu = \alpha_a^\nu. \tag{30}$$

We now introduce the vertical distance, vertical ball and and vertical neighborhood of a set:

$$\mathrm{dist}_y((x_0, y_0), (x, y)) := \begin{cases} |y - y_0| & \text{if } x = x_0, \\ \infty & \text{otherwise,} \end{cases}$$
$$B_y((x_0, y_0), r) := \{(x, y) : x = x_0 \text{ and } |y - y_0| < r\}.$$

Further, let

$$U_y(H, r) := \bigcup_{(x_0, y_0) \in H} B_y((x_0, y_0), r).$$

We define the vertical distance in the symbolic space as well:

$$\mathrm{dist}_y^s(\mathbf{i}, \mathbf{j}) := \begin{cases} b^{|\mathbf{i} \wedge \mathbf{j}|} & \text{if } \Pi_{\mathcal{H}}(\mathbf{i}) = \Pi_{\mathcal{H}}(\mathbf{j}), \\ \infty & \text{otherwise.} \end{cases}$$

$B_y^s(\mathbf{i}, r)$ and $U_y^s(H^s, r)$ are then defined analogously to $B_y((x, y), r)$ and $U_y(H, r)$.

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4. *The x-direction dominates*
In this case, by (6),

$$\dim_{\mathrm{aff}} \mathcal{S} = \begin{cases} \dfrac{\log N}{-\log c} & \text{if } Nc < 1, \\ 1 + \dfrac{\log(Nc)}{-\log b} & \text{if } Nc > 1. \end{cases} \tag{31}$$

As we have seen above, the $Nc < 1$ case is obvious. So from now on we may assume that $Nc > 1$.

Consider the IFS $\mathcal{H}$. If $Nc > 1$ then the affinity dimension of this system is larger than 1. For systems like that there are many results proving (under different assumptions) that their natural measure is not only absolutely continuous but also has L^q density for some $q > 1$. For example, for Bernoulli convolutions the natural measure has L^2 density for almost all parameters [**25**], has L^q density for some $q > 1$ for all parameters except a dimension-zero subset [**22**], and by the new preprint of Shmerkin [**21**] it has L^q density for all $q > 1$ for all parameters except a dimension-zero subset. It turns out that the knowledge of L^q properties of the density of natural measure is quite useful.

We will use the following assumptions:

Assumption A.
(A1) $c > 1/N$.
(A2) $b < 1/N$.
(A3) $\nu_x \ll \mathcal{L}\mathrm{eb}$ with L^q density, for some $q > 1$.
(A4) $\mathcal{S}$ satisfies transversality (Definition 1.1).

When we replace (A3) with a stronger assumption (B3), we relax our assumption about b:

Assumption B.
(B1) $c > 1/N$.
(B2) $b < c$.
(B3) $\nu_x \ll \mathcal{L}\mathrm{eb}$ with L^q density, for all $q > 1$.
(B4) $\mathcal{S}$ satisfies transversality.

If we consider the corresponding diagonal system (the one in (1)) then assumptions (A1, B1) say that the similarity dimension to the x-axis is greater than one in both cases, while (A2) postulates that the projection to the vertical axis of the corresponding diagonal system has similarity dimension less than one.

Our new results when $c > b$ are as follows.

THEOREM 4.1. *Let $\mathcal{S}$ be a self-affine IFS of the form (2) satisfying Assumption A. Then*

$$\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{H}}(\nu) = 1 + \frac{\log(Nc)}{-\log b}. \tag{32}$$

THEOREM 4.2. *Let $\mathcal{S}$ be a self-affine IFS of the form (2) satisfying Assumption B. Then*

$$\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{H}}(\nu) = \min\left(2,\, 1 + \frac{\log(Nc)}{-\log b}\right). \tag{33}$$

Observe that Assumption A guarantees that $1 + \log(Nc)/-\log b < 2$.

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4.1. *Proof of Theorem 4.1.* First we give the proof assuming a density assertion (Proposition 4.3). The proof of Proposition 4.3 is given in §§4.1.2 and 4.1.3.

4.1.1. *The proof of Theorem 4.1 modulo a density lemma.*

Proof of Theorem 4.1. Let $h_x(\mu)$ be the projection entropy which corresponds to the projection $\Pi_{\mathcal{H}}$ (that is, the entropy of $(\mathcal{H}, \Pi_{\mathcal{H}}(\mu))$). It follows from [**11**, Proposition 4.14] that

$$-\lim_{n\to\infty} \frac{\log \alpha^{\mu}_{\Pi_{\mathcal{H}}(\mathbf{i})}(B^s_y(\mathbf{i}, b^n))}{n} = h(\mu) - h_x(\mu) \quad \text{for } \mu \text{ a.a. } \mathbf{i}. \tag{34}$$

Observe that $h(\mu) = \log N$. On the other hand, by Assumption (A3), it follows from Corollary 2.3 that we have

$$h_x(\mu) = -\log c. \tag{35}$$

Putting these together, we obtain that

$$-\lim_{n\to\infty} \frac{\log \alpha^{\mu}_{\Pi_{\mathcal{H}}(\mathbf{i})}(B^s_y(\mathbf{i}, b^n))}{n} = \log(Nc) \quad \text{for } \mu \text{ a.a. } \mathbf{i}. \tag{36}$$

Our aim below is to prove the corresponding statement for $\alpha^{\nu}_{a_0}$:

$$-\lim_{n\to\infty} \frac{\log \alpha^{\nu}_{a_0}(B_y(\Pi_{a_0}(y_0), b^n))}{n} = \log(Nc) \quad \text{for } \nu \text{ a.a. } (a_0, y_0). \tag{37}$$

Namely, if Proposition 4.3 holds then (37) follows and this implies that for ν_x-almost all $a_0 \in [0, 1]$ we have

$$\dim_{\mathrm{H}}(\Lambda_{\mathcal{S}} \cap \xi_{a_0}) \geq \frac{\log(Nc)}{-\log b}. \tag{38}$$

Using that ν_x is absolutely continuous with respect to Lebesgue measure, we obtain that (38) holds for a set of Hausdorff dimension 1 of a_0. Then by [**9**, Theorem 5.8] we obtain that

$$\dim_{\mathrm{H}} \Lambda_{\mathcal{S}} \geq 1 + \frac{\log(Nc)}{-\log b}. \tag{39}$$

The opposite inequality is immediate since $1 + \log(Nc)/-\log b$ is the affinity dimension of $\mathcal{S}$ which is always an upper bound on the Hausdorff dimension of the attractor. So, to complete the proof of Theorem 4.1 it is enough to verify Proposition 4.3 below. □

4.1.2. *Density in the symbolic space versus on the attractor.*

PROPOSITION 4.3. *For μ-a.a. $\mathbf{i}$ we have*

$$\liminf_{n\to\infty} \frac{1}{n} \log \alpha^{\mu}_{\Pi_{\mathcal{H}}(\mathbf{i})}(B^s_y(\mathbf{i}, b^n)) = \liminf_{n\to\infty} \frac{1}{n} \log \alpha^{\nu}_{\Pi_{\mathcal{H}}(\mathbf{i})}(B_y(\Pi(\mathbf{i}), b^n)). \tag{40}$$

The same holds if we replace lim inf *with* lim sup *on both sides.*

We need the following notion.

Definition 4.4. (Definition of L) Let $\mathbf{i} \in \Sigma$. We write

$$Z_n(\mathbf{i}) := \{\mathbf{j} : \mathbf{j}|_n \neq \mathbf{i}|_n,\ \Pi_{\mathcal{H}}(\mathbf{j}) = \Pi_{\mathcal{H}}(\mathbf{i})\} = [\mathbf{i}|_n]^c \cap \Pi^{-1}_{\mathcal{H}}(\Pi_{\mathcal{H}}(\mathbf{i})), \tag{41}$$

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where $[\mathbf{i}|_n] := \{\mathbf{j} \in \Sigma : \mathbf{i}|_n = \mathbf{j}|_n\}$, and for a set A, we write A^c for the complement of A. We define the function L which is the vertical distance from the closest point having a different first cylinder in its symbolic representation:

$$L(\mathbf{i}) := \min\{\mathrm{dist}_y(\mathbf{i}, \mathbf{j}); \mathbf{j} \in Z_1(\mathbf{i})\}. \tag{42}$$

We say that an $\mathbf{i} \in \Sigma$ is *ε-good* if there exists $C = C(\mathbf{i}, \varepsilon) > 0$ such that for all $n \geq 0$ we have $L(\sigma^n \mathbf{i}) \geq C \cdot e^{-\varepsilon n}$. Moreover, $\mathbf{i} \in \Sigma$ is called *good* if it is ε-good for all $\varepsilon > 0$. That is, we write

$$G := \{\mathbf{i} \in \Sigma : \mathbf{i} \text{ is } \varepsilon\text{-good for all } \varepsilon > 0\}. \tag{43}$$

The geometric meaning of G is given by following observation.

CLAIM 1. *Assume that $\mathbf{i} \in \Sigma$ is ε-good for some $C, \varepsilon > 0$. Let $\boldsymbol{\omega} \in \{1, \ldots, N\}^n$ such that $\boldsymbol{\omega} \neq \mathbf{i}|_n$. Then*

$$\Lambda_{\boldsymbol{\omega}} \cap B_y(\Pi(\mathbf{i}), C \cdot b^n e^{-\varepsilon n}) = \emptyset. \tag{44}$$

Proof. Let $k := |\boldsymbol{\omega} \wedge \mathbf{i}| < n$. By assumption we have

$$\mathrm{dist}_y(\Pi(\sigma^k \mathbf{i}), \Lambda_{\boldsymbol{\omega}_{k+1}}) \geq C \cdot e^{-\varepsilon k}. \tag{45}$$

Using that $S_{\mathbf{i}|_k} = S_{\boldsymbol{\omega}|_k}$, $\Pi(\mathbf{i}) = S_{\mathbf{i}|_k}(\Pi(\sigma^k \mathbf{i}))$ and $S_{\boldsymbol{\omega}|_k}(\Lambda_{\omega_{k+1}}) = \Lambda_{\boldsymbol{\omega}|_{k+1}} \supset \Lambda_{\boldsymbol{\omega}}$, the statement follows from the fact that $S_{\mathbf{i}|_k}$ contracts in the vertical direction by a factor b^k. □

LEMMA 4.5. *Let $\mathbf{i} \in G$ and $\Pi(\mathbf{i}) = (a_0, y_0)$. Let η^s be a measure on $\xi^s_{a_0}$, and we write $\eta := (\Pi)_* \eta^s$. Then*

$$\liminf_{n\to\infty} \frac{1}{n} \log \eta^s(B^s_y(\mathbf{i}, b^n)) = \liminf_{n\to\infty} \frac{1}{n} \log \eta(B_y(\Pi(\mathbf{i}), b^n)). \tag{46}$$

The equality stays true if we replace lim inf *by* lim sup *on both sides.*

Proof. Clearly, η is supported on ξ_{a_0}. The direction '$\leq$' follows from the fact that Π restricted to $\xi^s_{a_0}$ is a Lipschitz map (with Lipschitz constant 1). This is true for every a_0, hence the inequality in this direction is true for all $\mathbf{i} \in \Sigma$.

Now we verify the '$\geq$' part. Fix an $\varepsilon > 0$ an n and an $\mathbf{i} \in \Sigma$ which is ε-good with a constant $C > 0$. As $\mathbf{i}$ is ε-good, by (44),

$$\eta(B_y(\Pi(\mathbf{i}), C \cdot b^n e^{-\varepsilon n})) \leq \eta^s(\Pi(\mathbf{i}), b^n).$$

Passing with n to infinity, we get

$$\left(1 - \frac{\varepsilon}{\log b}\right) \liminf_{n\to\infty} \frac{1}{n} \log \eta^s(B^s_y(\mathbf{i}, b^n)) \geq \liminf_{n\to\infty} \frac{1}{n} \log \eta(B_y(\Pi(\mathbf{i}), b^n)).$$

As $\varepsilon > 0$ can be chosen arbitrarily small, the assertion follows. □

4.1.3. *μ-almost every point is good.* Our goal is to prove the following proposition.

PROPOSITION 4.6. *Let μ be the uniform distribution on Σ and $G \subset \Sigma$ be defined by (43). Then*

$$\mu(G) = 1. \tag{47}$$

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Assuming this, the proof of Proposition 4.3 immediately follows from Lemma 4.5 and Proposition 4.6.

Recall that, as we discussed above, the proof of Theorem 4.1 follows from Proposition 4.3. So the only thing left is to prove Proposition 4.6. To do so we need the following lemma. Let

$$V_{\varepsilon,n} := \{\mathbf{j} : L(\mathbf{j}) < e^{-\varepsilon n}\}.$$

LEMMA 4.7. *There exist a constant $K_4 > 0$ and $r \in (0, 1)$ such that for all $n \geq 0$,*

$$\mu(V_{\varepsilon,n}) < K_4 \cdot r^n. \tag{48}$$

Proof. Let $\ell := \varepsilon/(-\log b) \cdot n$. Then $b^\ell = e^{-\varepsilon n}$. In the rest of this proof we always suppose that

$$\boldsymbol{\omega}, \boldsymbol{\tau} \in \Sigma_\ell, \quad \omega_1 \neq \tau_1.$$

Recall that by assumption there exists a $q > 1$ such that $M := \int \varphi^q(t)\, dt < \infty$, where $\varphi(t)$ is the density function of ν_x. It follows from Assumption (A2) that we can choose a $\delta \in (0, 1)$ such that

$$Nb \cdot (Nc)^{2\delta} < 1 \tag{49}$$

and let

$$\mathrm{Bad}^1_{\ell,\delta} := \{t \in [0, 1] : \varphi(t) > (Nc)^{\delta\ell}\}$$

and

$$\mathrm{Bad}^2_{\ell,\delta} := \left\{t \in [0, 1] : \sum_{|\boldsymbol{\tau}|=\ell} \mathbb{1}_{h_{\boldsymbol{\tau}}([0,1])}(t) > (Nc)^{(1+\delta)\ell}\right\}.$$

It follows from Corollary A.5 of the Appendix that

$$\nu_x(\mathrm{Bad}^2_{\ell,\delta}) \leq K_6 \cdot (Nc)^{-\ell\delta(q-1)}, \tag{50}$$

where $K_6 > 0$ is a constant. Furthermore, by the Markov inequality,

$$M = \int_0^1 \varphi^q(t)\, dt = \int_0^1 \varphi^{q-1}(t)\, d\nu_x(t) \geq \nu_x(\mathrm{Bad}^1_{\ell,\delta}) \cdot (Nc)^{(q-1)\delta\ell}$$

that is,

$$\nu_x(\mathrm{Bad}^1_{\ell,\delta}) \leq M \cdot (Nc)^{-\delta\ell(q-1)}. \tag{51}$$

Now we define

$$I_{\boldsymbol{\omega},\boldsymbol{\tau}} := \mathrm{proj}_x(S_{\boldsymbol{\omega}}([0, 1]^2) \cap U_y(S_{\boldsymbol{\tau}}([0, 1]^2), b^\ell))\backslash \mathrm{Bad}^2_{\ell,\delta}. \tag{52}$$

We define

$$R_{\boldsymbol{\omega},\boldsymbol{\tau}} := (I_{\boldsymbol{\omega},\boldsymbol{\tau}} \times [0, 1]) \cap S_{\boldsymbol{\omega}}([0, 1]^2). \tag{53}$$

That is, $R_{\boldsymbol{\omega},\boldsymbol{\tau}}$ consists of those elements of $S_{\boldsymbol{\omega}}$ which are 'bad' because of $S_{\boldsymbol{\tau}}([0, 1]^2)$. It follows from transversality (Definition 1.1) that

$$|I_{\boldsymbol{\omega},\boldsymbol{\tau}}| < 3K_3 \cdot b^\ell. \tag{54}$$

Let

$$\widetilde{R}_{\boldsymbol{\omega},\boldsymbol{\tau}} := [\boldsymbol{\omega}] \cap \Pi^{-1}(R_{\boldsymbol{\omega},\boldsymbol{\tau}}).$$

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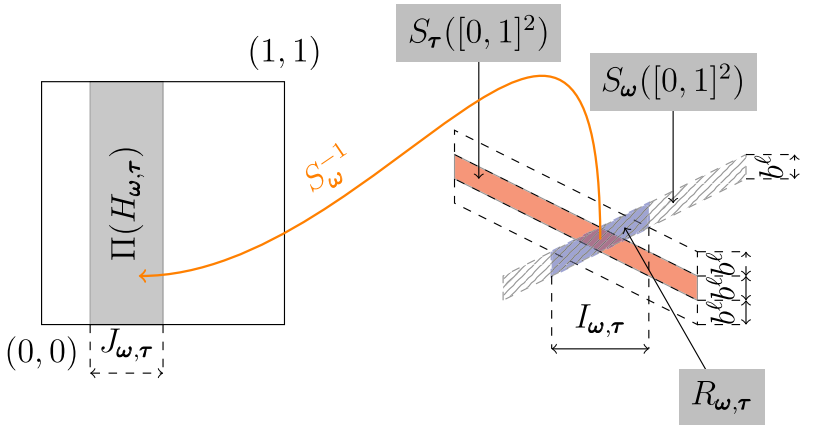


FIGURE 1. The idea of the proof of Lemma 4.7. ($\mathrm{Bad}^1_{\ell,\delta}$, $\mathrm{Bad}^2_{\ell,\delta}$ not included.)

The importance of $\widetilde{R}_{\boldsymbol{\omega},\boldsymbol{\tau}}$ is that it follows from elementary geometry that

$$\{\mathbf{j} \in [\boldsymbol{\omega}] : \mathrm{proj}_x(\mathbf{j}) \notin \mathrm{Bad}^2_{\ell,\delta}, \exists \mathbf{i} \in \boldsymbol{\tau}, \mathrm{dist}_y(\Pi(\mathbf{i}), \Pi(\mathbf{j})) < b^\ell\} \subset \widetilde{R}_{\boldsymbol{\omega},\boldsymbol{\tau}}. \tag{55}$$

It follows from the construction of $R_{\boldsymbol{\omega},\boldsymbol{\tau}}$ that there is a (possibly empty) interval $J_{\boldsymbol{\omega},\boldsymbol{\tau}}$ such that

$$S_{\boldsymbol{\omega}}^{-1}(R_{\boldsymbol{\omega},\boldsymbol{\tau}}) = (J_{\boldsymbol{\omega},\boldsymbol{\tau}} \setminus h_{\boldsymbol{\omega}}^{-1}(\mathrm{Bad}^2_{\ell,\delta})) \times [0,1],$$

where the length of the interval

$$|J_{\boldsymbol{\omega},\boldsymbol{\tau}}| \le 3K_3 \left(\frac{b}{c}\right)^\ell. \tag{56}$$

For a visualization of the relation between $R_{\boldsymbol{\omega},\boldsymbol{\tau}}$, $I_{\boldsymbol{\omega},\boldsymbol{\tau}}$ and $J_{\boldsymbol{\omega},\boldsymbol{\tau}}$, see Figure 1.

Let

$$H_{\boldsymbol{\omega},\boldsymbol{\tau}} := \Pi^{-1}((J_{\boldsymbol{\omega},\boldsymbol{\tau}} \setminus h_{\boldsymbol{\omega}}^{-1}(\mathrm{Bad}^2_{\ell,\delta})) \times [0,1]).$$

Hence the concatenation of $\boldsymbol{\omega}$ and $H_{\boldsymbol{\omega},\boldsymbol{\tau}}$ is

$$\widetilde{R}_{\boldsymbol{\omega},\boldsymbol{\tau}} = \boldsymbol{\omega} H_{\boldsymbol{\omega},\boldsymbol{\tau}}. \tag{57}$$

For simplicity, we define for an $\boldsymbol{\omega} \in \Sigma_\ell$,

$$D[\boldsymbol{\omega}] := \{\boldsymbol{\tau} \in \Sigma_\ell, \omega_1 \ne \tau_1\}. \tag{58}$$

Using (55), we can write

$$V_{\varepsilon,n} \subset (\Pi_{\mathcal{H}}^{-1}(\mathrm{Bad}^2_{\ell,\delta})) \bigcup \bigcup_{\boldsymbol{\omega}\in\Sigma_\ell} \bigcup_{\boldsymbol{\tau}\in D[\boldsymbol{\omega}]} \widetilde{R}_{\boldsymbol{\omega},\boldsymbol{\tau}}. \tag{59}$$

Hence, by (50) and (7) we get

$$\begin{aligned}\mu(V_{\varepsilon,n}) &= K_6(Nc)^{-\ell\delta(q-1)} + \sum_{\boldsymbol{\omega}\in\Sigma_\ell} \mu\Bigg(\bigcup_{\boldsymbol{\tau}\in D[\boldsymbol{\omega}]} \boldsymbol{\omega} H_{\boldsymbol{\omega},\boldsymbol{\tau}}\Bigg) \\ &= K_6(Nc)^{-\ell\delta(q-1)} + \frac{1}{N^\ell} \sum_{\boldsymbol{\omega}\in\Sigma_\ell} \mu\Bigg(\bigcup_{\boldsymbol{\tau}\in D[\boldsymbol{\omega}]} H_{\boldsymbol{\omega},\boldsymbol{\tau}}\Bigg).\end{aligned} \tag{60}$$

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To estimate $\mu(\bigcup_{\boldsymbol{\tau}\in D[\boldsymbol{\omega}]} H_{\boldsymbol{\omega},\boldsymbol{\tau}})$ we present

$$\bigcup_{\boldsymbol{\tau}\in D[\boldsymbol{\omega}]} H_{\boldsymbol{\omega},\boldsymbol{\tau}} = \Pi^{-1}\left(\left(\mathrm{Bad}^1_{\ell,\delta} \cup \bigcup_{\boldsymbol{\tau}\in D[\boldsymbol{\omega}]} \widetilde{J}_{\boldsymbol{\omega},\tau}\right) \times [0,1]\right),$$

where

$$\widetilde{J}_{\boldsymbol{\omega},\boldsymbol{\tau}} := J_{\boldsymbol{\omega},\boldsymbol{\tau}} \setminus (h_{\boldsymbol{\omega}}^{-1}(\mathrm{Bad}^2_{\ell,\delta} \cup \mathrm{Bad}^1_{\ell,\delta})).$$

It follows from the definition of the set $\mathrm{Bad}^1_{\ell,\delta}$ that

$$\text{if } t \in \widetilde{J}_{\boldsymbol{\omega},\boldsymbol{\tau}} \quad \text{then } \varphi(t) < (Nc)^{\ell\delta}. \tag{61}$$

On the other hand, it follows from the definition of $\mathrm{Bad}^2_{\ell,\delta}$ that

$$\sum_{\tau\in D[\omega]} \mathbb{1}_{\widetilde{J}_{\omega,\boldsymbol{\tau}}}(t) \le (Nc)^{(1+\delta)\ell} \quad \text{for all } t. \tag{62}$$

Putting these together and using (51) and (56), we get

$$\begin{aligned}
\mu\left(\bigcup_{\boldsymbol{\tau}\in D[\boldsymbol{\omega}]} H_{\boldsymbol{\omega},\boldsymbol{\tau}}\right) &\le \nu_x(\mathrm{Bad}^1_{\ell,\delta}) + \nu_x\left(\bigcup_{\boldsymbol{\tau}\in D[\boldsymbol{\omega}]} \widetilde{J}_{\boldsymbol{\omega},\boldsymbol{\tau}}\right) \\
&\le \nu_x(\mathrm{Bad}^1_{\ell,\delta}) + \int \sum_{\tau\in D[\omega]} \mathbb{1}_{\widetilde{J}_{\omega,\tau}}(t)\varphi(t)\,dt \\
&\le M\cdot(Nc)^{-\ell\delta(q-1)} + (Nc)^{(1+\delta)\ell}(Nc)^{\ell\delta} 3K_3\left(\frac{b}{c}\right)^{\ell} \\
&\le K_7((Nc)^{-\ell\delta(q-1)} + (Nb(Nc)^{2\delta})^{\ell}).
\end{aligned} \tag{63}$$

Putting this and (60) together, we obtain that

$$\mu(V_{\varepsilon,n}) < (K_6 + K_7) r^n,$$

where

$$r := \max\{(Nc)^{-\delta(q-1)},\, Nb(Nc)^{2\delta}\}^{\varepsilon/(-\log b)}. \qquad \square$$

COROLLARY 4.8. *Let* $R_1 := \{\mathbf{i} : \exists \mathbf{j} \ne \mathbf{i},\ \Pi(\mathbf{i}) = \Pi(\mathbf{j})\}$. *Then* $\mu(R_1) = 0$.

Proof. It follows from Lemma 4.7 that

$$\mu(\mathbf{i} : L(\mathbf{i}) = 0) = 0. \tag{64}$$

Clearly,

$$\begin{aligned}
R_1 &= \bigcup_{n=0}^{\infty} \{\mathbf{i} : \exists \mathbf{j},\ |\mathbf{i}\wedge\mathbf{j}| = n,\ \Pi(\mathbf{i}) = \Pi(\mathbf{j}).\} \\
&= \bigcup_{n=0}^{\infty} \sigma^{-n}\{\mathbf{i} : \exists \mathbf{j},\ i_1 \ne j_1,\ \Pi(\mathbf{i}) = \Pi(\mathbf{j}).\} \\
&= \bigcup_{n=0}^{\infty} \sigma^{-n}\{\mathbf{i} : L(\mathbf{i}) = 0\}.
\end{aligned}$$

This and (64) imply that $\mu(R_1) = 0$. $\square$

Proposition 4.6 is now immediate.

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Proof of Proposition 4.6. By definition,

$$\{\mathbf{i} : L(\sigma^n \mathbf{i}) < e^{-\varepsilon n}\} = \sigma^{-n}(V_{\varepsilon,n}).$$

From this and Lemma 4.7 we get that

$$\sum_n \mu(\{\mathbf{i} : L(\sigma^n \mathbf{i}) < e^{-\varepsilon n}\}) < \infty.$$

Let $R_2 := \{\mathbf{i} : \exists N_0, \forall n > N_0\ L(\sigma^n \mathbf{i}) \geq e^{-\varepsilon n}\}$. Then $\mu(R_2) = 1$. It is immediate to see that every $\mathbf{i} \in R_2 \setminus R_1$ is ε-good. This means that the ε-good $\mathbf{i}$ form a set of full measure. By taking a countable intersection we get that $\mu(G) = 1$. □

4.2. *$L^q \forall q$ density case.* In this section we will give the proof of Theorem 4.2. In this subsection we assume that ν_x is absolutely continuous with L^q density for all $q > 1$. We also assume the transversality condition.

In the previous subsection we proved Lemma 4.7. In the calculation of $\mu(V_{\varepsilon,\ell})$ we were not able to make use of the following fact: if the intervals $\mathrm{proj}_x S_{\boldsymbol{\omega}}([0,1]^2)$ and $\mathrm{proj}_x S_{\boldsymbol{\tau}}([0,1]^2)$ intersect for some $\boldsymbol{\omega}, \boldsymbol{\tau} \in \Sigma_\ell$, it does not necessarily mean that the parallelograms $S_{\boldsymbol{\omega}}([0,1]^2)$ and $S_{\boldsymbol{\tau}}([0,1]^2)$ intersect as well. Under the $L^q \forall q$ assumption this distinction can be made.

4.2.1. *Number of pairs of intersecting cylinder parallelograms.* In this subsection we give an upper bound on the number of intersecting (or close-by) cylinders of level ℓ with distinct first coordinates. To state the lemma we need some preparation.

We write

$$\Sigma_\ell^{2,\mathrm{diff}} := \{(\boldsymbol{\omega}, \boldsymbol{\tau}) \in \Sigma_\ell \times \Sigma_\ell : \omega_1 \neq \tau_1\}. \tag{65}$$

For an $\boldsymbol{\omega} \in \Sigma_\ell$ and $L > 0$ let

$$P_{\boldsymbol{\omega}}^L := U_y(S_{\boldsymbol{\omega}}([0,1]^2),\, Lb^\ell) \quad \text{and} \quad I_{\boldsymbol{\omega}} := \mathrm{proj}_x S_{\boldsymbol{\omega}}([0,1]^2) = \mathrm{proj}_x P_{\boldsymbol{\omega}}. \tag{66}$$

Further, for $\boldsymbol{\omega}, \boldsymbol{\tau} \in \Sigma_\ell^{2,\mathrm{diff}}$ we write

$$P_{\boldsymbol{\omega},\boldsymbol{\tau}}^L := P_{\boldsymbol{\omega}}^L \cap P_{\boldsymbol{\tau}}^L \supset R_{\boldsymbol{\omega},\boldsymbol{\tau}}, \tag{67}$$

where $R_{\boldsymbol{\omega},\boldsymbol{\tau}}$ was defined in (53) and

$$I_{\boldsymbol{\omega},\boldsymbol{\tau}}^L := \mathrm{proj}_x P_{\boldsymbol{\omega},\boldsymbol{\tau}}^L. \tag{68}$$

Note that by transversality

$$|I_{\boldsymbol{\omega},\boldsymbol{\tau}}^L| < (L+2) K_3 \cdot b^\ell. \tag{69}$$

Let

$$B_\ell^L := \#\{(\boldsymbol{\omega}, \boldsymbol{\tau}) \in \Sigma_\ell^{2,\mathrm{diff}} : P_{\boldsymbol{\omega},\boldsymbol{\tau}}^L \neq \emptyset\}. \tag{70}$$

LEMMA 4.9. *Assume that Conditions (B1), (B3) and (B4) hold. Then, for every $L > 0$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log B_n^L \leq \log(N^2 c^2).$$

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Proof. Choose some $\widetilde{n} > 1$. Let $n \geq \widetilde{n}$. Let us define a finite sequence n_i as follows: $n_0 := n$ and

$$n_k := \left\lfloor \left(\frac{\log c}{\log b}\right)^k n \right\rfloor. \tag{71}$$

Let $\mathcal{K} := \max\{k \geq 0 : n_k \geq \widetilde{n}\}$. Naturally,

$$B^L_{n_\mathcal{K}} \leq N^{2n_\mathcal{K}}.$$

We show that for every $q > 1$ there exists $K = K(q, L)$ such that for every $0 \leq k \leq \mathcal{K}$,

$$B^L_{n_k} \leq B^L_{n_{k+1}} \cdot (K(Nc)^{n_k - n_{k+1}} c^{-(n_k - n_{k+1})/q})^2. \tag{72}$$

Let $\boldsymbol{\omega}, \boldsymbol{\tau} \in \Sigma_{n_k}$. Assume that

$$P^L_{\boldsymbol{\omega}} \cap P^L_{\boldsymbol{\tau}} \neq \emptyset. \tag{73}$$

Observe that this is possible only if both

(a) $P^L_{\boldsymbol{\omega}|_{n_{k+1}}} \cap P^L_{\boldsymbol{\tau}|_{n_{k+1}}} \neq \emptyset$, and

(b) $I_{\boldsymbol{\omega}} \cap I^L_{\boldsymbol{\omega}|_{n_{k+1}}, \boldsymbol{\tau}|_{n_{k+1}}} \neq \emptyset$ and $I_{\boldsymbol{\tau}} \cap I^L_{\boldsymbol{\omega}|_{n_{k+1}}, \boldsymbol{\tau}|_{n_{k+1}}} \neq \emptyset$

hold. By (69) and (71),

$$|I_{\boldsymbol{\omega}|_{n_{k+1}}, \boldsymbol{\tau}|_{n_{k+1}}}| \leq (L+2) K_3 b^{n_{k+1}} \approx c^{n_k}.$$

Hence by Corollary A.6, for every $\boldsymbol{\omega}|_{n_{k+1}}, \boldsymbol{\tau}|_{n_{k+1}}$ there are at most $C(L, q)(Nc)^{n_k - n_{k+1}} c^{-(n_k - n_{k+1})/q}$ words $\boldsymbol{\omega} \in \Sigma_{n_k}$ such that $I_{\boldsymbol{\omega}} \cap I^L_{\boldsymbol{\omega}|_{n_{k+1}}, \boldsymbol{\tau}|_{n_{k+1}}} \neq \emptyset$. In the same way there are at most $C(L, q)(Nc)^{n_k - n_{k+1}} c^{-(n_k - n_{k+1})/q}$ words $\boldsymbol{\tau} \in \Sigma_{n_k}$ such that $I_{\boldsymbol{\tau}} \cap I^L_{\boldsymbol{\omega}|_{n_{k+1}}, \boldsymbol{\tau}|_{n_{k+1}}} \neq \emptyset$. So, taking into consideration condition (a) above, we obtain (72).

Thus, by induction

$$B^L_{n_0} \leq (K)^{2\mathcal{K}} N^{2n_\mathcal{K}} (Nc)^{2(n_0 - n_\mathcal{K})} c^{-2(n_0 - n_\mathcal{K})/q}.$$

But by definition of the sequence $\{n_k\}$, $n_0 = n$, $n_\mathcal{K} \approx \widetilde{n}$, and there exist constants $c_1, c_2 \in \mathbb{R}$ such that $\mathcal{K} \leq c_1 \log n + c_2$. Therefore,

$$B^L_n \leq C(q, L) \cdot K(q, L)^{2c_1 \log n} \cdot c^{-n/q} \cdot (Nc)^{2n},$$

and passing with q to infinity proves the assertion. □

4.2.2. *The correlation dimension.* First we recall the definition of the correlation dimension (see [**17**]).

Definition 4.10. Let $\mathfrak{m}$ be a positive bounded regular Borel measure on $\mathbb{R}^d$ with bounded support $\mathrm{spt}(\mathfrak{m})$. For every $r > 0$, let $\{B^{(r)}_i\}_i$ be the r-mesh cubes that intersect $\mathrm{spt}(\mathfrak{m})$. For a $q > 0$, $q \neq 1$, we define

$$\tau(q) := \liminf_{r \to 0^+} \frac{\log \sum_i \mathfrak{m}(B^{(r)}_i)^q}{\log r}. \tag{74}$$

Equivalently we could define $\tau(q)$ (see [**17**]) in either of the following ways. Let

$$I_r(q) := \int_{\mathbb{R}^d} \mathfrak{m}(B(x, r))^q \, dx \quad \text{and} \quad \widetilde{I}_r(q) := \int_{\mathbb{R}^d} \mathfrak{m}(B(x, r))^{q-1} \, d\mathfrak{m}(x). \tag{75}$$

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Then

$$\tau(q) = \liminf_{r\to 0^+} \frac{\log I_r(q)}{\log r} - d = \liminf_{r\to 0^+} \frac{\log \widetilde{I}_r(q)}{\log r}. \tag{76}$$

For $q > 1$, the L^q-dimension of $\mathfrak{m}$ is defined as

$$\underline{\dim}_{\mathrm{q}}(\mathfrak{m}) := \frac{\tau(q)}{q-1}. \tag{77}$$

It was proved by Hunt and Kaloshin [**15**, Proposition 2.1] that

$$\begin{aligned}\underline{\dim}_{\mathrm{q}}(\mathfrak{m}) &= \sup\left\{t \ge 0 : \int \left(\int \frac{d\mathfrak{m}(y)}{|x-y|^t}\right)^{q-1} d\mathfrak{m}(x) < \infty\right\} \\ &= \inf\left\{t \ge 0 : \int \left(\int \frac{d\mathfrak{m}(y)}{|x-y|^t}\right)^{q-1} d\mathfrak{m}(x) = \infty\right\} \\ &= \sup\left\{s : \int_0^\infty r^{-s(q-1)-1}\widetilde{I}_r(q)\,dr < \infty\right\}. \end{aligned} \tag{78}$$

If we apply this for $q = 2$ we get the correlation dimension of the measure $\mathfrak{m}$:

$$\dim_{\mathrm{C}}(\mathfrak{m}) := \underline{\dim}_2(\mathfrak{m}).$$

It follows from (78) that

$$\dim_{\mathrm{C}}(\mathfrak{m}) \le \dim_{\mathrm{H}}(\mathfrak{m}). \tag{79}$$

Proof of Theorem 4.2. We will prove that under Assumption B,

$$\dim_C \nu \ge \min\left(2,\, 1 + \frac{\log(Nc)}{-\log b}\right).$$

Let $\ell > 0$ and consider the grid of size $r = 2b^\ell$. Let $B = [x - b^\ell, x + b^\ell]\times [y - b^\ell, y + b^\ell]$ be one of the r-mesh cubes. Let $|\boldsymbol{\tau}| = \ell$ be such that B intersects $S_{\boldsymbol{\tau}}([0,1]^2)$. Then $S_{\boldsymbol{\tau}}^{-1}(B \cap S_{\boldsymbol{\tau}}([0,1]^2))$ is contained in some vertical strip of width b^ℓ/c^ℓ, hence by Corollary A.6,

$$\nu(S_{\boldsymbol{\tau}}^{-1}(B \cap S_{\boldsymbol{\tau}}([0,1]^2))) \le C(q)(b/c)^{\ell(1-1/q)}.$$

Let $L := (c-b)^{-1}\cdot \max_{i\le m}|d_i|$ be an upper bound on the maximal slope of (the principal axis of) our cylinders. Then if the cylinder $S_{\boldsymbol{\tau}}([0,1]^2)$, $|\boldsymbol{\tau}| = \ell$, intersects B, it must also intersect at least one of vertical intervals: either $\{x - b^\ell\}\times[y-(2L+1)b^\ell, y+(2L+1)b^\ell]$ or $\{x + b^\ell\}\times[y-(2L+1)b^\ell, y+(2L+1)b^\ell]$, see Figure 2. Let us denote the number of cylinders $S_{\boldsymbol{\tau}}([0,1]^2)$ intersecting $\{x - b^\ell\}\times[y-(L+1)b^\ell, y+(L+1)b^\ell]$ by $Z_1(x,y)$ and the number of cylinders intersecting $\{x + b^\ell\}\times[y-(L+1)b^\ell, y+(L+1)b^\ell]$ by $Z_2(x,y)$. Using a estimate similar to that in the proof of Lemma 4.7, in equation (60) we can write

$$\nu(B) \le C(q)N^{-\ell}\cdot(b/c)^{\ell(1-1/q)}\cdot(Z_1(x,y) + Z_2(x,y)),$$

hence

$$(\nu(B))^2 \le C(q)^2 N^{-2\ell}\cdot(b/c)^{2\ell(1-1/q)}\cdot(2Z_1^2(x,y) + 2Z_2^2(x,y)).$$

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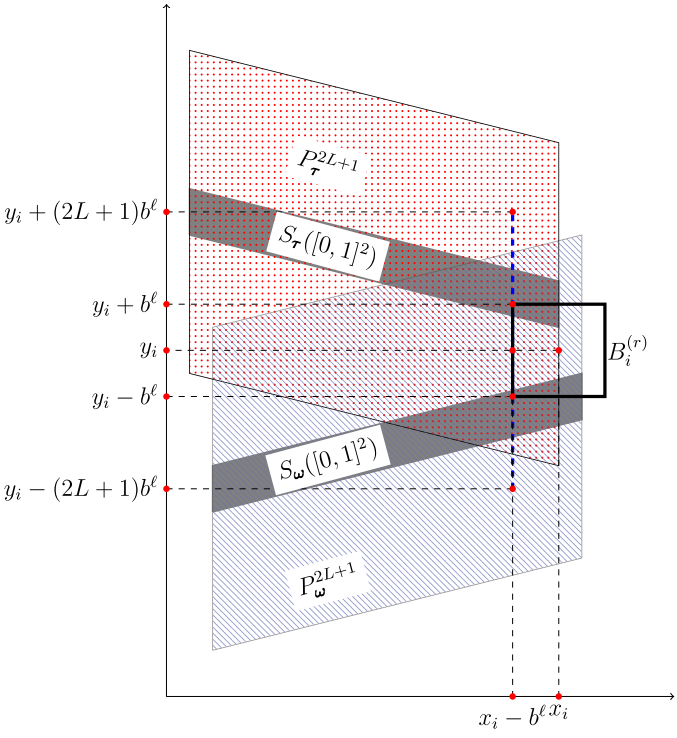


FIGURE 2. Geometric position of stripes.

Thus,

$$\sum \nu(B_i^{(r)})^2 \le C(q)^2 N^{-2\ell} (b/c)^{2\ell(1-1/q)} \cdot \left(\sum Z_1^2(x_i, y_i) + \sum Z_2^2(x_i, y_i) \right),$$

where $B_i^{(r)}$ is the ith r-mesh cube. It is enough to estimate the first sum, the second is analogous. The interval $\{x - b^\ell\} \times [y - (2L+1)b^\ell, y + (2L+1)b^\ell]$ intersects $S_{\boldsymbol{\tau}}([0,1]^2)$ if and only if the point $(x - b^\ell, y)$ is contained in $S_{\boldsymbol{\tau}}([0,1] \times [-2L-1, 2L+2])$. Hence, $Z_1^2(x, y)$ is equal to the number of pairs $\boldsymbol{\omega}, \boldsymbol{\tau} \in \Sigma_\ell \times \Sigma_\ell$ such that $P_{\boldsymbol{\omega},\boldsymbol{\tau}}^{2L+1}$ contains $(x - b^\ell, y)$.

For every $\boldsymbol{\omega}, \boldsymbol{\tau} \in \Sigma_\ell$ we write $k := |\boldsymbol{\omega} \wedge \boldsymbol{\tau}|$. Then for $\boldsymbol{\alpha} := \boldsymbol{\omega} \wedge \boldsymbol{\tau} \in \Sigma_k$, there exists $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \Sigma_{\ell-k}$ such that

$$\boldsymbol{\omega} = \boldsymbol{\alpha}\boldsymbol{\beta} \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\alpha}\boldsymbol{\gamma}.$$

With this notation we present this sum as

$$\sum Z_1^2(x_i, y_i) = \sum_{k=0}^{\ell} \sum_{\boldsymbol{\alpha} \in \Sigma_k} \sum_{(\boldsymbol{\beta},\boldsymbol{\gamma}) \in \Sigma_{\ell-k}^{2,\mathrm{diff}}} |\{i;\, (x_i - b^\ell, y_i) \in P_{\boldsymbol{\omega},\boldsymbol{\tau}}^{2L+1}\}|.$$

Given $\boldsymbol{\alpha}$, the last sum can be estimated by

$$\sum_{(\boldsymbol{\beta},\boldsymbol{\gamma}) \in \Sigma_{\ell-k}^{2,\mathrm{diff}}} |\{i;\, (x_i - b^\ell, y_i) \in P_{\boldsymbol{\omega},\boldsymbol{\tau}}^{2L+1}\}| \le C B_{\ell-k}^{2L+1} \left(\frac{c}{b}\right)^k. \tag{80}$$

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Namely, by the transversality condition we can apply (69) which yields that $|I^{2L+1}_{\boldsymbol{\beta},\boldsymbol{\gamma}}| \le (2L+5)K_3 b^{\ell-k}$ and $|I^{2L+1}_{\boldsymbol{\omega},\boldsymbol{\tau}}| = c^k \cdot |I^{2L+1}_{\boldsymbol{\beta},\boldsymbol{\gamma}}|$. Thus we have only $C \cdot (c/b)^k$ different $(x_i - b^\ell)$s in $I^{2L+1}_{\boldsymbol{\omega},\boldsymbol{\tau}}$ and for each of them we have at most $2L+2$ different y_i in $P^{L+1}_{\boldsymbol{\omega},\boldsymbol{\tau}}$. Then (80) follows from (70).

Applying Lemma 4.9 and noting that $\boldsymbol{\alpha}$ can take N^k values, we get

$$\sum Z_1^2(x_i, y_i) \le CN^{2\ell} c^{2\ell} o(N^{\ell\varepsilon}) \cdot \sum_{k=0}^{\ell} (Ncb)^{-k}.$$

The last sum is a geometric series, hence it is bounded by a constant when $Ncb > 1$ and by $(Ncb)^{-\ell}$ when $Ncb < 1$. When $Ncb = 1$ this sum equals $\ell + 1 = o(N^{\ell\varepsilon})$. Thus,

$$\sum \nu(B_i^{(r)})^2 \le Cb^{\ell \cdot \min(2, 1+\log(Nc)/-\log b)} (b/c)^{-2\ell/q} o(N^{2\ell\varepsilon}).$$

Passing with q to infinity and ε to 0 we get

$$\tau(2) \ge \min\left(2,\, 1 + \frac{\log(Nc)}{-\log b}\right). \qquad \square$$

5. *The y-direction dominates*

Finally, in this section we turn to the case where the the y-direction dominates. In this direction, we have only a mild development on the way of understanding overlapping self-affine systems. The result can be considered as an extension of [**6**, Theorem B] and [**2**, Theorems 4.8, 4.11].

Similarly to §1.5, we define the backward Furstenberg–Kifer measure and IFS. This measure is supported on the set of the strong-stable directions. We note that in the case where the x-direction dominates, the backward Furstenberg–Kifer measure is supported on the singleton $\{(0, 1)\}$.

Consider again the vertical line $\xi := \{(1, z) \in \mathbb{R}^2 : z \in \mathbb{R}\}$ on the plane and identify $(1, z) \in \xi$ with $\widetilde{z} \in \mathbb{R}$. Moreover, let $\mathcal{B}$ be the self-similar IFS on ξ defined by

$$\mathcal{B} := \left\{ g_i(\widetilde{z}) := \frac{c}{b}\widetilde{z} - \frac{d_i}{b} \right\}_{i=1}^{N}. \tag{81}$$

Let us define the natural projection by $\Pi_{BF} : \Sigma \to \xi$ in the usual way:

$$\Pi_{BF}(\mathbf{i}) := -\frac{d_{i_1}}{b} - \sum_{k=2}^{\infty} \frac{d_{i_k}}{b} \cdot \left(\frac{c}{b}\right)^{k-1}. \tag{82}$$

Similarly to (12), we have that the action of $\{T_i^{-1}\}_{i=1}^N$ on the projective line is described by the maps $\widehat{T}_i : \xi \to \xi$,

$$\widehat{T}_i(\widetilde{z}) := c \cdot T_i^{-1} \cdot \begin{pmatrix} 1 \\ z \end{pmatrix},$$

where $\widetilde{z} \in \xi$ is $\widetilde{z} = (1, z)$.

Assumption C. We make the following assumptions.
(C1) $c < 1/N$.
(C2) $b > c$.

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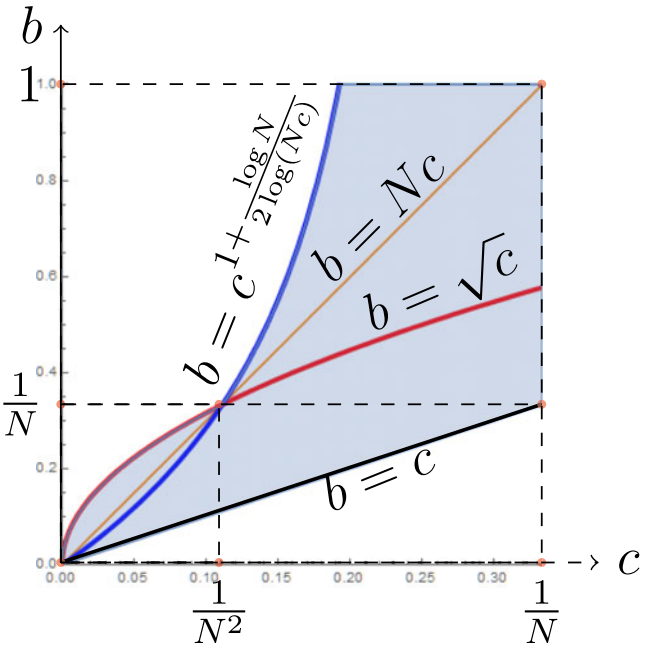


FIGURE 3. The region where (C5) holds.

(C3) The backward Furstenberg–Kifer IFS $\mathcal{B}$ satisfies Hochman's exponential separation condition.

(C4) $\mathcal{H}$ satisfies Hochman's exponential separation condition.

(C5) $$\frac{\log N}{\log(b/c)} \geq \min\left\{1, \frac{\log N}{-\log b}, 2\left(1 - \frac{\log N}{-\log c}\right)\right\}.$$

Conditions (C1) and (C4) are required in order to be able to handle the projection entropy (defined in §1.7.2). Condition (C3) allows us to calculate the dimension of the backward Furstenberg–Kifer measure. Condition (C5) ensures that this dimension is larger than some possible exceptional set of orthogonal projections, for which the dimension drops, see Figure 3.

THEOREM 5.1. *Let $\mathcal{S}$ be a self-affine IFS of the form (2) satisfying Assumption C. Then*

$$\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{H}}(\nu) = \min\left\{\frac{\log N}{-\log b}, 1 + \frac{\log(Nb)}{-\log c}\right\}. \tag{83}$$

As previously, let μ be the uniform Bernoulli measure on the symbolic space and let ν be its projection by the mapping Π, defined in §1.4.

For θ a proper subspace of $\mathbb{R}^2$, let us denote the orthogonal projection from $\mathbb{R}^2$ to the subspace $\theta^{\perp}$ (orthogonal to θ) by Proj_{θ}. By [**3**, Theorem 2.2],

$$\dim \nu = \frac{h_{\Pi}(\mu)}{-\log c} + \left(1 - \frac{-\log b}{-\log c}\right)\dim(\mathrm{Proj}_{\theta})_{*}\nu \quad \text{for } (\Pi_{BF})_{*}\mu\text{-almost every (a.e.) } \theta, \tag{84}$$

where $h_{\Pi}(\mu)$ is the projection entropy, defined in §1.7.2.

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LEMMA 5.2. *If (C1), (C2) and (C4) in Assumption C hold then* $h_\Pi(\mu) = h(\mu) = \log N$.

Proof. Let us define a lifted IFS on $[0, 1]^3$ and a derived IFS on $\{0\} \times [0, 1]^2$, by

$$\widehat{\Phi} := \{\widehat{S}_i(x, y, z) = (cx + u_i, by + d_i x + v_i, \rho z + w_i)\}_{i=1}^m \quad \text{and}$$
$$\widetilde{\Phi} := \{\widetilde{S}_i(y, z) = (cx + u_i, \rho z + w_i)\}_{i=1}^m,$$

where $0 < \rho < \min\{|c|, |b|\}$ and $w_i \in \mathbb{R}$ are chosen such that

$$\widehat{S}_i([0, 1]^3) \cap \widehat{S}_j([0, 1]^3) = \emptyset \quad \text{and} \quad \widetilde{S}_i([0, 1]^2) \cap \widetilde{S}_j([0, 1]^2) = \emptyset \quad \text{for every } i \neq j. \tag{85}$$

Denote the natural projections of $\widehat{\Phi}$ and $\widetilde{\Phi}$ by $\widehat{\Pi}$ and $\widetilde{\Pi}$ respectively. Let us define $\widehat{\nu} = \widehat{\Pi}_* \mu$ and $\widetilde{\nu} = \widetilde{\Pi}_* \mu$ the push-down measures.

We note that the Lyapunov exponents coincide for every measure $\widehat{\nu}$, $\widetilde{\nu}$, and ν for the appropriate directions. Applying [**3**, Corollary 2.9] and [**11**, Theorem 2.11], we have

$$\dim_H \widehat{\nu} = \frac{h(\mu)}{-\log \rho} + \left(\frac{-\log \rho + \log c}{-\log \rho}\right) \dim \nu \tag{86}$$
$$+ \left(\frac{-\log c + \log b}{-\log \rho}\right) \dim(\mathrm{Proj}_\theta)_* \nu \quad \text{for } (\Pi_{BF})_* \mu\text{-a.e. } \theta, \tag{87}$$
$$\dim_H \widetilde{\nu} = \frac{h_{\Pi_{\mathcal{H}}}(\mu)}{-\log c} + \frac{h(\mu) - h_{\Pi_{\mathcal{H}}}(\mu)}{-\log \rho}. \tag{88}$$

By (84) and (86) we have

$$\dim_H \widehat{\nu} = \frac{h(\mu) - h_\Pi(\mu)}{-\log \rho} + \dim \nu. \tag{89}$$

Let us introduce measurable partitions of $[0, 1]^3$ by $\xi(a, y) := \{a\} \times \{y\} \times [0, 1]$ and $\tau(a) := \{a\} \times [0, 1] \times [0, 1]$. Moreover, define a measurable partition of $[0, 1] \times \{0\} \times [0, 1]$ by $\zeta(a) = \{a\} \times \{0\} \times [0, 1]$ and a measurable partition of $[0, 1]^2 \times \{0\}$ by $\eta(a) = \{a\} \times [0, 1] \times \{0\}$.

By Rokhlin's theorem there are families of conditional measures $\widehat{\nu}^\xi_{a,y}$, $\widehat{\nu}^\tau_a$, $\widetilde{\nu}^\zeta_a$ and ν^η_a respectively on the partitions, uniquely defined up to zero-measure sets.

By definition of conditional measures and the partition τ, $\widehat{\nu} = \int \widehat{\nu}^\tau_a d\nu_x(a)$, where $\nu_x = (\Pi_{\mathcal{H}})_* \mu$ (see (29)). On the other hand, $\widehat{\nu} = \int \widehat{\nu}^\xi_{a,y}\, d\nu(a, y) = \iint \widehat{\nu}^\xi_{a,y}\, d\nu^\eta_a(y)\, d\nu_x(a)$. Thus,

$$\widehat{\nu}^\tau_a = \int \widehat{\nu}^\xi_{a,y}\, d\nu^\eta_a(y) \quad \text{for } \nu_x\text{-a.e. } a.$$

Let $\mathrm{proj} : [0, 1]^3 \mapsto \{0\} \times [0, 1]^2$ be the orthogonal projection to the (y, z)-coordinate plane. Since $(\mathrm{proj})_* \widehat{\nu}^\tau_a = \widetilde{\nu}^\zeta_a$ for ν_x-a.e. a, we get that

$$\widetilde{\nu}^\zeta_a = \int (\mathrm{proj})_* \widehat{\nu}^\xi_{a,y}\, d\nu^\eta_a(y) \quad \text{for } \nu_x\text{-a.e. } a. \tag{90}$$

Applying [**3**, Theorem 7.1] we have

$$\dim \widehat{\nu}^\xi_{a,y} = \frac{h(\mu) - h_\Pi(\mu)}{-\log \rho} \quad \text{for } \nu\text{-a.e. } (a, y),$$
$$\dim \widetilde{\nu}^\zeta_a = \frac{h(\mu) - h_{\Pi_{\mathcal{H}}}(\mu)}{-\log \rho} \quad \text{for } \nu\text{-a.e. } a.$$

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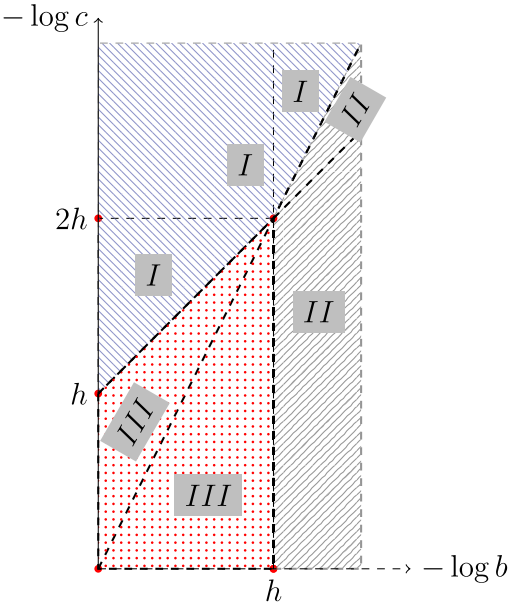


FIGURE 4. Regions I, II and III correspond to the area where the minimum in (91) is attained at the first, second or third expression, respectively.

By assumptions (C1) and (C4), we may apply [**11**, Theorem 2.8] and [**13**, Theorem 1.1], and therefore

$$\dim \nu_x = \frac{h_{\Pi_{\mathcal{H}}}(\mu)}{-\log c} = \frac{h(\mu)}{-\log c}.$$

Therefore $\dim \widetilde{\nu}_a^{\zeta} = 0$ for ν_x-a.e. a.

By (90), if $\widetilde{\nu}_a^{\zeta}(R) = 0$ for a Borel set $R \subseteq \{a\} \times \{0\} \times [0, 1]$ then $(\mathrm{proj})_* \widehat{\nu}_{a,y}^{\xi}(R) = 0$ for ν_a^{η}-a.e y. Thus, by the definition of the Hausdorff dimension $\dim_H \widetilde{\nu}_a^{\zeta} \geq \dim_H (\mathrm{proj})_* \widehat{\nu}_{a,y}^{\xi} = \dim_H \widehat{\nu}_{a,y}^{\xi}$ for ν-a.e (a, y). Hence $\dim_H \widehat{\nu}_{a,y}^{\xi} = 0$ for ν-a.e. (a, y), which implies that $h(\mu) = h_{\Pi}(\mu)$. □

LEMMA 5.3. *Let $\mathcal{S}$ be a self-affine IFS of the form (2) satisfying the assumptions of Assumption C. Then*

$$\dim \nu \geq \min \left\{ 2\frac{h(\mu)}{-\log c}, \frac{h(\mu)}{-\log b}, 1 + \frac{h(\mu) + \log b}{-\log c} \right\}. \tag{91}$$

The geometric interpretation of formula (91) is presented in Figure 4.

Proof. By [**13**, Theorem 1.1],

$$\dim(\Pi_{BF})_* \mu = \min \left\{ 1, \frac{h(\mu)}{\log b - \log c} \right\}.$$

By [**2**, Lemma 4.3],

$$\dim(\mathrm{Proj}_\theta)_* \nu \geq \min\{\dim(\Pi_{BF})_* \mu, \dim \nu\} \quad \text{for } (\Pi_{BF})_* \mu\text{-a.e. } \theta. \tag{92}$$

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Let us define a sequence $\{x_n\}_{n=0}^{\infty}$ inductively as follows. Let $x_0 = h(\mu)/(-\log c)$ and $x_n = r(x_{n-1})$ for $n \geq 1$, where

$$r(x) = \frac{h(\mu)}{-\log c} + \left(1 - \frac{\log b}{\log c}\right) \min\left\{1, \frac{h(\mu)}{\log(b/c)}, x\right\}.$$

It is easy to see that

$$\lim_{n\to\infty} x_n = \min\left\{2\frac{h(\mu)}{-\log c}, \frac{h(\mu)}{-\log b}, 1 + \frac{h(\mu) + \log b}{-\log c}\right\},$$

which is the fixed point of $x \mapsto r(x)$. By applying (84) and Lemma 5.2, one can show by induction that $\dim \mu \geq x_n$ for every $n \geq 0$, as required. □

Proof of Theorem 5.1. By [**19**, Proposition 6.1] and (92), if

$$\dim(\Pi_{BF})_*\mu > \min\{\dim \nu, 2 - \dim \nu\} \tag{93}$$

then

$$\dim(\mathrm{Proj}_\theta)_*\nu = \min\{1, \dim \nu\} \quad \text{for } (\Pi_{BF})_*\mu\text{-a.e. } \theta. \tag{94}$$

But by [**13**, Theorem 1.1],

$$\dim(\Pi_{BF})_*\mu = \min\left\{1, \frac{h(\mu)}{\log b - \log c}\right\},$$

and by Lemma 5.3,

$$\min\left\{2\frac{h(\mu)}{-\log c}, \frac{h(\mu)}{-\log b}, 1 + \frac{h(\mu) + \log b}{-\log c}\right\} \leq \dim \nu,$$

which together with assumption (C5) implies (93). Thus, (84), Lemma 5.2 and (94) verify the statement. □

6. *Examples*

6.1. *Examples for the case where the x-direction dominates.* We present three examples. In the first and second case we can apply Theorem 4.1 and in the third example we can apply Theorem 4.2. In all examples c can be chosen as an arbitrary element of a parameter interval except for a small exceptional set which will be $E_1 \subset (0.5, 1)$, $E_2 \subset (1/3, 1)$ and $E_3 \subset (1/\sqrt{3}, 1)$ in the examples respectively with the properties:

$$\dim_{\mathrm{H}}(E_1) = \dim_{\mathrm{H}}(E_2) = 0 \quad \text{and} \quad \mathcal{L}\mathrm{eb}(E_3) = 0. \tag{95}$$

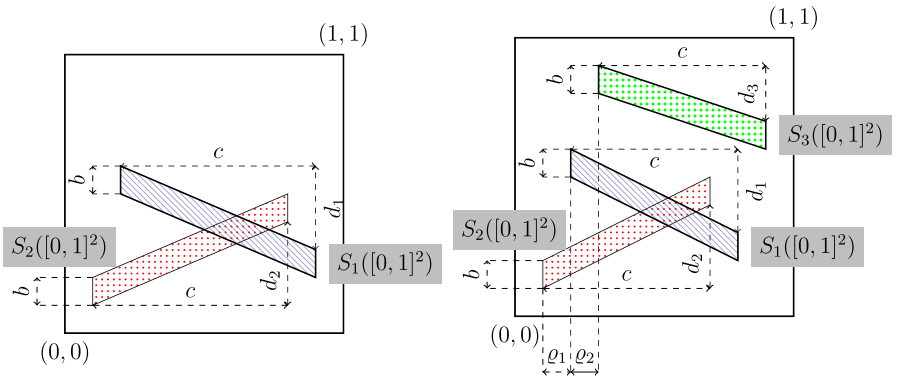
The precise definition of these exceptional sets is given in §6.1.2.

Example 6.1. Let $\mathcal{S}$ be an IFS of the form (2), where $N = 2$. We choose an arbitrary $c \in (\frac{1}{2}, 1)\backslash E_1$ (see (105)) and $0 < b < c/2$. We assume that $u_1 \neq u_2$, $d_1 \neq d_2$ and v_1, v_2 are selected in such a way that the IFS $\mathcal{S} := \{S_1, S_2\}$ satisfies $S_i([0, 1]^2) \subset [0, 1]^2$. Then by Theorem 4.1 we have

$$\dim_{\mathrm{H}}(\Lambda) = 1 + \frac{\log(2c)}{-\log b} \tag{96}$$

(see Figure 5(a)).

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(A) In Example 6.1; (B) and in Example 6.2 and 6.3.

FIGURE 5. Negative values of d_i correspond to 'decreasing' parallelograms. (In Example 6.2, $\rho_1 = \rho_2$.)

We can apply Theorem 4.1 in Example 6.1 because Assumption A holds. Namely, it is obvious that Assumptions (A1)–(A2) hold. Assumption (A3) follows from the choice of E_1 (see (105)). Assumption (A4) (the transversality condition) follows from the fact that the associated three-dimensional IFS $\widetilde{\mathcal{S}}$, defined in (14), satisfies the SSP. That is, the third coordinate of $\widetilde{\mathcal{S}}$ is an IFS $\mathcal{F}$ on the line, defined in (9). This consists of two maps with distinct fixed points and the sum of their contraction ratios is less than 1. This means that the SSP holds for $\mathcal{F}$. Consequently, the SSP holds for $\widetilde{\mathcal{S}}$. Hence the transversality condition (A4) also holds.

Example 6.2. (a) $N = 3$.

(b) We fix an arbitrary $c \in (\frac{1}{3}, 1) \setminus E_2$ (see (106)).

(c) Let $b \in (0, c/2) \cap (0, \frac{1}{3})$.

(d) u_1, u_2, u_3 are pairwise different.

(e) We assume that $d_1 < d_2$ and we choose d_3 from the interval

$$d_3 \in \left(d_2\left(2 - \frac{c}{b}\right) + d_1\left(\frac{c}{b} - 1\right), d_1\left(2 - \frac{c}{b}\right) + d_2\left(\frac{c}{b} - 1\right)\right).$$

Note that this holds for example if $d_1 < d_3 < d_2$.

(f) We choose the vertical translation parameters (as on the right-hand side of Figure 5(b)) so that

$$S_3([0, 1]^2) \cap (S_1([0, 1]^2) \cup S_2([0, 1]^2)) = \emptyset, \tag{97}$$

and $S_i([0, 1]^2) \subset [0, 1]^2$ for $i = 1, 2, 3$.

Then we have

$$\dim_{\mathrm{H}}(\Lambda) = 1 + \frac{\log(3c)}{-\log b} \tag{98}$$

(see Figure 5(b)).

Example 6.3. (a) $N = 3$.

(b) We fix an arbitrary $c \in (1/\sqrt{3}, 1) \setminus E_3$ (see (106)).

(c) Let $b \in (0, c/2)$.

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(d) u_1, u_2, u_3 are consecutive elements of an arithmetic progression (not necessarily in this order).

(e) We assume that $d_1 < d_2$ and we choose d_3 from the interval

$$d_3 \in \left(d_2\left(2 - \frac{c}{b}\right) + d_1\left(\frac{c}{b} - 1\right), d_1\left(2 - \frac{c}{b}\right) + d_2\left(\frac{c}{b} - 1\right)\right).$$

Note that this holds for example if $d_1 < d_3 < d_2$.

(f) We choose the vertical translation parameters (as on the right-hand side of Figure 5(b)) so that

$$S_3([0,1]^2) \cap (S_1([0,1]^2) \cup S_2([0,1]^2)) = \emptyset, \tag{99}$$

and $S_i([0,1]^2) \subset [0,1]^2$ for $i = 1, 2, 3$.

Then we have

$$\dim_{\mathrm{H}}(\Lambda) = \min\left\{2, 1 + \frac{\log(3c)}{-\log b}\right\} \tag{100}$$

(see Figure 5(b)).

Combining (c), (e) and (f) (note that these assumptions appear both in Examples 6.2 and 6.3) we get that the transversality condition holds. Namely, with the open set

$$V := (0,1)^2 \times \left(\frac{\max(d_2, d_3)}{b - c}, \frac{\min(d_1, d_3)}{b - c}\right)$$

the IFS $\widetilde{\mathcal{S}}$ satisfies the SSP. Hence by Lemma 1.2 the transversality condition holds. Hence, we can apply Theorem 4.1 for Example 6.2 in the same way we applied it for Example 6.1.

For Example 6.3 we will apply Theorem 4.2 here. The combination of (b) and Corollary 6.5 implies that ν_x has bounded density. Hence, Theorem 4.2 applies.

6.1.1. *Phase transition in Example 6.3.* Note that in Example 6.3 the parameter interval $I = (1/\sqrt{3}, 1)$ can be partitioned naturally into

$$I_1 := \left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right] \quad \text{and} \quad I_2 := \left(\sqrt{\frac{2}{3}}, 1\right].$$

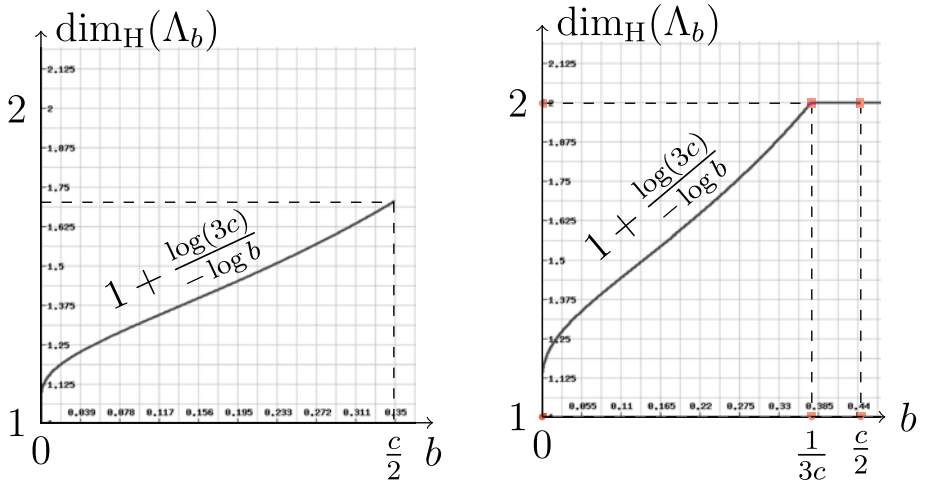
Namely, the affinity dimension

$$A(b, c) := 1 + \frac{\log(3c)}{-\log b}$$

is monotone increasing in both b and c. So for a fixed c it takes its biggest value for $b = c/2$. Clearly $c = \sqrt{2/3}$ is the solution of $A(c/2, c) = 2$. That is, for $c \in I_1 \setminus E_3$ we have $\dim_{\mathrm{H}}(\Lambda) = A(b, c)$ for every $0 < b \le c/2$. However, for $c \in I_2 \setminus E_3$ it depends on the choice of b if $\dim_{\mathrm{H}}(\Lambda) = A(b, c)$ or $\dim_{\mathrm{H}}(\Lambda) = 2$. Now we fix a $c \in I_2 \setminus E_3$, u_1, u_2, u_3 and d_1, d_2, d_3 satisfying the conditions of Example 6.3. We vary only b. So the attractor will be denoted by Λ_b. To study the function

$$b \mapsto \dim_{\mathrm{H}}(\Lambda_b), \quad b \in \left[0, \frac{c}{2}\right], \tag{101}$$

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(A) $c = 0.7 < \sqrt{2/3}$. (B) $c = 8/9 > \sqrt{2/3}$.

FIGURE 6. The graphs of the continuous functions with which the dimension coincides almost everywhere in Example 6.3. There is no phase transition if $c < \sqrt{2/3}$ and there is a phase transition at $b = c/3$ for $\sqrt{2/3} < c < 1$.

first observe that for a $c \in I_2$ the function $A(\cdot, c)$ is monotone increasing and

$$A\left(\frac{1}{3c}, c\right) = 2. \tag{102}$$

Then by Theorems 4.2, for a $c \in I_2 \backslash E_3$ we have

$$\dim_{\mathrm{H}} \Lambda_b = \begin{cases} 1 + \dfrac{\log(3c)}{-\log b} & \text{if } 0 < b \le \dfrac{1}{3c}, \\ 2 & \text{if } \dfrac{1}{3c} \le b \le \dfrac{c}{2}. \end{cases} \tag{103}$$

This means that there is a phase transition at $b = c/3$, where the graph of $b \mapsto \dim_{\mathrm{H}} \Lambda_b$ is non-differentiable, although it is continuous on the interval $b \in (0, c/2]$, see Figure 6.

6.1.2. *The definition of exceptional parameter sets E_1, E_2, E_3.* Let E_1 be the set of exceptional parameters

$$E_1 := \left\{c \in \left(\tfrac{1}{2}, 1\right) : \text{(A3) does not hold for the IFS } \{cx, cx + 1\}\right\}. \tag{104}$$

Let

$$E_2 := \left\{c \in \left(\tfrac{1}{3}, 1\right) : \text{(A3) does not hold for the IFS } \{cx + u_1, cx + u_2, cx + u_3\}\right\}. \tag{105}$$

It follows from [**22**, Theorem A] that E_1 and E_2 are sets of Hausdorff dimension zero. Note concerning E_1 that as long as $u_1 \neq u_2$ the exceptional set is the same for all IFSs $\{cx + u_1, cx + u_2\}$.

To define the third exceptional set, first we need to state the following theorem [**24**, Theorem 2].

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THEOREM 6.4. (Simon and Tóth) *For a natural number $N > 1$ let $\mathfrak{m}_\lambda^N$ be the self-similar measure corresponding to the IFS $\{\lambda x + i\}_{i=0}^{N-1}$ and the uniform distribution of weights $\{1/N, \dots, 1/N\}$. Then there exists an exceptional set $R_N \subset (1/N, 1)$ having Lebesgue measure zero such that $\mathfrak{m}_\lambda^N$ is absolutely continuous with L^2 density whenever $\lambda \in (1/N, 1)\backslash R_N$.*

From this is immediate to see that the following corollary holds.

COROLLARY 6.5. *Here we use the notation of Theorem 6.4. Let*

$$E_3 := \left\{ c \in \left(\frac{1}{\sqrt{3}}, 1 \right) : c^2 \in R_3 \right\}. \tag{106}$$

Then the Lebesgue measure of E_3 is zero and $\mathfrak{m}_c^N$ is absolutely continuous with continuous (consequently bounded) density for each $c \in (1/\sqrt{3}, 1)\backslash E_3$.

6.2. *Example for the case where the y-direction dominates.* Before we show an example for Theorem 5.1, we show a family of IFSs of similarities, which satisfies Hochman's exponential separation condition.

LEMMA 6.6. *Let $\Psi := \{x \mapsto \alpha x + \tau_1, x \mapsto \alpha x + \tau_2, x \mapsto \alpha x + \tau_3\}$ be an IFS on the real line such that $\alpha, \tau_1, \tau_2 \in \mathbb{Q}$, and $\tau_3 \notin \mathbb{Q}$. Then Ψ satisfies Hochman's exponential separation condition.*

The proof is similar to the proof of [**13**, Theorem 1.6].

Proof. Without loss of generality, we may assume that $\tau_1 = 0$, $\tau_2 = 1$, and $0 < \tau_3 < 1$ irrational. For any finite length word $\bar{\imath} \in \{1, 2, 3\}^*$, let $\pi(\bar{\imath}) = \sum_{k=0}^{|\bar{\imath}|} \tau_{i_k} \alpha^k$.

By (21), it is enough to show that there exist $\varepsilon > 0$ and a sequence n_k such that for any $\bar{\imath} \neq \bar{\jmath}$ with $|\bar{\imath}| = |\bar{\jmath}| = n_k$, $|\pi(\bar{\imath}) - \pi(\bar{\jmath})| > \varepsilon^{n_k}$. Suppose that this is not the case. That is, for every $\varepsilon > 0$ there exists a K such that for every $k \geq K$ there exist finite words $\bar{\imath}_k, \bar{\jmath}_k$ with $|\bar{\imath}_k| = |\bar{\jmath}_k| = k$ such that

$$|\pi(\bar{\imath}_k) - \pi(\bar{\jmath}_k)| < \varepsilon^k. \tag{107}$$

By definition, $\alpha = p/q$, with $p, q \in \mathbb{N}$ relatively prime and $q \neq 0$. Thus, for any $\bar{\imath} \neq \bar{\jmath}$ with $|\bar{\imath}| = |\bar{\jmath}| = k$,

$$\pi(\bar{\imath}) - \pi(\bar{\jmath}) = \frac{p_1(\bar{\imath}, \bar{\jmath})}{q^k} + \tau_3 \frac{p_2(\bar{\imath}, \bar{\jmath})}{q^k},$$

where $p_1(\bar{\imath}, \bar{\jmath})$, $p_2(\bar{\imath}, \bar{\jmath}) \in \mathbb{N}$. Observe that

$$|p_i(\bar{\imath}, \bar{\jmath})| \leq (2q)^k$$

for every $k \geq 1$ and $\bar{\imath}, \bar{\jmath}$ with $|\bar{\imath}| = |\bar{\jmath}| = k$. On the other hand, if $p_2(\bar{\imath}, \bar{\jmath}) = 0$ and $\bar{\imath} \neq \bar{\jmath}$ then $p_1(\bar{\imath}, \bar{\jmath}) \neq 0$, because of the rational root test of polynomials with integer coefficients. Thus, $|\pi(\bar{\imath}) - \pi(\bar{\jmath})| \geq 1/q^k$.

Hence, by taking $\varepsilon = 1/(16q^3)$ and the sequence of words $\bar{\imath}_k, \bar{\jmath}_k$ with $|\bar{\imath}_k| = |\bar{\jmath}_k| = k$, given in equation (107), we get

$$\left| \frac{p_1(\bar{\imath}_k, \bar{\jmath}_k)}{p_2(\bar{\imath}_k, \bar{\jmath}_k)} - \tau_3 \right| = \left| \pi(\bar{\imath}_k) - \pi(\bar{\jmath}_k) \right| \frac{q^k}{|p_2(\bar{\imath}_k, \bar{\jmath}_k)|} \leq \frac{1}{(16q^2)^k}. \tag{108}$$

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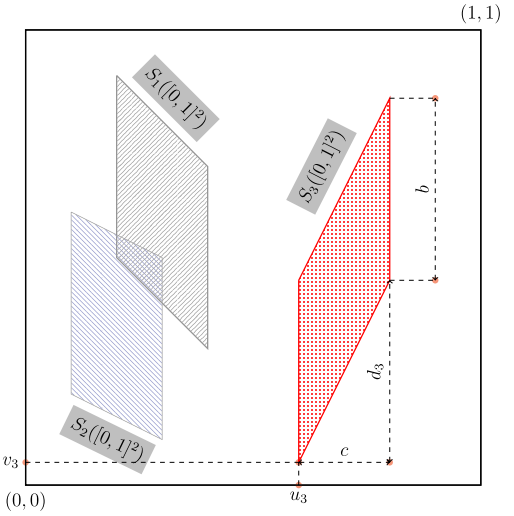


FIGURE 7. Illustration of Example 6.7.

Therefore

$$\left|\frac{p_1(\overline{\imath}_{k+1},\overline{\jmath}_{k+1})}{p_2(\overline{\imath}_{k+1},\overline{\jmath}_{k+1})}-\frac{p_1(\overline{\imath}_k,\overline{\jmath}_k)}{p_2(\overline{\imath}_k,\overline{\jmath}_k)}\right|\le\frac{2}{(16q^2)^k}.$$

But since the left-hand side is a rational number, either

$$\left|\frac{p_1(\overline{\imath}_{k+1},\overline{\jmath}_{k+1})}{p_2(\overline{\imath}_{k+1},\overline{\jmath}_{k+1})}-\frac{p_1(\overline{\imath}_k,\overline{\jmath}_k)}{p_2(\overline{\imath}_k,\overline{\jmath}_k)}\right|>\frac{1}{(2q)^{2k+1}}$$

or

$$\left|\frac{p_1(\overline{\imath}_{k+1},\overline{\jmath}_{k+1})}{p_2(\overline{\imath}_{k+1},\overline{\jmath}_{k+1})}-\frac{p_1(\overline{\imath}_k,\overline{\jmath}_k)}{p_2(\overline{\imath}_k,\overline{\jmath}_k)}\right|=0.$$

Thus, for every k large enough $p_1(\overline{\imath}_k,\overline{\jmath}_k)/p_2(\overline{\imath}_k,\overline{\jmath}_k)\equiv P/Q$, and by (108), $\tau_3=P/Q$, which contradicts the assumption that τ_3 is irrational. □

Example 6.7. Let $\mathcal{S}$ be an IFS of the form (2), where:

- $N=3$;
- $c, b\in\mathbb{Q}$ such that $c<\min\{1/3, b\}$, $b<1$ and
$$b<\min\left\{\sqrt{c}, c^{1+\log 3/(2\log(3c))}\right\};$$
- $d_{i \bmod 3}, d_{i+1 \bmod 3}\in\mathbb{Q}$ and $d_{i+2 \bmod 3}\notin\mathbb{Q}$ for an $i\in\mathbb{N}$;
- $u_{j \bmod 3}, u_{j+1 \bmod 3}\in\mathbb{Q}$ and $u_{j+2 \bmod 3}\notin\mathbb{Q}$ for an $j\in\mathbb{N}$

(see Figure 7).

Lemma 6.6 implies that $\mathcal{H}$ (defined in (4)) and the backward Furstenberg–Kifer IFS $\mathcal{B}$ (defined in (81)) satisfy Hochman's exponential separation condition. Moreover, the assumptions on c, b verify that (C1), (C2) and (C5) hold in Assumption C.

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A. *Appendix*

Here we gather together some consequences of the fact that a measure on the line is absolutely continuous with L^q density for a $q > 1$.

Given a measure $\mu \ll \mathcal{L}_1$ supported by [0, 1], we assume that there exists a $q > 1$ such that the density φ of μ satisfies

$$C_q := \int_0^1 \varphi(t)^q \, dt < \infty. \tag{A.1}$$

LEMMA A.1. *Let $\varphi(t)$ be the density of ν_x. Assume that (A.1) holds. Then for any interval I.*

$$\mu(I) \le C_q^{1/q} |I|^{1-1/q}.$$

Proof. By Jensen's inequality we have

$$\int_I \varphi^q(t) \, dt \ge \Bigg(\underbrace{\int_I \varphi(t) \, dt}_{\mu(I)} \Bigg)^q \cdot |I|^{1-q}. \tag{A.2}$$

This completes the proof after some obvious algebraic manipulations. □

Definition A.2. Given a real number $R > 1$, we say that a not necessarily countable family $\mathcal{I} := \{I_i\}_{i \in \Gamma}$, $I_i \subset [0, 1]$, of closed intervals is an *R-bad family* if

$$\frac{\mu(I_i)}{|I_i|} > R \quad \text{for all } i \in \Gamma. \tag{A.3}$$

We say that $\mathcal{I}$ is a *disjoint R-bad family* of intervals if $\mathcal{I}$ is an R-bad family and the intervals in $\mathcal{I}$ are pairwise disjoint.

LEMMA A.3. *There exists a constant $c_1 > 0$ independent of everything such that for every $R > 1$ if $\mathcal{I}$ is an R-bad family of intervals then*

$$\mu\left\{ \bigcup_{i \in \Gamma} I_i \right\} < c_1 \cdot R^{-(q-1)}. \tag{A.4}$$

Proof for disjoint R-bad families. Fix an $R > 1$. By Lemma A.1 and equation (A.5), for every $i \in \Gamma$ we have

$$C_q \ge \mu(I_i) \cdot \left(\frac{\mu(I_i)}{|I_i|} \right)^{q-1}. \tag{A.5}$$

Using (A.3) and the fact that we temporarily assumed that $\mathcal{I}$ is a disjoint family, we obtain that

$$\mu\left(\bigcup_{i \in \Gamma} I_i \right) \le C_q \cdot R^{-(q-1)}. \tag{A.6}$$

□

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Proof for non-disjoint R-bad families. Fix an $R > 1$. For every $i \in \Gamma$ and for every $x \in I_i$ let

$$\widetilde{I}_i(x) := [x - |I_i|, x + |I_i|].$$

Let $A := \bigcup_i I_i$, and we form the family $\mathcal{B}$ of closed (one-dimensional) balls $\{\widetilde{I}_i(x)\}_{i, x \in I_i}$. Then by definition we can apply Besicovitch's covering theorem (using the notation of [**18**, p. 30, Theorem 2.7 part (2)]) for A and $\mathcal{B}$. From this theorem there is a constant Q independent of everything such that we can find Q sub-families $\mathcal{B}_1, \ldots, \mathcal{B}_Q$ of $\mathcal{B}$ such that:

(a) for every k, $\mathcal{B}_k$ is a disjoint family. That is, for any two distinct intervals $I, J \in B_k$ we have $I \cap J = \emptyset$;

(b) if $J \in \mathcal{B}_k$ then $J = [x - |I_i|, x + |I_i|]$ for some $I_i \in \mathcal{I}$ and $x \in I_i$. Then

$$\frac{\mu(J)}{|J|} \geq \frac{\mu(I_i)}{2|I_i|} \geq \frac{1}{2}\frac{\mu(I_i)}{|I_i|} > \frac{1}{2} \cdot R. \tag{A.7}$$

(c) $A \subset \bigcup_{k=1}^{Q} \bigcup_{J \in \mathcal{B}_k} J$.

Using (a) and (b) for every k, $\mathcal{B}_k$ is a disjoint $R/2$-bad family. Since we have already verified the assertion of the lemma for disjoint families we can apply (A.6) for $\mathcal{B}_k$ (writing $R/2$ instead of R) to get that for all $k = 1, \ldots, Q$,

$$\mu\left\{ \bigcup_{J \in \mathcal{B}_k} J \right\} < C_q \cdot 2^{q-1} \cdot R^{-(q-1)}. \tag{A.8}$$

Using this and (c) above, we obtain that

$$\mu(A) \leq \underbrace{QC_q 2^{q-1}}_{c_1} \cdot R^{-(q-1)}. \tag{A.9}$$

□

Hence we get the following two corollaries.

COROLLARY A.4. *Let $\mathcal{I}_n^\delta = \{I_i^{n,\delta}\}$ be a family of intervals of $|I_i^{(n,\delta)}| = c^n$ and $\mu(I_i^{(n,\delta)}) > c^{n(1-\delta)}$. Then $R = c^{-n\delta}$. So*

$$\mu\left(\bigcup_i I_i^{(n,\delta)} \right) < c_1 \cdot c^{n\delta(q-1)}. \tag{A.10}$$

COROLLARY A.5. *Let $n \geq 1$ be arbitrary. Let $\boldsymbol{\omega} \in \{1, \ldots, N\}^n$, and we write $I_{\boldsymbol{\omega}}$ for the interval which supports $\Pi_x(\boldsymbol{\omega})$ on the x-axis. We assume that $Nc > 1$. Let*

$$L_{n,\delta} := \left\{ t \in [0, 1] : \sum_{|\boldsymbol{\omega}|=n} \mathbb{1}_{I_{\boldsymbol{\omega}}}(t) > (Nc)^{n(1+\delta)} \right\}.$$

For every $t \in L_{n,\delta}$ let

$$J_t := [t - c^n, t + c^n].$$

Then

$$\frac{\mu(J_t)}{|J_t|} > \frac{1}{2}(Nc)^{n\delta}.$$

That is, we may apply (A.4) for $\{J_t\}_{t \in L_{n,\delta}}$ to get that

$$\mu(L_{n,\delta}) \leq \mu\left(\bigcup_{t \in L_{n,\delta}} J_t \right) \leq c_1 2^{q-1} (Nc)^{-n\delta(q-1)}. \tag{A.11}$$

Recall that the constant c_1 was defined in (A.9), so it is independent of δ.

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COROLLARY A.6. *Assume that ν_x has L^q density for all $q > 1$. Then for every $K > 0$ and $q > 1$ there exists $C = C(K, q)$ such that for every $\ell > 0$ every interval I of length Kc^ℓ intersects at most $C(Nc)^\ell c^{-\ell/q}$ intervals $h_{\boldsymbol{\tau}}([0, 1])$, $|\boldsymbol{\tau}| = \ell$.*

Proof. Denote by Z the number of intervals $h_{\boldsymbol{\tau}}([0, 1])$, $|\boldsymbol{\tau}| = \ell$, intersecting I. Each of those intervals is contained in the interval $I' = B_{c^\ell}(I)$. Hence,

$$\nu_x(I') \geq Z \cdot N^{-\ell}.$$

Applying Lemma A.1 to the interval I', we get the assertion. □

REFERENCES

[1] B. Bárány. Dimension of the generalized 4-corner set and its projections. *Ergod. Th. & Dynam. Sys.* **32**(04) (2012), 1190–1215.
[2] B. Bárány. On the Ledrappier–Young formula for self-affine measures. *Math. Proc. Cambridge Philos. Soc.* **159**(3) (2015), 405–432.
[3] B. Bárány and A. Käenmäki. Ledrappier-Young formula and exact dimensionality of self-affine measures. *Adv. in Math.* **318** (2017), 88–129.
[4] B. Bárány, M. Pollicott and K. Simon. Stationary measures for projective transformations: The Blackwell and Furstenberg measures. *J. Stat. Phys.* **148**(3) (2012), 393–421.
[5] B. Bárány and M. Rams. Dimension maximizing measures and local dimension spectrum for self-affine systems. *Trans. Amer. Math. Soc.* (2017), to appear. https://doi.org/10.1090/tran/7103.
[6] B. Bárány, M. Rams and K. Simon. On the dimension of self-affine sets and measures with overlaps. *Proc. Amer. Math. Soc.* **144** (2016), 4427–4440.
[7] K. Falconer and T. Kempton. The dimension of projections of self-affine sets and measures. *Ann. Acad. Sci. Fenn. Math.* **42**(1) (2017), 473–486.
[8] K. Falconer and J. Miao. Dimensions of self-affine fractals and multifractals generated by upper-triangular matrices. *Fractals* **15**(03) (2007), 289–299.
[9] K. J. Falconer. *The Geometry of Fractal Sets*. Cambridge University Press, Cambridge, 1986.
[10] K. J. Falconer. The Hausdorff dimension of self-affine fractals. *Math. Proc. Cambridge Philos. Soc.* **103**(3) (1988), 339–350.
[11] D.-J. Feng and H. Hu. Dimension theory of iterated function systems. *Comm. Pure Appl. Math.* **62**(11) (2009), 1435–1500.
[12] H. Furstenberg and Y. Kifer. Random matrix products and measures on projective spaces. *Israel J. Math.* **46**(1–2) (1983), 12–32.
[13] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy. *Ann. of Math. (2)* **180**(2) (2014), 773–822.
[14] M. Hochman and B. Solomyak. On the dimension of Furstenberg measure for $\mathrm{SL}_2(\mathbb{R})$ random matrix products. *Invent. Math.* (2017), to appear. https://doi.org/10.1007/s00222-017-0740-6.
[15] B. R. Hunt and V. Yu. Kaloshin. How projections affect the dimension spectrum of fractal measures. *Nonlinearity* **10**(5) (1997), 1031–1046.
[16] T. Jordan, M. Pollicott and K. Simon. Hausdorff dimension for randomly perturbed self affine attractors. *Comm. Math. Phys.* **270**(2) (2007), 519–544.
[17] K.-S. Lau. Self-similarity, L^p-spectrum and multifractal formalism. *Fractal Geometry and Stochastics (Finsterbergen, 1994) (Progress in Probability, 37)*. Birkhäuser, Basel, 1995, pp. 55–90.
[18] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*. Cambridge University Press, Cambridge, 1999.
[19] Y. Peres and W. Schlag. Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions. *Duke Math. J.* **102**(2) (2000), 193–251.
[20] A. Rapaport. On self-affine measures with equal Hausdorff and Lyapunov dimension. *Trans. Amer. Math. Soc.* (2017), to appear. https://doi.org/10.1090/tran/7099.
[21] P. Shmerkin. On Furstenberg's intersection conjecture, self-similar measures, and the L^q norms of convolutions. *Preprint*, 2016, arXiv:1609.07802.
[22] P. Shmerkin and B. Solomyak. Absolute continuity of self-similar measures, their projections and convolutions. *Trans. Amer. Math. Soc.* **368**(7) (2016), 5125–5151.

https://doi.org/10.1017/etds.2017.110 Published online by Cambridge University Press

[**23**] K. Simon. The dimension theory of almost self-affine sets and measures. *Fractals, Wavelets, and their Applications (Springer Proceedings in Mathematics & Statistics, 92)*. Springer, Cham, 2014, pp. 103–127.
[**24**] K. Simon and H. R. Tóth. The absolute continuity of the distribution of random sums with digits $\{0, 1, \ldots, m-1\}$. *Real Anal. Exchange* **30**(1) (2004/05), 397–409.
[**25**] B. Solomyak. On the random series $\sum \pm\lambda^n$ (an Erdös problem). *Ann. of Math. (2)* **142**(3) (1995), 611–625.
[**26**] B. Solomyak. Measure and dimension for some fractal families. *Math. Proc. Cambridge Philos. Soc.* **124** (1998), 531–546.
[**27**] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, New York, 1982.

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