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# The sequential topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not regular

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The compact-open topology on the set of continuous functionals from the Baire space to the natural numbers is well known to be zero-dimensional. We prove that the closely related sequential topology on this set is not even regular. The sequential topology arises naturally as the topology carried by the exponential $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ formed in various cartesian closed categories of topological spaces. Moreover, we give an example of an effectively open subset of $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ that violates regularity. The topological properties of $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ are known to be closely related to an open problem in Computable Analysis. We also show that the sequential topology on the space of continuous real-valued functions on a Polish space need not be regular.

## 1. Introduction

The Kleene–Kreisel continuous functionals over the natural numbers play an important role in mathematical logic and in higher type computability (Kleene 1959; Kreisel 1959). There are several ways to define the hierarchy of continuous functionals (Hyland 1979; Normann 1980). One important approach is to construct the hierarchy via a sequence of exponentials in an appropriate cartesian closed category by applying the recursion formula $\mathsf{N}\langle 0\rangle := \mathbb{N}$ and $\mathsf{N}\langle k+1\rangle := \mathbb{N}^{\mathsf{N}\langle k\rangle}$.

Among the cartesian closed categories of topological spaces (Escardó *et al.* 2004), both the category kHaus of Hausdorff Kelley spaces and the category Seq of sequential spaces can be used to establish the continuous functionals. An application of the above recursion formula in either of these categories produces the same sequence of topological spaces (*cf.* Section 2.5). The underlying set of the topological space $\mathsf{N}\langle k\rangle$ obtained in this way consists exactly of the Kleene–Kreisel continuous functionals of type $k$. The topology of $\mathsf{N}\langle k\rangle$ is referred to as the *sequential topology* on the continuous functionals of type $k$.

For $k \geqslant 2$, the sequential topology on $\mathsf{N}\langle k\rangle$ differs from the ordinary compact-open topology. Indeed, there are continuous functionals of type $k+1$ that are discontinuous with respect to the compact-open topology on $\mathsf{N}\langle k\rangle$. So, in contrast to the sequential topology, the compact-open topology fails to describe the topological behaviour of the continuous functionals appropriately.

As our main result, we prove that the sequential topology on $\mathsf{N}\langle k\rangle$ is not regular for $k \geqslant 2$ and thus not zero-dimensional. This is surprising, since the compact-open topology on $\mathsf{N}\langle k\rangle$ is zero-dimensional and its kelleyfication (that is, its k-space coreflection) yields the

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sequential topology. Moreover, the sequential topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}} = \mathsf{N}\langle 2\rangle$ was conjectured to be zero-dimensional in Bauer *et al.* (2002).

We also construct an example of an effectively open subset of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ that violates regularity. As an intermediary result we prove the existence of a countable Polish space $\mathsf{S}$ with the property that the sequential topology on the space of continuous real-valued functions on $\mathsf{S}$ is not regular, again unlike the case for the corresponding compact-open topology. The space $\mathsf{S}$ can be used to deduce the non-regularity of further function spaces constructed in either of the categories $\mathsf{Seq}$ and $\mathsf{kHaus}$.

The topological properties of the Kleene–Kreisel continuous functionals are related to an open problem in Computable Analysis, which is the theory of computability over the space of real numbers and related spaces. The problem is whether or not two natural hierarchies of continuous functionals over the reals (called the intensional hierarchy and the extensional hierarchy) coincide. Bauer, Escardó and Simpson were the first to discover this relationship (Bauer *et al.* 2002). They showed that zero-dimensionality of the sequential topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ would imply that the two hierarchies agree on level 3. Normann (2005) gave a sufficient and necessary condition for the two hierarchies to coincide on each level.

In Section 2 we recall the definitions and basic properties of sequential spaces and related spaces. In Section 3 we give the proof of the main result and some of its consequences. In Section 4 we show that $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ contains an effectively open set that is not open in the completely regular reflection of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ and is thus not functionally open – an open set is *effectively open* if its characteristic function is computable in some sense (*cf.* Section 4). Finally, we discuss the results in Section 5.

## 2. Preliminaries

### 2.1. *Notation*

We use sans serif letters like $\mathsf{T}, \mathsf{X}, \mathsf{Y}$ to denote topological spaces. We write $\mathcal{O}(\mathsf{X})$ for the topology of the space $\mathsf{X}$ and $\to_{\mathsf{X}}$ for the induced convergence relation on sequences. By abuse of notation, we will often use $\mathsf{X}$ to denote the carrier set of a space $\mathsf{X}$.

We write $\mathbb{N}$ for the set of natural numbers (including 0) and also for the discrete topological space with carrier set $\mathbb{N}$. We use $\mathsf{2}$ to denote the two point discrete topological space with carrier set $\{0, 1\}$. We use:

— $\Sigma^{\mathbb{N}}$ to denote set of infinite sequences over a set $\Sigma$
— $\Sigma^*$ to denote the set of finite words over $\Sigma$
— $\Sigma^k$ to denote the set of words of length $k \in \mathbb{N}$ over $\Sigma$.

We use:

— $0^\omega$ to denote the constant zero-sequence
— $0^k$ to denote the word consisting of $k$ zeroes
— $p^{<k}$ to denote the prefix of $p \in \Sigma^{\mathbb{N}}$ of length $k$
— $\sqsubseteq$ to denote the prefix relation on $\Sigma^* \cup \Sigma^{\mathbb{N}}$.

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We use '.' to denote concatenation of finite and infinite words, but will use it rather sloppily: for example, $\{0,1\}^k.0^l.1.\{0,1\}^{\mathbb{N}}$ stands for the set

$$\{p \in \{0,1\}^{\mathbb{N}} \mid u.0^l.1 \sqsubseteq p \text{ for some } u \in \{0,1\}^k\}$$
$$= \{p \in \{0,1\}^{\mathbb{N}} \mid p(i) = 0 \text{ for all } i \in \{k, \ldots, k+l-1\} \text{ and } p(k+l) = 1\}.$$

For a word $u \in \Sigma^*$, we use $\lg(u)$ to denote the length of $u$. Finally, we use $\langle \cdot, \cdot \rangle$ to denote a computable bijection between $\mathbb{N}^2$ and $\mathbb{N}$.

### 2.2. *Sequential spaces*

In this section we summarise some basic facts about the category $\mathsf{Seq}$ of sequential spaces – more details can be found in Engelking (1989), Escardó *et al.* (2004) and Hyland (1979).

A *sequential space* is a topological space $\mathsf{X}$ such that every sequentially open subset of $\mathsf{X}$ is open in $\mathsf{X}$. Remember that a subset $U$ of $\mathsf{X}$ is *sequentially open* if every sequence $(x_n)_n$ that converges to some element in $U$ is eventually in $U$ (meaning that there is some $n_0$ with $x_n \in U$ for all $n \geqslant n_0$). Morphisms in $\mathsf{Seq}$ are the *sequentially continuous* functions (that is, functions that preserve convergent sequences). Importantly, sequential continuity coincides with topological continuity for functions between sequential spaces. Given a topological space $\mathsf{T}$, the topological space $\mathrm{seq}(\mathsf{T})$, which is topologised by the family of all sequentially open sets of $\mathsf{T}$, is the unique sequential space having the same convergence relation as $\mathsf{T}$. This functor seq is right-adjoint to the inclusion functor of $\mathsf{Seq}$ into the category $\mathsf{Top}$ of topological spaces (Hyland 1979). So we refer to $\mathrm{seq}(\mathsf{T})$ as the *sequential coreflection* of $\mathsf{T}$.

In contrast to $\mathsf{Top}$, the category $\mathsf{Seq}$ is cartesian closed (Escardó *et al.* 2004); (Hyland 1979). The product $\mathsf{X} \times \mathsf{Y}$ of two sequential spaces $\mathsf{X}$ and $\mathsf{Y}$ is given by the sequential coreflection $\mathrm{seq}(\mathsf{X} \times_{\mathsf{Top}} \mathsf{Y})$ of the usual topological product $\mathsf{X} \times_{\mathsf{Top}} \mathsf{Y}$. The exponential of $\mathsf{X}$ and $\mathsf{Y}$, which we denote by $\mathsf{Y}^{\mathsf{X}}$, has the set of all continuous functions from $\mathsf{X}$ to $\mathsf{Y}$ as its carrier set. Its convergence relation $\to_{\mathsf{Y}^{\mathsf{X}}}$ on sequences is *continuous convergence*: a sequence $(f_n)_n$ of continuous functions converges in $\mathsf{Y}^{\mathsf{X}}$ to $f_\infty$ if and only if $(x_n)_n \to_{\mathsf{X}} x_\infty$ implies $(f_n(x_n))_n \to_{\mathsf{Y}} f_\infty(x_\infty)$ for every sequence $(x_n)_n$ and every point $x_\infty$ in $\mathsf{X}$ (Escardó *et al.* 2004; Hyland 1979). By Schröder (2003, Lemma 4.2.2), the compact-open topology on $\mathsf{Y}^{\mathsf{X}}$ induces $\to_{\mathsf{Y}^{\mathsf{X}}}$, but it is not sequential in general. The exponential $\mathbb{N}^{\mathbb{N}}$ formed in $\mathsf{Seq}$ is indeed equal to the Baire space, and $2^{\mathbb{N}}$ is equal to the Cantor space, which has $\{W.\{0,1\}^{\mathbb{N}} \mid W \subseteq \{0,1\}^*\}$ as its topology. Both spaces are zero-dimensional, countably based and completely metrisable.

Let $e : \mathsf{X} \to \mathsf{Y}$ be a function between sequential spaces $\mathsf{X}, \mathsf{Y}$. We say that $e$ *reflects convergent sequences* if it satisfies $(x_n)_n \to_{\mathsf{X}} x_\infty \iff (e(x_n))_n \to_{\mathsf{Y}} e(x_\infty)$. Note that this property implies continuity. Moreover, $e$ is called a *homeomorphic embedding of* $\mathsf{X}$ *into* $\mathsf{Y}$ if $e$ is injective and the corestriction of $e$ to its image $e(\mathsf{X})$ is a homeomorphism between $\mathsf{X}$ and $e(\mathsf{X})$ endowed with the subspace topology.

**Lemma 2.1.** Let $\mathsf{X}, \mathsf{Y}$ be sequential spaces and let $e : \mathsf{X} \to \mathsf{Y}$ be an injection that reflects convergent sequences and has a closed image. Then $e$ is a homeomorphic embedding of $\mathsf{X}$ into $\mathsf{Y}$.

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*Proof.* Since $e$ is sequentially continuous and $\mathsf{X}$ is sequential, $e$ is topologically continuous by Engelking (1989, Proposition 1.6.15). As $e(\mathsf{X})$ is closed, the subspace topology on $e(\mathsf{X})$ is sequential (Engelking 1989, 2.1.H). Therefore, the sequentially continuous inverse of $e$ from $e(\mathsf{X})$ back to $\mathsf{X}$ is topologically continuous, so $e$ is a homeomorphic embedding. □

**Lemma 2.2.** Let $\mathsf{T}, \mathsf{X}, \mathsf{Y}$ be sequential spaces such that $\mathsf{X}$ is homeomorphic to a closed subspace of $\mathsf{Y}$. Then $\mathsf{X}^{\mathsf{T}}$ is homeomorphic to a closed subspace of $\mathsf{Y}^{\mathsf{T}}$.

*Proof.* Let $e$ be a homeomorphism between $\mathsf{X}$ and some closed subspace of $\mathsf{Y}$. Then $e$ reflects convergent sequences. We define $e^{\mathsf{T}} : \mathsf{X}^{\mathsf{T}} \to \mathsf{Y}^{\mathsf{T}}$ by $e^{\mathsf{T}}(h)(t) := e(h(t))$. It is easy to verify that $e^{\mathsf{T}}$ is injective and reflects convergent sequences. Let $(h_n)_n$ be a sequence in $\mathsf{X}^{\mathsf{T}}$ such that $(e(h_n))_n$ converges to some $f \in \mathsf{Y}^{\mathsf{T}}$. For every $t \in \mathsf{T}$, the sequence $(eh_n(t))_n$ converges to $f(t)$. Since $e(\mathsf{X})$ is closed and $e$ is injective, there is a unique element $h_\infty(t) \in \mathsf{X}$ with $eh_\infty(t) = f(t)$. Let $(t_n)_n$ converge to $t_\infty$ in $\mathsf{T}$. Then $(eh_\infty(t_n))_n$ converges to $eh_\infty(t_\infty)$ by continuity of $f$. Since $e$ reflects convergent sequences, $(h_\infty(t_n))_n$ converges to $h_\infty(t_\infty)$, which shows that $h_\infty \in \mathsf{X}^{\mathsf{T}}$. Thus $e^{\mathsf{T}}(\mathsf{X}^{\mathsf{T}})$ is closed. By Lemma 2.1, $e^{\mathsf{T}}$ is a homeomorphic embedding of $\mathsf{X}^{\mathsf{T}}$ into $\mathsf{Y}^{\mathsf{T}}$. □

A topological space $\mathsf{X}$ is called a *retract* of another space $\mathsf{Y}$ if there exist two continuous functions $s : \mathsf{X} \to \mathsf{Y}$ (called the *section*) and $r : \mathsf{Y} \to \mathsf{X}$ (called the *retraction*) satisfying $r \circ s = \mathrm{id}_{\mathsf{X}}$. Clearly, $s$ is a homeomorphic embedding of $\mathsf{X}$ into $\mathsf{Y}$. If $\mathsf{Y}$ is a Hausdorff space, $s(\mathsf{X})$ is closed. We can then get the following lemma from category theory.

**Lemma 2.3.** Let $\mathsf{X}_1$ and $\mathsf{X}_2$ be sequential retracts of sequential spaces $\mathsf{Y}_1$ and $\mathsf{Y}_2$, respectively. Then the function space $\mathsf{X}_2^{\mathsf{X}_1}$ formed in $\mathsf{Seq}$ is a retract of $\mathsf{Y}_2^{\mathsf{Y}_1}$.

*Proof.* Let $s_1 : \mathsf{X}_1 \to \mathsf{Y}_1, s_2 : \mathsf{X}_2 \to \mathsf{Y}_2, r_1 : \mathsf{Y}_1 \to \mathsf{X}_1, r_2 : \mathsf{Y}_2 \to \mathsf{X}_2$ be the respective sections and retractions. One can easily verify that the functions $S : \mathsf{X}_2^{\mathsf{X}_1} \to \mathsf{Y}_2^{\mathsf{Y}_1}$ and $R : \mathsf{Y}_2^{\mathsf{Y}_1} \to \mathsf{X}_2^{\mathsf{X}_1}$ defined by $S(f)(y) := s_2 f r_1(y)$ and $R(g)(x) := r_2 g s_1(x)$ are (sequentially) continuous functions such that $R \circ S$ is the identity on $\mathsf{X}_2^{\mathsf{X}_1}$. □

### 2.3. *Qcb-spaces and pseudobases*

An important subclass of the sequential spaces is formed by the qcb-spaces – see Simpson (2003). A *qcb-space* is defined to be a topological *q*uotient of a *c*ountably-*b*ased space. Qcb-spaces can be characterised as those sequential spaces that have a countable pseudobase (Escardó *et al.* 2004; Schröder 2003; Simpson 2003). A *pseudobase* for a topological space $\mathsf{X}$ is a family $\mathscr{B}$ of subsets of $\mathsf{X}$ such that for all open sets $U$ and all sequences $(x_n)_n$ that converge to an element in $U$ there is a pseudobase set $B \in \mathscr{B}$ satisfying $B \subseteq U$ and $x_n \in B$ for almost all $n \in B$.

The category $\mathsf{QCB}$ of qcb-spaces is cartesian closed and inherits its cartesian closed structure from many interesting cartesian closed supercategories: for example, from the category $\mathsf{Seq}$ of sequential spaces and the category $\mathsf{kTop}$ of compactly generated spaces without the separation axiom, *cf.* Escardó *et al.* (2004).

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Spaces with a countable pseudobase (hence qcb-spaces) are hereditarily Lindelöf. Thus regularity, complete regularity and perfect normality coincide for countably pseudobased spaces. So a space with a countable pseudobase is regular if and only if every open set is functionally open (= co-zero-set). The sequential coreflection seq(T) of a Hausdorff space T with a countable pseudobase is a qcb-space and equals the kelleyfication of T by Schröder (2003, Proposition 3.3.2). The *kelleyfication* k(T) of a Hausdorff space T carries as its topology the family of all sets $V \subseteq \mathsf{T}$ such that for all compact subspaces K of T, the set $\mathsf{K} \cap V$ is open in K.

### 2.4. *Limit spaces and $\mathscr{L}^*$-spaces*

A *limit space* (Engelking 1989; Hyland 1979; Schröder 2003) is a set $X$ equipped with a *convergence relation* $\to_X$ between sequences $(x_n)_n$ and points $x$ of $X$ subject to the following axioms:

(L1) $(x)_n \to_X x$.
(L2) If $(x_n)_n \to_X x_\infty$, then $(y_n)_n \to_X x_\infty$ for every subsequence $(y_n)_n$ of $(x_n)_n$.
(L3) If $(x_n)_n \not\to_X x_\infty$, then $(x_n)_n$ has a subsequence $(y_n)_n$ such that $(z_n)_n \not\to_X x_\infty$ for every subsequence $(z_n)_n$ of $(y_n)_n$.

A limit space such that every sequence has at most one limit is called an *$\mathscr{L}^*$-space*. A subset $U$ of a limit space is *sequentially open* if $U$ contains almost all elements of any sequence that converges to some element of $U$. Clearly, the family of sequentially open subsets forms a sequential topology on $X$. We know from Kisyński (1960) that $\mathscr{L}^*$-spaces can be regarded as sequential spaces – see also Engelking (1989, Ex. 1.7.19).

**Lemma 2.4.** The convergence relation of an $\mathscr{L}^*$-space is induced by the topology of sequentially open subsets.

### 2.5. *The Kleene–Kreisel continuous functionals over* $\mathbb{N}$

Using the cartesian closed structure of Seq, we define for $k \in \mathbb{N}$ the sequential space $\mathsf{N}\langle k\rangle$ inductively by $\mathsf{N}\langle 0\rangle := \mathbb{N}$ and $\mathsf{N}\langle k+1\rangle := \mathbb{N}^{\mathsf{N}\langle k\rangle}$. The elements of the carrier set of $\mathsf{N}\langle k\rangle$, which consists of all *continuous* functions from $\mathsf{N}\langle k-1\rangle$ to $\mathbb{N}$, are called the *Kleene–Kreisel continuous functionals* over $\mathbb{N}$ of type $k$. Clearly, $\mathsf{N}\langle 1\rangle$ is simply the Baire space. The sequential topology on $\mathsf{N}\langle 2\rangle = \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ differs from the compact-open topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$: the set

$$\{f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N} \text{ continuous} \mid f\big(f(0^\omega).f(1^\omega).f(2^\omega)\ldots\big) = 0\}$$

is a well-known example of a sequentially clopen set that does not belong to the compact-open topology. Since $\mathbb{N}$ is a Hausdorff qcb-space and Hausdorff qcb-spaces are closed under exponentiation in Seq, each space $\mathsf{N}\langle k\rangle$ is a Hausdorff qcb-space and therefore a Hausdorff–Kelley space. As the category of Hausdorff qcb-spaces inherits its cartesian closed structure from both the category of Hausdorff sequential spaces and the category kHaus of Hausdorff–Kelley spaces, one can equivalently define $\mathsf{N}\langle k\rangle$ using exponentiation in kHaus (or using exponentiation in QCB).

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Further information and proofs can be found in Escardó *et al.* (2004), Hyland (1979), Normann (1980) and Normann (2005).

## 3. Proof of the main result

In this section we prove the non-regularity of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ by constructing a simple non-regular space $\mathsf{Z}$ and embedding it into $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ as a closed subspace. As $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ (and thus any of its subspaces) is the sequential coreflection of a zero-dimensional space, this space $\mathsf{Z}$ is necessarily the sequential coreflection of some zero-dimensional topological space. The embedding is constructed in two steps: first we construct a countable space $\mathsf{S}$ such that $2^{\mathbb{N}\times\mathsf{S}}$ contains a closed copy of $\mathsf{Z}$, and then we show that $2^{\mathbb{N}\times\mathsf{S}}$ is itself homeomorphic to a closed subspace of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. This space $\mathsf{S}$ is interesting in its own right in that it is a countable Polish space such that $\mathbb{R}^{\mathsf{S}}$ is not regular in $\mathsf{Seq}$.

### 3.1. *A non-regular sequential coreflection of a zero-dimensional space*

We define a sequential space $\mathsf{Z}$ such that, on the one hand, its convergence relation is induced by a zero-dimensional topology, but, on the other hand, the topology of sequentially open sets is not even regular. This space is inspired by Michael's example of a regular space whose kelleyfication is not regular (Michael 1973, Example 1.2).

We choose $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ as the underlying set of $\mathsf{Z}$. We topologise $\mathsf{Z}$ by defining a sequence convergence relation $\to_{\mathsf{Z}}$ on $\mathsf{Z}$ and taking the topology of sequentially open sets pertaining to $\to_{\mathsf{Z}}$. Let $P$ and $Q$ be the following subsets of $\{0,1\}^{\mathbb{N}}$:

$$Q := \{w.0^{\omega} \mid w \in \{0,1\}^{*}\} \text{ and } P := \{0,1\}^{\omega} \setminus Q.$$

It is known that $Q$ is a countable dense subset of the Cantor space $2^{\mathbb{N}}$. For $(x_\infty, y_\infty) \in \mathsf{Z}$ and a sequence $(x_n, y_n)_n$ in $\mathsf{Z}$, we define the convergence relation of $\mathsf{Z}$ by

$$(x_n,y_n)_n \to_{\mathsf{Z}} (x_\infty,y_\infty) :\Longleftrightarrow \begin{cases} (x_n)_n \to_{2^{\mathbb{N}}} x_\infty \text{ and } (y_n)_n \to_{2^{\mathbb{N}}} y_\infty \text{ and} \\ (x_\infty,y_\infty) \in Q \times Q \Rightarrow \\ \qquad\qquad \exists n_0 \forall n \geqslant n_0 : (x_n = x_\infty \vee y_n = y_\infty), \\ (x_\infty,y_\infty) \in Q \times P \;\Rightarrow\; \exists n_0 \forall n \geqslant n_0 : x_n = x_\infty, \\ (x_\infty,y_\infty) \in P \times Q \;\Rightarrow\; \exists n_0 \forall n \geqslant n_0 : y_n = y_\infty. \end{cases} \tag{1}$$

It is easy to verify that $\mathsf{Z}$ satisfies the axioms of a limit space. As $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is a Hausdorff space, $\to_{\mathsf{Z}}$ assigns at most one limit to every sequence. So by Lemma 2.4, the space $\mathsf{Z}$ is endowed with a topology that induces the convergence relation defined in (1). We will see in Corollary 3.8 that $\to_{\mathsf{Z}}$ is induced by some zero-dimensional topology, which, by Proposition 3.4, differs from the sequential topology.

We will now work towards showing that $\mathsf{Z}$ is not regular.

**Lemma 3.1.** The set $P^2$ is closed in $\mathsf{Z}$.

*Proof.* A sequence $(x_n, y_n)_n$ that converges to a point $(x_\infty, y_\infty)$ in $\mathsf{Z} \setminus P^2$ has the property that eventually one component is in $Q$. This means that $(x_n, y_n)_n$ is eventually in $\mathsf{Z} \setminus P^2$. So $\mathsf{Z} \setminus P^2$ is sequentially open in $\mathsf{Z}$, and thus open, since $\mathsf{Z}$ is sequential. □

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**Lemma 3.2.** Let $U$ be an open subset of $\mathsf{Z}$, and let $(p,q) \in U \cap Q^2$. Then there are $n,m \in \mathbb{N}$ with $(p^{<n}.\{0,1\}^{\mathbb{N}}) \times \{q\} \subseteq U$ and $\{p\} \times (q^{<m}.\{0,1\}^{\mathbb{N}}) \subseteq U$.

*Proof.* Assume that for every $n$ there is some $x_n \in p^{<n}.\{0,1\}^{\mathbb{N}}$ with $(x_n,q) \notin U$. As $(x_n)_n$ converges to $p$ in $2^{\mathbb{N}}$, we have that $(x_n,q)_n$ converges to $(p,q)$ in $\mathsf{Z}$. Since $U$ is open, $(x_n,q)_n$ is eventually in $U$, which gives a contradiction. So there must be some $n$ such that $(p^{<n}.\{0,1\}^{\mathbb{N}}) \times \{q\} \subseteq U$. The second statement follows similarly. □

**Lemma 3.3.** The point $(0^\omega, 0^\omega)$ and the set $P^2$ cannot be separated by two disjoint open sets of $\mathsf{Z}$.

*Proof.* Assume that $U, V$ are open sets of $\mathsf{Z}$ with $(0^\omega,0^\omega) \in U$ and $P^2 \subseteq V$. In order to show that there is a sequence in $U$ converging to some point in $P^2$, we construct inductively two $\sqsubseteq$-increasing sequences $(u_n)_n$, $(v_n)_n$ of words in $\{0,1\}^*$ with $(u_n.0^\omega, v_n.0^\omega) \in U$, where we use $\sqsubseteq$ to denote the prefix-relation on words over $\{0,1\}$.

— $n = 0$:
Let $u_0$ and $v_0$ be the empty word. Then $(u_0.0^\omega, v_0.0^\omega) \in U$.

— $n \to n+1$:
Since $(u_n.0^\omega, v_n.0^\omega) \in U$, by Lemma 3.2, there is some $i \in \mathbb{N}$ such that we have $(u_n.0^i.\{0,1\}^{\mathbb{N}}) \times \{v_n.0^\omega\} \subseteq U$. Set $u_{n+1} := u_n.0^i.1$.
Since $(u_{n+1}.0^\omega, v_n.0^\omega) \in U$, by Lemma 3.2 there exists some $j \in \mathbb{N}$ such that $\{u_{n+1}.0^\omega\} \times (v_n.0^j.\{0,1\}^{\mathbb{N}}) \subseteq U$. Set $v_{n+1} := v_n.0^j.1$. Then $(u_{n+1}.0^\omega, v_{n+1}.0^\omega) \in U$.

Since $(u_n)_n$ and $(v_n)_n$ are strictly increasing with respect to the prefix relation, we have that $(u_n.0^\omega, v_n.0^\omega)_n$ converges in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ to some $(x_\infty, y_\infty)$. Both $x_\infty$ and $y_\infty$ belong to $P$ since they contain infinitely many 1's. By the definition of $\mathsf{Z}$, we have that $(u_n.0^\omega, v_n.0^\omega)_n$ converges in $\mathsf{Z}$ to $(x_\infty, y_\infty) \in P^2 \subseteq V$. So $(u_n.0^\omega, v_n.0^\omega)_n$ must be eventually in $V$, contradicting $U \cap V = \varnothing$. □

Lemmas 3.1 and 3.3 imply the following proposition.

**Proposition 3.4.** The sequential space $\mathsf{Z}$ is not regular.

### 3.2. *Construction of a Polish space* $\mathsf{S}$ *such that* $2^{\mathbb{N} \times \mathsf{S}}$ *is not regular*

We construct a countable Polish space $\mathsf{S}$ with the property that $2^{\mathbb{N} \times \mathsf{S}}$ contains a closed copy of $\mathsf{Z}$. This implies that $2^{\mathbb{N} \times \mathsf{S}}$ is not regular.

The underlying set of $\mathsf{S}$ is $\mathbb{N}^2 \cup \{(\infty,\infty)\}$, and its topology is defined as the one induced by the unique metric $d_{\mathsf{S}}$ that satisfies

$$d_{\mathsf{S}}\big((a,b),(\infty,\infty)\big) = 2^{-a} \quad \text{and} \quad d_{\mathsf{S}}\big((a,b),(a',b')\big) = 2^{-a} + 2^{-a'}$$

for $a,b,a',b' \in \mathbb{N}$ with $(a,b) \neq (a',b')$. Clearly, the metric space $(\mathsf{S}, d_{\mathsf{S}})$ is separable and complete. Moreover, $(\infty,\infty)$ is the only point in $\mathsf{S}$ to have no compact neighbourhood, whereas all the other points are isolated.

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Now we show that $\mathsf{Z}$ is homeomorphic to a closed subspace of $2^{\mathbb{N}\times\mathsf{S}}$. To this end, we define functions $e_i : \mathsf{Z}\times\mathbb{N}\times\mathsf{S}\to\{0,1\}$ and $e:\mathsf{Z}\to 2^{\mathbb{N}\times\mathsf{S}}$ as follows:

$$\begin{aligned}
e_0(x,y,k,s) &:= x(k),\\
e_1(x,y,k,s) &:= y(k),\\
e_2(x,y,k,\infty,\infty) &:= e_3(x,y,k,\infty,\infty) := 0,\\
e_2(x,y,k,a,\langle c,d\rangle) &:= \begin{cases} 1 & \text{if } x\in\{0,1\}^k.0^{a+c}.1.\{0,1\}^{\mathbb{N}} \text{ and } y(a+d)=1\\ 0 & \text{otherwise,}\end{cases}\\
e_3(x,y,k,a,\langle c,d\rangle) &:= \begin{cases} 1 & \text{if } y\in\{0,1\}^k.0^{a+c}.1.\{0,1\}^{\mathbb{N}} \text{ and } x(a+d)=1\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

and

$$e(x,y)(4k+i,s) := e_i(x,y,k,s)$$

for all $(x,y)\in\mathsf{Z}$, $a,c,d,k\in\mathbb{N}$, $i\in\{0,1,2,3\}$ and $s\in\mathsf{S}$, where we use $\langle\cdot,\cdot\rangle$ to denote a computable bijection between $\mathbb{N}^2$ and $\mathbb{N}$.

The idea behind the above functions is as follows: $e_0$ and $e_1$ encode the two components of an element of $\mathsf{Z}$ in such a way that convergence of $(e(x_n,y_n))_n$ in $2^{\mathbb{N}\times\mathsf{S}}$ implies convergence of $(x_n,y_n)_n$ in $2^{\mathbb{N}}\times 2^{\mathbb{N}}$. The functions $e_2$ and $e_3$ ensure that $(x_n)_n$ and $(y_n)_n$ converge to elements in $P$ if $(x_n,y_n)_n$ is a sequence in $P^2$ such that $(e(x_n,y_n))_n$ converges in $2^{\mathbb{N}\times\mathsf{S}}$ to some point.

**Lemma 3.5.** The mappings $e_0, e_1, e_2, e_3$, and $e$ are continuous.

*Proof.* As we are dealing with sequential spaces, it suffices to show sequential continuity. Sequential continuity of $e_0$ and $e_1$ follows from the fact that convergence in $\mathsf{Z}$ implies convergence in $2^{\mathbb{N}}\times 2^{\mathbb{N}}$.

Now let $(x_n,y_n)_n$ converge to $(x_\infty,y_\infty)_n$ in $\mathsf{Z}$, and let $(k_n,s_n)_n$ converge to $(k,s_\infty)$ in $\mathbb{N}\times\mathsf{S}$. Without loss of generality, we can assume $k_n=k$ for all $n$ because $k_n$ is equal to $k$ for almost all $n$. We consider the following cases:

(1) Let $s_\infty\neq(\infty,\infty)$.
Then there are $a,c,d\in\mathbb{N}$ with $s_\infty=(a,\langle c,d\rangle)$. As $(s_n)_n$ is eventually equal to $s_\infty$ and convergence in $\mathsf{Z}$ implies convergence in $2^{\mathbb{N}}\times 2^{\mathbb{N}}$, there is some $n_0\in\mathbb{N}$ with

$$\begin{aligned}
&x_n(0)=x_\infty(0),\ldots,x_n(k+a+c)=x_\infty(k+a+c),\\
&y_n(a+d)=y_\infty(a+d),\\
&s_n=(a,\langle c,d\rangle)
\end{aligned}$$

for all $n\geqslant n_0$. Clearly, we have $e_2(x_n,y_n,k,s_n)=e_2(x_\infty,y_\infty,k,s_\infty)$ for all $n\geqslant n_0$.

(2) Let $s_\infty=(\infty,\infty)$.
Without loss of generality, we can assume $s_n\in\mathbb{N}^2$ for all $n\in\mathbb{N}$, as $e_2(z,k,\infty,\infty)=0=e_2(x_\infty,y_\infty,k,s_\infty)$ for all $z\in\mathsf{Z}$. Then there exist $a_n,c_n,d_n\in\mathbb{N}$ with $s_n=(a_n,\langle c_n,d_n\rangle)$. We consider the following subcases:

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(a) Let $x_\infty \in \{0,1\}^k.0^m.1.\{0,1\}^{\mathbb{N}}$ for some $m \in \mathbb{N}$.
Then there is some $n_0 \in \mathbb{N}$ such that $x_n \in \{0,1\}^k.0^m.1.\{0,1\}^{\mathbb{N}}$ and $a_n > m$ for all $n \geqslant n_0$. Clearly, we have $e_2(x_n, y_n, k, s_n) = 0 = e_2(x_\infty, y_\infty, k, s_\infty)$ for all $n \geqslant n_0$.

(b) Let $x_\infty = u.0^\omega$ and $y_\infty = v.0^\omega$ for some $u \in \{0,1\}^k$ and $v \in \{0,1\}^*$.
Then there is some $n_0$ such that for all $n \geqslant n_0$ we have

$$(x_n = u.0^\omega \text{ or } y_n = v.0^\omega) \quad \text{and} \quad a_n \geqslant \lg(v).$$

Then every $n \geqslant n_0$ satisfies either $x_n \notin \{0,1\}^k.0^{a_n}.1.\{0,1\}^{\mathbb{N}}$ or $y_n(a_n + d_n) \neq 1$, implying $e_2(x_n, y_n, k, s_n) = 0 = e_2(x_\infty, y_\infty, k, s_\infty)$.

(c) Let $x_\infty = u.0^\omega$ for some $u \in \{0,1\}^k$ and $y_\infty \in P$.
Then there is some $n_0$ with $x_n = u.0^\omega$ for all $n \geqslant n_0$. Clearly, for all $n \geqslant n_0$ we have $e_2(x_n, y_n, k, s_n) = 0 = e_2(x_\infty, y_\infty, k, s_\infty)$.

We conclude that $e_2$ is sequentially continuous, and hence continuous. By symmetry, $e_3$ is continuous as well.

Since the functions $e_i$ are sequentially continuous, the function $\hat{e}\colon \mathsf{Z} \times (\mathbb{N} \times \mathsf{S}) \to \mathsf{2}$ defined by $\hat{e}(z,t) := e(z)(t)$ is sequentially continuous. Cartesian closedness of $\mathsf{Seq}$ and sequential continuity of $\hat{e}$ then imply that $e(z)$ is an element of $\mathsf{2}^{\mathbb{N}\times\mathsf{S}}$ and that $e$ is continuous. □

The following lemma is instrumental in showing that $e$ is a homeomorphic embedding of $\mathsf{Z}$ into the function space $\mathsf{2}^{\mathbb{N}\times\mathsf{S}}$.

**Lemma 3.6.** Let $(x_n, y_n)_n$ be a sequence in $\mathsf{Z}$ such that $(e(x_n, y_n))_n$ converges in $\mathsf{2}^{\mathbb{N}\times\mathsf{S}}$ to some function $f$. Then $(x_n, y_n)_n$ converges in $\mathsf{Z}$ and its limit is mapped by $e$ to $f$.

*Proof.* We define $x_\infty, y_\infty \in \{0,1\}^{\mathbb{N}}$ by $x_\infty(k) := f(4k, 0, 0)$ and $y_\infty(k) := f(4k+1, 0, 0)$ for $k \in \mathbb{N}$ and show that $(x_n, y_n)_n$ converges to $(x_\infty, y_\infty)$ in $\mathsf{Z}$. We begin by observing that $(x_n, y_n)_n$ converges to $(x_\infty, y_\infty)$ in $\mathsf{2}^{\mathbb{N}} \times \mathsf{2}^{\mathbb{N}}$, since

$$x_\infty(k) = \lim_{n\to\infty} e_0(x_n, y_n, k, 0, 0) \text{ and } y_\infty(k) = \lim_{n\to\infty} e_1(x_n, y_n, k, 0, 0).$$

Now we consider the following cases:

(1) Let $(x_\infty, y_\infty) \in P^2$.
Then $(x_n, y_n)_n$ converges to $(x_\infty, y_\infty)$ in $\mathsf{Z}$ by converging to $(x_\infty, y_\infty)$ in $\mathsf{2}^{\mathbb{N}} \times \mathsf{2}^{\mathbb{N}}$.

(2) Let $(x_\infty, y_\infty) \in Q^2$.
Then there are $u, v \in \{0,1\}^*$ with $x_\infty = u.0^\omega$ and $y = v.0^\omega$.
Suppose $(x_n \neq x_\infty \wedge y_n \neq y_\infty)$ holds for infinitely many $n$. Without loss of generality, we can assume this for all $n$, because otherwise we could consider an appropriate subsequence. Since $(x_n, y_n)_n$ converges in $\mathsf{2}^{\mathbb{N}} \times \mathsf{2}^{\mathbb{N}}$ to $(u.0^\omega, v.0^\omega)$, there are some $n_0 \in \mathbb{N}$ and two sequences $(s(n))_n$ and $(t(n))_n$ of natural numbers, both converging to $\infty$, such that $x_n \in u.0^{s(n)}.1.\{0,1\}^{\mathbb{N}}$ and $y_n \in v.0^{t(n)}.1.\{0,1\}^{\mathbb{N}}$ for all $n \geqslant n_0$. We define

$$k := \lg(u),\ a_n := \min\{s(n), t(n)\},\ c_n := s(n) - a_n,\ d_n := \lg(v) + t(n) - a_n$$

for all $n \in \mathbb{N}$. Then $(4k+2, a_n, \langle c_n, d_n\rangle)_n$ converges to $(4k+2, \infty, \infty)$ in $\mathbb{N} \times \mathsf{S}$. Moreover, we have $e(x_n, y_n)(4k+2, a_n, \langle c_n, d_n\rangle) = 1$ for all $n \geqslant n_0$ by the definition of $e_2$. Since

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$(e(x_n, y_n))_n$ converges continuously to $f$, this implies $f(4k+2, \infty, \infty) = 1$. On the other hand, by the definition of $e_2$, we have $e(x_n, y_n)(4k+2, \infty, \infty) = 0$ for all $n \in \mathbb{N}$, implying $f(4k+2, \infty, \infty) = 0$, again by continuous convergence, which gives a contradiction.

(3) Let $(x_\infty, y_\infty) \in Q \times P$.
Then there exist some $u \in \{0,1\}^*$ with $x_\infty = u.0^\omega$ and some strictly increasing function $t : \mathbb{N} \to \mathbb{N}$ with $y_\infty(t(n)) = 1$ for all $n \in \mathbb{N}$.
Suppose $x_n \neq x_\infty$ holds for infinitely many $n$. Without loss of generality, we can assume $x_n \neq x_\infty$ for all $n$, because otherwise we could consider an appropriate subsequence. Since $(x_n)_n$ converges to $x_\infty$ in $2^{\mathbb{N}}$, there is some $n_0 \in \mathbb{N}$ and a sequence $(s(n))_n$ with $x_n \in u.0^{s(n)}.1.\{0,1\}^{\mathbb{N}}$ for all $n \geqslant n_0$. Moreover, there is a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ with $y_{\varphi(n)}(t(n)) = y_\infty(t(n)) = 1$ for all $n \in \mathbb{N}$. We define

$$k := \lg(u),\ a_n := \min\{s\varphi(n), t(n)\},\ c_n := s\varphi(n) - a_n,\ d_n := t(n) - a_n$$

for all $n \in \mathbb{N}$. Then $(4k+2, a_n, \langle c_n, d_n\rangle)_n$ converges to $(4k+2, \infty, \infty)$ in $\mathbb{N} \times \mathsf{S}$. By the definition of $e_2$, we have $e(x_{\varphi(n)}, y_{\varphi(n)})(4k+2, a_n, \langle c_n, d_n\rangle) = 1$ for all $n \geqslant n_0$. This implies $f(4k+2, \infty, \infty) = 1$ by convergence of $(e(x_{\varphi(n)}, y_{\varphi(n)}))_n$ to $f$. But, as in (2), $f(4k+2, \infty, \infty)$ is equal to 0 by being the limit of $(e_2(x_n, y_n, k, \infty, \infty))_n$, which gives a contradiction.

(4) Let $(x_\infty, y_\infty) \in P \times Q$.
This case can be argued in the same way as for case (3) but using the function $e_3$ in place of $e_2$.

We conclude that $(x_n, y_n)_n$ converges to $(x_\infty, y_\infty)$ in $\mathsf{Z}$. As $e$ is sequentially continuous by Lemma 3.5, $(e(x_n, y_n))_n$ converges to $e(x_\infty, y_\infty)$. Since $2^{\mathbb{N}\times\mathsf{S}}$ is a Hausdorff space, this implies $f = e(x_\infty, y_\infty)$. □

As a consequence, we have that $\mathsf{Z}$ embeds topologically into $2^{\mathbb{N}\times\mathsf{S}}$.

**Corollary 3.7.** The space $\mathsf{Z}$ is homeomorphic to a closed subspace of $2^{\mathbb{N}\times\mathsf{S}}$. The corresponding homeomorphic embedding is $e$.

*Proof.* The functions $e_0, e_1$ ensure that $e$ is injective. By Lemma 3.6, $(e(z_n))_n \to_{2^{\mathbb{N}\times\mathsf{S}}} e(z_\infty)$ implies $(z_n)_n \to_{\mathsf{Z}} z_\infty$. Taken together with Lemma 3.5, this implies that $e$ reflects convergent sequences. Moreover, by Lemma 3.6, convergent sequences of elements in $e(\mathsf{Z})$ converge to elements in $e(\mathsf{Z})$. Since $2^{\mathbb{N}\times\mathsf{S}}$ is a Hausdorff space, this means that $e(\mathsf{Z})$ is sequentially closed, and hence closed in $2^{\mathbb{N}\times\mathsf{S}}$. The claim then follows from Lemma 2.1. □

The convergence relation of function spaces formed in $\mathsf{Seq}$ is induced by the compact-open topology, *cf.* Schröder (2003, Lemma 4.2.2). As the compact-open topology on $2^{\mathbb{N}\times\mathsf{S}}$ is zero-dimensional (because the codomain space $2$ is zero-dimensional) and $e$ reflects convergent sequences, the family of sets $e^{-1}(C)$, where $C$ is clopen in the compact-open topology, is a base of a zero-dimensional topology $\tau$ that induces the convergence relation of $\mathsf{Z}$. So $\mathsf{Z}$ is the sequential coreflection of the space topologised by $\tau$. Since the compact-open topology induces the convergence relation of the qcb-space $2^{\mathbb{N}\times\mathsf{S}}$, it has a countable pseudobase. By Schröder (2002, Theorem 14), this implies that $\tau$ has a countable pseudobase as well.

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**Corollary 3.8.** The space $\mathsf{Z}$ is the sequential coreflection of a zero-dimensional space with a countable pseudobase.

Michael (1973, Lemma 4.9) can be equivalently expressed as stating that kelleyfication of countably pseudobased spaces does not preserve regularity. The kelleyfication of a countably pseudobased Hausdorff space is known to be equal to its sequential coreflection (*cf.* Schröder (2003)). Proposition 3.4 and Corollary 3.8 imply the following generalisation.

**Corollary 3.9.** The sequential coreflection of a zero-dimensional space with a countable pseudobase need not be regular.

By Corollory 3.7 and Proposition 3.4, we have that $2^{\mathbb{N}\times\mathsf{S}}$ contains a non-regular topological subspace. By cartesian closedness of $\mathsf{Seq}$, we have that $(2^{\mathbb{N}})^{\mathsf{S}}$ is homeomorphic to $2^{\mathbb{N}\times\mathsf{S}}$. So we have the following proposition.

**Proposition 3.10.** The sequential topologies on $2^{\mathbb{N}\times\mathsf{S}}$ and on $(2^{\mathbb{N}})^{\mathsf{S}}$ are not regular.

It is well known that $2^{\mathbb{N}}$ is homeomorphic to Cantor's ternary set, which is a closed subspace of the unit interval $\mathbb{I}$ endowed with the Euclidean topology. Hence $(2^{\mathbb{N}})^{\mathsf{S}}$ is homeomorphic to a closed subspace of $\mathbb{I}^{\mathsf{S}}$ by Lemma 2.2. This shows that we have the following proposition.

**Proposition 3.11.** The sequential topologies on $\mathbb{I}^{\mathsf{S}}$ and on $\mathbb{R}^{\mathsf{S}}$ are not regular.

Since $\mathbb{N}$, $\mathbb{I}$, $\mathbb{R}$, $\mathsf{S}$, $2$ are qcb-spaces, the function spaces $2^{\mathbb{N}\times\mathsf{S}}$, $(2^{\mathbb{N}})^{\mathsf{S}}$, $\mathbb{I}^{\mathsf{S}}$ and $\mathbb{R}^{\mathsf{S}}$ formed in $\mathsf{Seq}$ agree with the corresponding ones formed in $\mathsf{kHaus}$, the category of Hausdorff k-spaces. So exponentiation in $\mathsf{kHaus}$ does not preserve regularity.

### 3.3. *The main results*

Now we are ready to prove our main result stating that $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not regular. In view of Proposition 3.10, it suffices to show that the space $2^{\mathbb{N}\times\mathsf{S}}$ is a retract of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. Then non-regularity of $2^{\mathbb{N}\times\mathsf{S}}$ transfers to $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$.

**Theorem 3.12.** The sequential topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not regular.

*Proof.* The space $\mathbb{N}\times\mathsf{S}$ is a retract of $\mathbb{N}^{\mathbb{N}}$ since it is a zero-dimensional Polish space: a section $s\colon \mathbb{N}\times\mathsf{S}\to\mathbb{N}^{\mathbb{N}}$ and a retraction $r\colon \mathbb{N}^{\mathbb{N}}\to\mathbb{N}\times\mathsf{S}$ can be defined by

$$s(k,a,b) := \begin{cases} k.0^{\omega} & \text{if } (a,b)=(\infty,\infty) \\ k.0^{a}.(b+1).0^{\omega} & \text{otherwise,} \end{cases}$$

$$r(p) := \begin{cases} \big(p(0), m-1, p(m)-1\big) & \text{if } m := \min\{j\geqslant 1 \mid p(j)\neq 0\} \text{ exists} \\ \big(p(0),\infty,\infty\big) & \text{otherwise} \end{cases}$$

for $p\in\mathbb{N}^{\mathbb{N}}$, $k\in\mathbb{N}$ and $(a,b)\in\mathsf{S}$. It is not difficult to verify that both functions are sequentially continuous and that $r\circ s$ is the identity on $\mathbb{N}\times\mathsf{S}$. Clearly, $2$ is a retract of the discrete space $\mathbb{N}$. So $2^{\mathbb{N}\times\mathsf{S}}$ is indeed a retract of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ by Lemma 2.3 and hence

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homeomorphic to a (closed) subspace of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. Therefore $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ cannot be regular, because, by Proposition 3.10, $2^{\mathbb{N}\times\mathbf{S}}$ is not regular. □

An easy consequence of Theorem 3.12 is that no space in the hierarchy of Kleene–Kreisel continuous functionals (except for $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$) is zero-dimensional.

**Theorem 3.13.** For $k \geqslant 2$, the sequential topology on the space of Kleene–Kreisel continuous functionals of type $k$ is not regular, normal, perfectly normal or zero-dimensional.

*Proof.* The discrete space $\mathbb{N}$ is a retract of $\mathbb{N}^{\mathbb{N}}$ through the retraction $p \mapsto p(0)$. By an induction using Lemma 2.3, we can show that $\mathsf{N}\langle k\rangle$ is a retract of $\mathsf{N}\langle k+1\rangle$ for every $k \in \mathbb{N}$. So $\mathsf{N}\langle k\rangle$ contains a subspace homeomorphic to $\mathsf{N}\langle 2\rangle = \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ and, by Theorem 3.12, is therefore not regular for $k \geqslant 2$. Since either zero-dimensionality or perfect normality imply regularity, $\mathsf{N}\langle k\rangle$ is neither zero-dimensional nor perfectly normal. □

### 3.4. *The sequential hierarchy of continuous functionals over the reals*

The sequential hierarchy of continuous functionals over the reals is constructed inside the category of sequential spaces by applying the recursion formula $\mathsf{R}\langle 0\rangle := \mathbb{R}$ and $\mathsf{R}\langle k+1\rangle := \mathbb{R}^{\mathsf{R}\langle k\rangle}$. By Normann (2005, Corollary 4), $\mathsf{N}\langle k\rangle$ is homeomorphic to a closed subspace of $\mathsf{R}\langle k\rangle$. Hence no space in the sequential hierarchy (apart from the Polish spaces $\mathbb{R}$ and $\mathbb{R}^{\mathbb{R}} = \mathsf{R}\langle 1\rangle$) is regular.

**Corollary 3.14.** For $k \geqslant 2$, the sequential topology on $\mathsf{R}\langle k\rangle$ is not regular.

Note that the underlying sets of the spaces $\mathsf{R}\langle k\rangle$ form the extensional hierarchy over the reals mentioned in Section 1.

## 4. An effectively open, but not functionally open, subset of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$

In this section we assume some familiarity with the representation-based approach to Computable Analysis, the Type Two Theory of Effectivity (TTE). The basic idea of TTE is to represent the objects of a given space by elements of the Baire space and to compute on these names rather than on the objects themselves. Details can be found in Weihrauch (2000), Schröder (2002) and Schröder (2003).

Let $\mathsf{X}$ be a qcb-space endowed with a standard representation $\delta_{\mathsf{X}} : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathsf{X}$. In order to define the effectively open subsets of $\mathsf{X}$, we use the Sierpiński space $\mathbf{S}$. The Sierpiński space has $\{\top, \bot\}$ as its carrier set and $\{\bot\}$ as its only closed singleton. A standard representation $\delta_{\mathbf{S}} : \mathbb{N}^{\mathbb{N}} \to \mathbf{S}$ of $\mathbf{S}$ is given by $\delta_{\mathbf{S}}(p) = \bot \iff p = 0^{\omega}$.

An open set $U$ in $\mathsf{X}$ is said to be *effectively open* if its characteristic function $\mathrm{cf}_U$ from $\mathsf{X}$ to $\mathbf{S}$ defined by $\mathrm{cf}_U(x) = \top \iff x \in U$ is computable in the sense of TTE, meaning that there is a computable partial function $g : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$ satisfying $\delta_{\mathbf{S}}(g(p)) = \mathrm{cf}_U(\delta_{\mathsf{X}}(p))$ for all $p \in \mathrm{dom}(\delta_{\mathsf{X}})$.

We show that $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ contains an effectively open subset $V$ that does not belong to the largest regular topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ inducing the same convergence relation as

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both the sequential topology and the compact-open topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. The underlying representation $\delta_{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}}$ of $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is constructed as the usual function space representation in TTE (Schröder 2003; Weihrauch 2000).

**Theorem 4.1.** There exists an effectively open neighbourhood $V$ of the constant zero-function $\mathbf{0}: \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ in $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ that contains no closed neighbourhood of $\mathbf{0}$.

*Proof.* Using the section–retraction pair $(s, r)$ from the proof of Theorem 3.12, we define the functions $S: 2^{\mathbb{N}\times\mathsf{S}} \to \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ and $R: \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} \to 2^{\mathbb{N}\times\mathsf{S}}$ by

$$S(g)(p) := gr(p) \quad \text{and} \quad R(f)(t) := \min\{1, fs(t)\}$$

for $p \in \mathbb{N}^{\mathbb{N}}$, $t \in \mathbb{N} \times \mathsf{S}$, $f \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ and $g \in 2^{\mathbb{N}\times\mathsf{S}}$. Since $(S, R)$ is a section–retraction pair (*cf.* the proof of Lemma 2.3) and $e(P^2)$ is closed by Corollary 3.7 and Lemma 3.1, the set

$$V := \{f \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}} \mid R(f) \notin e(P^2)\}$$

is an open neighbourhood of $Se(0^\omega, 0^\omega)$. As $e(0^\omega, 0^\omega)$ is the constant zero-function on $\mathbb{N} \times \mathsf{S}$, $Se(0^\omega, 0^\omega)$ is the constant zero-function on $\mathbb{N}^{\mathbb{N}}$. By the injectivity of $e$, we have $(Se)^{-1}(V) = \mathsf{Z} \setminus P^2$. Taken together with Lemma 3.3, this implies that $V$ does not contain any closed neighbourhood of $\mathbf{0} = Se(0^\omega, 0^\omega)$.

For any $p \in \mathrm{dom}(\delta_{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}})$, we let $f_p := \delta_{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}}(p)$ and define $x_p, y_p \in \{0,1\}^{\mathbb{N}}$ by $x_p(k) := R(f_p)(4k, 0, 0)$ and $y_p(k) := R(f_p)(4k+1, 0, 0)$. Clearly, $R(f_p) \in e(\mathsf{Z})$ implies $R(f_p) = e(x_p, y_p)$.

We now show that $\delta_{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}}(p) \in V$ holds if and only if there is some $(k, a, b) \in \mathbb{N}^3$ satisfying at least one of the following conditions:

(1) $R(f_p)(4k, a, b) \neq x_p(k)$
(2) $R(f_p)(4k+1, a, b) \neq y_p(k)$
(3) $R(f_p)(4k+2, a, b) \neq e_2(x_p, y_p, k, a, b)$
(4) $R(f_p)(4k+3, a, b) \neq e_3(x_p, y_p, k, a, b)$
(5) $x_p \in \{0,1\}^k.0^a.\{0,1\}^{\mathbb{N}}$ and $0 \sqsubseteq \sigma\big(p^{<a}, (4k+2).0^a\big)$
(6) $y_p \in \{0,1\}^k.0^a.\{0,1\}^{\mathbb{N}}$ and $0 \sqsubseteq \sigma\big(p^{<a}, (4k+3).0^a\big)$.

Here $\sigma: \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ denotes a computable, total and $\sqsubseteq$-monotone word function that satisfies the following properties for all $z \in \mathrm{dom}(\delta_{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}})$ and $q \in \mathbb{N}^{\mathbb{N}}$:

(a) For all $i, j \in \mathbb{N}$, if $\lg(\sigma(z^{<i}, q^{<j})) > 0$, then $\delta_{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}}(z)(q) \sqsubseteq \sigma(z^{<i}, q^{<j})$.
(b) There is some $n \in \mathbb{N}$ such $\lg(\sigma(z^{<n}, q^{<n+1})) > 0$.

Such a function $\sigma$, which realises the evaluation function for the function space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ on prefixes of names, can be obtained from an effective representation $\eta$ of the set of partial continuous functions on $\mathbb{N}^{\mathbb{N}}$ with a $\mathcal{G}_\delta$-domain, *cf.* Weihrauch (2000) and Schröder (2003).

— *Only if direction*:
Assume $f_p \in V$. If $R(f_p) \notin e(\mathsf{Z})$, there are obviously some $k, a, b \in \mathbb{N}$ satisfying one of the first four conditions. Otherwise we have $R(f_p) = e(x_p, y_p)$ and thus $x_p \in Q$ or $y_p \in Q$; without loss of generality, we can assume $x_p \in \{0,1\}^k.\{0^\omega\}$ for some $k \in \mathbb{N}$. Since

$$0 = e(x_p, y_p)(4k+2, \infty, \infty) = R(f_p)(4k+2, \infty, \infty) = f_p\big((4k+2).0^\omega\big),$$

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by Property (b) there is some $a \in \mathbb{N}$ such that $0 \sqsubseteq \sigma\big(p^{<a}, (4k+2).0^a\big)$. Therefore $(k, a, 0)$ satisfies Condition (5).

— *If direction*:
Assume $f_p \notin V$ and let $(k, a, b) \in \mathbb{N}^3$. Since $R(f_p) \in e(P^2) \subseteq e(\mathsf{Z})$, the triple $(k, a, b)$ satisfies none of the first four conditions. As $R(f_p) = e(x_p, y_p)$, there are some $m, d \in \mathbb{N}$ with $x_p \in \{0,1\}^k.0^m.1.\{0,1\}^{\mathbb{N}}$ and $y_p(a+d) = 1$. If $m < a$, then the first part of (5) is not fulfilled. Otherwise we consider $q := s(4k+2, a, \langle m-a, d\rangle)$. Since

$$f_p(q) \geqslant R(f_p)(4k+2, a, \langle m-a, d\rangle) = e_2(x_p, y_p, k, a, \langle m-a, d\rangle) = 1$$

and $q \in (4k+2).0^a.\mathbb{N}^{\mathbb{N}}$, the second part of Condition (5) is not fulfilled due to Property (a). We can show that $(k, a, b)$ does not satisfy Condition (6) using similar reasoning.

This equivalence shows that the partial computable function $g : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$ defined by

$$g(p)\big(\langle k, \langle a, b\rangle\rangle\big) := \begin{cases} 1 & \text{if } (k,a,b) \text{ satisfies one of the conditions (1) to (6)} \\ 0 & \text{otherwise} \end{cases}$$

realises the characteristic function $\mathrm{cf}_V$, meaning that $\mathrm{cf}_V(\delta_{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}}(p)) = \delta_{\mathbb{S}}(g(p))$ holds for every $p \in \mathrm{dom}(\delta_{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}})$. Hence $\mathrm{cf}_V$ is computable in the sense of TTE. Therefore $V$ is effectively open. □

This theorem implies that the topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ induced by the family of effectively open sets as a base is not regular.

## 5. Discussion

The main result of this paper gives a negative answer to the question posed in Normann (2005) asking whether there exists some $k \geqslant 2$ such that the sequential topology on the continuous functionals of type $k$ is zero-dimensional or at least perfectly normal, *cf.* Normann (2005, Problems 6.1 and 6.3). Also, it refutes Bauer *et al.* (2002, Conjecture 1), which claims zero-dimensionality for $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$. If $\mathsf{N}\langle k\rangle$ were zero-dimensional, we would know that the intensional hierarchy and the extensional hierarchy over the reals would agree up to level $k+1$. Yet both hierarchies may still be equal: Normann (2005) proved that the two hierarchies coincide if and only if for all $k \geqslant 2$ the completely regular reflection of $\mathsf{N}\langle k\rangle$ is zero-dimensional. As $\mathsf{N}\langle k\rangle$ is a qcb-space, the topology of the completely regular reflection of $\mathsf{N}\langle k\rangle$ is equal to the largest regular topology between the compact-open topology and the sequential topology on $\mathsf{N}\langle k\rangle$, and, moreover, equal to the topology of functionally open sets of $\mathsf{N}\langle k\rangle$. By Theorem 3.13, this topology on $\mathsf{N}\langle k\rangle$ differs from the sequential topology.

The countable Polish space $\mathsf{S}$ demonstrates that even in simple cases it is not guaranteed that exponentiation of regular spaces in the category $\mathsf{Seq}$ of sequential spaces yields a regular space. Sequential spaces and qcb-spaces arise naturally in Computable Analysis, since the final topology of TTE-representations and domain representations are qcb-topologies and therefore sequential.

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Also, exponentiation in other familiar cartesian closed categories of topological spaces, like the category kHaus of Hausdorff Kelley spaces or the category of quotients of exponentiable spaces, fails preservation of regularity and of normality: as QCB is a sub-ccc of these categories (*cf.* Escardó *et al.* (2004)), all counterexamples for the category QCB transfer to these categories.

## References

Asperti, A. and Longo, G. (1991) *Categories, Types, and Structures: An Introduction to Category Theory for the Working Computer Scientist*, MIT Press.

Bauer, A., Escardó, M. and Simpson, A. (2002) Comparing Functional Paradigms for Exact Real-number Computation. In: Proc. ICALP '02. *Springer-Verlag Lecture Notes in Computer Science* **2380** 488–500.

Engelking, R. (1989) *General Topology*, Heldermann.

Escardó, M., Lawson, J. and Simpson, A. (2004) Comparing Cartesian Closed Categories of (Core) Compactly Generated Spaces. *Topology and its Applications* **143** 105–145.

Hyland, J. M. E. (1979) Filter Spaces and Continuous Functionals. *Annals of Mathematical Logic* **16** 101–143.

Kisyński, J. (1960) Convergence du Type L. *Colloquium Mathematicum* **7** 205–211.

Kleene, S. E. (1959) Countable Functionals. In: *Constructivity in Mathematics*, North-Holland 81–100.

Kreisel, G. (1959) Interpretation of Analysis by means of Functionals of Finite Type. In: *Constructivity in Mathematics*, North-Holland 101–128.

Michael, E. (1973) On $k$-spaces, $k_R$-spaces and $k(X)$. *Pacific Journal of Mathematics* **47** (2) 487–498.

Normann, D. (1980) Recursion on the Countable Functionals. *Springer-Verlag Lecture Notes in Mathematics* **811**.

Normann, D. (2005) Comparing Hierarchies of Total Functionals. *Logical Methods in Computer Science* **1** (2:4) 1–28.

Schröder, M. (2002) Extended Admissibility. *Theoretical Computer Science* **284** 519–538.

Schröder, M. (2003) *Admissible Representations for Continuous Computations*, Ph.D. Thesis, Fernuniversität Hagen.

Simpson, A. (2003) Towards a Convenient Category of Topological Domains. In: *Proceedings of thirteenth ALGI Workshop*, RIMS, Kyoto University.

Weihrauch, K. (2000) *Computable Analysis*, Springer-Verlag.

Willard, S. (1970) *General Topology*, Addison-Wesley.

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