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# Various representations of a generalized hypergeometric function through statistical techniques

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*Abstract*

This paper deals with a statistical technique of deriving new results on Meijer's $G$-function. Multiple integral as well as computable series representations are given. The method used is to derive the density of a product of independent beta-distributed random variables by using different statistical techniques, and then compare the expressions to get the desired results. The results obtained in this article do not seem to be available through standard mathematical techniques.

## 1. *Introduction*

Meijer's $G$-function appears in very many places in applied areas, especially in statistical distribution problems and in the physical sciences. The theory of G-functions, as well as their various fields of application, are available in Mathai and Saxena [1]. Most of the work on $G$-functions is in the area of its properties, integral transforms, and other related theoretical aspects. In order to make use of these functions in practical problems, one needs computable representations for these functions. The standard mathematical technique for getting a computable series representation is to evaluate the $G$-function as a sum of residues by using residue calculus. This technique is used with $G$-functions involved with statistical problems, and a number of statistical distribution problems connected with multivariate statistical analysis have been solved, accounts of which are given in Mathai and Saxena [1].

The $G$-function (or Meijer's $G$-function) is defined as a Mellin–Barnes contour integral as follows:

$$G(z) = G^{m,n}_{p,q}(z|^{a_1,\dots a_p}_{b_1,\dots b_q}) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j+s)\prod_{j=1}^n \Gamma(1-a_j-s)}{\prod_{j=m+1}^q \Gamma(1-b_j-s)\prod_{j=n+1}^p \Gamma(a_j+s)} z^{-s}ds, \quad (1{\cdot}1)$$

where $i = \sqrt{-1}$; $m$, $n$, $p$, and $q$ are integers with $0 \leqslant n \leqslant p$ and $0 \leqslant m \leqslant q$; $\Gamma(\cdot)$ is the gamma function, and the poles of $\Gamma(b_j+s)$, $j = 1,\dots,m$ and those of $\Gamma(1-a_j-s)$, $j = 1,\dots,n$ are assumed to be separated. Various types of contour exist, but since the definition and properties of $G$-functions are available in many books, details will not be given here.

In this article, we will be considering only $G$-functions of the type $G^{p,0}_{p,p}$. Such a $G$-function exists for $|z| < 1$ and the contour of interest in this case is the straight-line contour from $c-i\infty$ to $c+i\infty$. This will be seen from the discussion to follow. In

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particular, we will be considering the problem of finding multiple integral as well as computable series representations for $G$-functions of the following type:

$$G(x) = G^{p,0}_{p,p}(x|^{\alpha_1+\beta_1,\ldots,\alpha_p+\beta_p}_{\alpha_1,\ldots,\alpha_p}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{p}\frac{\Gamma(\alpha_j+h)}{\Gamma(\alpha_j+\beta_j+h)}\right\}x^{-h}dh, \tag{1·2}$$

where $\mathrm{Re}\,(\alpha_j) > 0$, $\mathrm{Re}\,(\beta_j) > 0$, $c > -\mathrm{Re}\,(\alpha_j)$, $j = 1,\ldots,p$; where $\mathrm{Re}\,(\cdot)$ denotes the real part.

## 2. *A statistical problem*

Consider a random variable $x_j$ with the following density function:

$$f_j(x_j) = \begin{cases} \dfrac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\,\Gamma(\beta_j)}x_j^{\alpha_j-1}(1-x_j)^{\beta_j-1} & (0 < x_j < 1), \\ 0 \quad \text{otherwise}, \end{cases} \tag{2.1}$$

where $\mathrm{Re}\,(\alpha_j) > 0$ and $\mathrm{Re}\,(\beta_j) > 0$, $j = 1,\ldots,p$. Then $x_j$ is said to be beta-distributed, or a beta random variable with parameters $(\alpha_j, \beta_j)$. Now consider a set of $p$ statistically independent beta random variables $x_1, x_2, \ldots, x_p$ with parameters $(\alpha_1,\beta_1), \ldots, (\alpha_p,\beta_p)$ respectively, and let $y = x_1x_2\ldots x_p$. Then the $h$th moment of $y$, that is $E(y^h)$ where $E$ denotes the operator 'mathematical expectation', is the product of the $h$th moments of $x_1, \ldots, x_p$. That is,

$$E(y^h) = E((x_1x_2\ldots x_p)^h) = \prod_{i=1}^{p}E(x_i^h). \tag{2·2}$$

This is a trivial consequence of the statistical independence of $x_1, \ldots, x_p$. The $h$th moment of $x_j$ is defined as

$$\begin{aligned} E(x_j^h) &= \int_{-\infty}^{+\infty} x_j^h f_j(x_j)\,dx_j = \int_0^1 x_j^h f_j(x_j)\,dx_j \\ &= \frac{\Gamma(\alpha_j+h)\,\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\,\Gamma(\alpha_j+\beta_j+h)}, \quad \mathrm{Re}\,(h) > -\mathrm{Re}\,(\alpha_j). \end{aligned} \tag{2.3}$$

But it can be seen from (2·3) that $E(x_j^h)$ is nothing but the Mellin transform of $f_j(x_j)$ for $h = s-1$. Since the right hand side of (2·3) goes to zero as $|h| \to \infty$ from the uniqueness of the Mellin transform, $f_j(x_j)$ is uniquely determined by $E(x_j^h)$. If $g(y)$ denotes the density of $y$, it is seen from (2·2) that

$$E(y^h) = \prod_{j=1}^{p}E(x_j^h) = \prod_{j=1}^{p}\frac{\Gamma(\alpha_j+h)\,\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\,\Gamma(\alpha_j+\beta_j+h)} \tag{2·4}$$

and then $g(y)$ is uniquely determined by the right hand side of (2·4). Even without knowing $g(y)$ we were able to evaluate the $h$th moment of $y$ using the property of statistical independence of $x_1, \ldots, x_p$. By taking the inverse Mellin transform of (2·4) and then indentifying with (1·2), one gets the density of $Y$ in the following form:

$$g(y) = \left\{\begin{cases} C_p\,G^{p,0}_{p,p}(y|^{\alpha_j+\beta_j-1,\,j=1,\ldots,p}_{\alpha_j-1,\,j=1,\ldots,p}) & (0 < y < 1) \\ 0 \quad \text{elsewhere}, \end{cases}\right\} \tag{2·5}$$

where

$$C_p = \prod_{j=1}^{p}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}. \tag{2·6}$$

Now the idea is to work out the density $g(y)$ using other statistical techniques and compare these results with (2·5) to get various representations for the $G$-function, because $g(y)$ in this case is unique whichever way it is derived.

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### 3. *Transformation of variables*

One method which is commonly applied in the statistical literature is the method of transformation of variables. This will be illustrated step-by-step by working out the products of $x_1, x_2, \ldots, x_p$. Consider a one-to-one transformation of $x_1$ and $x_2$ to $y_1 = x_1$ and $y_2 = x_1 x_2$. The differentials are then $\partial(y_1, y_2) = x_1 \partial(x_1, x_2)$. Due to independence, the joint density of $x_1$ and $x_2$, denoted by $f(x_1, x_2)$ is

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) = \begin{cases} D_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} (1 - x_1)^{\beta_1 - 1} (1 - x_2)^{\beta_2 - 1} \\ \qquad \text{if} \quad 0 < x_1 < 1, 0 < x_2 < 1, \\ 0 \text{ elsewhere,} \end{cases} \tag{3.1}$$

where

$$D_p = \prod_{j=1}^{p} \frac{\Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j)\,\Gamma(\beta_j)}. \tag{3·2}$$

If $g(y_1, y_2)$ denotes the joint density of $y_1 = x_1$ and $y_2 = x_1 x_2$, then

$$g(y_1, y_2)\,\partial(y_1, y_2) = f(x_1, x_2)\,\partial(x_1, x_2)$$

and the density of $y_2$, denoted by $g_2(y_2)$, is available by integrating out $y_1$ from $g(y_1, y_2)$. That is,

$$\begin{aligned} g_2(y_2) &= D_2 y_2^{\alpha_2 - 1} \int_{y_2}^{1} y_1^{\alpha_1 - \alpha_2 - \beta_2} (1 - y_1)^{\beta_1 - 1} (y_1 - y_2)^{\beta_2 - 1} dy_1 \\ &= D_2 y_2^{\alpha_2 - 1} (1 - y_2)^{\beta_1 + \beta_2 - 1} \int_0^1 y_1^{\beta_1 - 1} (1 - y_1)^{\beta_2 - 1} [1 - y_1(1 - y_2)]^{-(\alpha_2 + \beta_2 - \alpha_1)} dy_1. \end{aligned} \tag{3·3}$$

This integral can be easily evaluated by expanding the last factor, integrating out $y_1$, and then summing up, or by using a standard result on the hypergeometric function ${}_2F_1(\cdot)$. The result is the following:

$$g_2(y_2) = \begin{cases} D_2 \dfrac{\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(\beta_1 + \beta_2)} y_2^{\alpha_2 - 1} (1 - y_2)^{\beta_1 + \beta_2 - 1}\, {}_2F_1(\beta_1, \alpha_2 + \beta_2 - \alpha_1; \beta_1 + \beta_2; 1 - y_2) \\ \qquad \text{if} \quad 0 < y_2 < 1, \\ 0 \quad \text{elsewhere.} \end{cases} \tag{3·4}$$

Consider $y_3 = x_1 x_2 x_3, \ldots, y_p = x_1 x_2 \ldots x_p$. Proceeding as in (3·3) and by induction, one obtains the density of $y_p = y$, denoted by $g_p(y_p) = g(y)$, as follows:

$$g(y) = \begin{cases} D_p y^{\alpha_p - 1} (1 - y)^{\delta_p - 1} \int_0^1 \cdots \int_0^1 \prod_{j=1}^{p-1} t_j^{\delta_j - 1} (1 - t_j)^{\beta_{j+1} - 1} \\ \qquad \times [1 - t_{p-1} t_{p-2} \ldots t_{p-j}(1 - y)]^{-\gamma_{p-j}} dt_j \} \quad (0 < y < 1), \\ 0 \quad \text{elsewhere,} \end{cases} \tag{3·5}$$

where $D_p$ is defined in (3·2), $\delta_j = \beta_1 + \beta_2 + \ldots + \beta_j$, and $\gamma_j = \alpha_{j+1} + \beta_{j+1} - \alpha_j$. Comparing (3·5) and (2·5), we have the following result.

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THEOREM 1. *For* $\mathrm{Re}(\alpha_j) > 0, \mathrm{Re}(\beta_j) > 0, j = 1, \ldots, p$, *and* $0 < y < 1$,

$$G^{p,0}_{p,p}(y|^{\alpha_j+\beta_j-1,\, j=1,\ldots,p}_{\alpha_j-1,\, j=1,\ldots,p}) = [\prod_{j=1}^{p}\Gamma(\beta_j)]^{-1} y^{\alpha_p-1}(1-y)^{\delta_p-1} \times \int_0^1 \cdots \int_0^1 \{\prod_{j=1}^{p-1} t_j^{\delta_j-1}(1-t_j)^{\beta_{j+1}-1} \times [1-t_{p-1}t_{p-2}\cdots t_{p-j}(1-y)]^{-\gamma_{p-j}}\, dt_j\} \tag{3·6}$$

*where* $\delta_j$ *and* $\gamma_j$ *are defined following* (3·5).

One can look upon (2·4) as an integral equation

$$\int_0^1 y^h g(y)\, dy = C_p \prod_{j=1}^{p} \frac{\Gamma(\alpha_j+h)}{\Gamma(\alpha_j+\beta_j+h)}, \tag{3·7}$$

where $g(y)$ is unknown, $C_p$ is defined in (2·6), and $\mathrm{Re}(h) > -\mathrm{Re}(\alpha_j)$, $\mathrm{Re}(\alpha_j) > 0$, $\mathrm{Re}(\beta_j) > 0$ for $j = 1, \ldots, p$. This integral equation in the statistical literature is known as the Wilks' type $B$ integral equation. Since it is connected with many statistical problems, its solution has been under investigation since the 1930's. See for example, Wilks [5], Wald and Brookner [3], Tretter and Walster [2], and Walster and Tretter [4]. Wilks [5] has given the following representation for $g(y)$:

$$g(y) = D_p y^{\alpha_p-1}(1-y)^{\beta_1+\ldots+\beta_p-1} \times \int_0^1 \cdots \int_0^1 \{\prod_{j=1}^{p-1} v_j^{\beta_j-1}(1-v_j)^{\beta_p+\beta_{p-1}+\ldots+\beta_{j+1}-1} \xi_j^{\alpha_j-\alpha_{j+1}-\beta_{j+1}}\, dv_j\} \quad (0 < y < 1) \tag{3·8}$$

where

$$\xi_j = \{v_1 + v_2(1-v_1) + \ldots + v_j(1-v_1)(1-v_2)\ldots(1-v_{j-1})\}(1-y).$$

Now by comparing (2·5) and (3·8), we have the following result.

THEOREM 2. *For* $\mathrm{Re}(\alpha_j) > 0, \mathrm{Re}(\beta_j) > 0, j = 1, \ldots, p$, *and* $0 < y < 1$,

$$G^{p,0}_{p,p}(y|^{\alpha_j+\beta_j-1,\, j=1,\ldots,p}_{\alpha_j-1,\, j=1,\ldots,p}) = [\prod_{j=1}^{p}\Gamma(\beta_j)]^{-1} y^{\alpha_p-1}(1-y)^{\beta_1+\ldots+\beta_p-1} \times \int_0^1 \cdots \int_0^1 \{\prod_{j=1}^{p-1} v_j^{\beta_j-1}(1-v_j)^{\beta_p+\beta_{p+1}+\ldots+\beta_{j+1-1}} \xi_j^{\alpha_j-\alpha_{j+1}-\beta_{j+1}}\, dv_j\} \tag{3·9}$$

*where* $\xi_j$ *is defined in* (3·8).

Tretter and Walster [2] have given the following series expansion for $g(y)$ by using the result (3·8):

$$g(y) = D_p \sum_{j=0}^{\infty} \sigma_j y^{\alpha_p-1}(1-y)^{\delta_p+j-1} \quad (0 < y < 1), \tag{3·10}$$

where

$$\sigma_j = \Sigma_J [\prod_{i=1}^{p-1} B_{j_i}] \times \sum_{r_{p-1}=0}^{j_{p-1}} C(p-1, j_{p-1}, r_{p-1}) \times \sum_{r_{p-2}=0}^{j_{p-2}+r_{p-1}} C(p-2, j_{p-2}+r_{p-1}, r_{p-2}) \vdots \times \sum_{r_2=0}^{j_2-r_3} C(2, j_2+r_3, r_2)\, C(1, j_1+r_2, 0),$$

$\Sigma_J$ denotes the sum over all $j_i$ such that $\sum_{i=1}^{p-1} j_i = j$,

$$B_{j_i} = \frac{\Gamma(\gamma_i+j_i)}{\Gamma(\gamma_i)\,\Gamma(j_i+1)},$$

$$C(i, j_i, r_i) = \binom{j_i}{r_i} \frac{\Gamma(\beta_i+j_i-r_i)\,\Gamma(\delta_p-\delta_i+r_i)}{\Gamma(\beta_i+j_i+\delta_p-\delta_i)},$$

and $D_p$, and $\delta_i$ and $\gamma_i$, are given in (3·2) and (3·5) respectively.

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See the remarks in [2] concerning the availability of numerically stable computer programs for computing the percentage points from (3·10). Comparing (2·5) and (3·10), we have the following result.

THEOREM 3. *For* $\mathrm{Re}\,(\alpha_j) > 0$, $\mathrm{Re}\,(\beta_j) > 0, j = 1, \ldots, p$, *and* $0 < y < 1$,

$$G^{p,0}_{p,p}(y|^{\alpha_j+\beta_j-1,\, j-1,\ldots,p}_{\alpha_j-1,\, j=1,\ldots,p}) = [\textstyle\prod_{j=1}^{p}\Gamma(\beta_j)]^{-1}\sum_{j=0}^{\infty}\sigma_j y^{\alpha_p-1}(1-y)^{\delta_p+j-1}, \qquad (3\cdot11)$$

*where* $\sigma_j$ *is given following* (3·10).

Expanding the last factor in (3·6) by using a binomial expansion, and integrating term by term (note that the expansion is valid since $0 < t_j < 1$, $j = 1, \ldots, p$ and $0 < y < 1$), one gets the following series form for the density of $y$:

$$g(y) = D_p[\textstyle\prod_{j=2}^{p}\Gamma(\beta_j)]\, y^{\gamma_p-1}(1-y)^{\delta_p-1}\sum_{r=0}^{\infty}\Sigma_R\left\{\prod_{j=1}^{p-1}\frac{(\gamma_j)_{r_j}\,\Gamma(\delta_j+R_j)}{\Gamma(\delta_{j+1}+R_j)\,r_j!}\right\}(1-y)^r, \qquad (3\cdot12)$$

for $0 < y < 1$, where $\gamma_j$ and $\delta_j$ are defined in (3·5), $R_j = r_1 + \ldots + r_j$, and $R = (r_1, \ldots, r_{p-1})$ is a partitioning of $r$ (i.e. $r_i \geqslant 0$, $i = 1, \ldots, p-1$ and $r_1 + \ldots + r_{p-1} = r$). By comparing (2·5) and (3·11) we have the following result.

THEOREM 4. *For* $\mathrm{Re}\,(\alpha_j) > 0$, $\mathrm{Re}\,(\beta_j) > 0, j = 1, \ldots, p$, *and* $0 < y < 1$,

$$G^{p,0}_{p,p}(y|^{\alpha_j+\beta_j-1,\, j=1,\ldots,p}_{\alpha_j-1,\, j=1,\ldots,p}) = [\Gamma(\beta_1)]^{-1}\, y^{\gamma_p-1}(1-y)^{\delta_p-1}\sum_{r=0}^{\infty}\Sigma_R \times\left\{\prod_{j=1}^{p-1}\frac{(\gamma_j)_{r_j}\,\Gamma(\delta_j+R_j)}{\Gamma(\delta_{j+1}+R_j)\,r_j!}\right\}(1-y)^r. \qquad (3\cdot13)$$

## 4. *Particular cases*

From the results given above, one can get some interesting particular cases. From (3·4), we have the following result:

$$G^{2,0}_{2,2}(y|^{\alpha_1+\beta_1-1,\,\alpha_2+\beta_2-1}_{\alpha_1-1,\,\alpha_2-1}) = [\Gamma(\beta_1+\beta_2)]^{-1}\, y^{\alpha_2-1}(1-y)^{\beta_1+\beta_2-1} \times {}_2F_1(\beta_1, \alpha_2+\beta_2-\alpha_1;\, \beta_1+\beta_2;\, 1-y) \quad (0 < y < 1). \qquad (4\cdot1)$$

This result is also available through standard mathematical techniques.

Consider the case $p = 2$, $\alpha_2 = \alpha_1 + (\frac{1}{2})$, $\beta_1 = \beta_2$. Then by using the duplication formula for gamma functions, namely

$$\Gamma(2z) = (2\pi)^{-(\frac{1}{2})}\, 2^{2z-(\frac{1}{2})}\Gamma(z)\,\Gamma(z+(\tfrac{1}{2})), \qquad (4\cdot2)$$

one can simplify the moment expression in (2·4) as follows:

$$E(y^h) = \frac{\Gamma(2\alpha_1+2h)\,\Gamma(2\alpha_1+2\beta_1)}{\Gamma(2\alpha_1)\,\Gamma(2\alpha_1+2\beta_1+2h)}. \qquad (4\cdot3)$$

Since $E(y^h) = E((y^{\frac{1}{2}})^{2h}) = E((y^{\frac{1}{2}})^s)$, $s = 2h$, one has the result from (4·3) that $y^{\frac{1}{2}}$ is beta distributed with parameters $(2\alpha_1, 2\beta_1)$. The density of $u = y^{\frac{1}{2}}$ is

$$f(u) = \left\{\begin{array}{ll} \dfrac{\Gamma(2\alpha_1+2\beta_1)}{\Gamma(2\alpha_1)\,\Gamma(2\beta_1)}\, u^{2\alpha_1-1}(1-u)^{2\beta_1-1}, & 0 < u < 1, \\ 0 \quad \text{elsewhere.} & \end{array}\right\} \qquad (4\cdot4)$$

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By making a transformation $u = y^{\frac{1}{2}}$ or $du = (\frac{1}{2})y^{-\frac{1}{2}}dy$, the density of $y$ is given by

$$g(y) = \begin{cases} \dfrac{\Gamma(2\alpha_1 + 2\beta_1)}{2\Gamma(2\alpha_1)\,\Gamma(2\beta_1)} y^{\alpha_1 - 1}(1 - y^{\frac{1}{2}})^{2\beta_1 - 1}, & 0 < y < 1, \\ 0 \quad \text{elsewhere.} \end{cases} \tag{4·5}$$

Now comparing (4·5) with (2·5) for $p = 2$, $\alpha_2 = \alpha_1 + \frac{1}{2}$, $\beta_1 = \beta_2$, one has the following result.

THEOREM 5. *For* $\mathrm{Re}\,(\alpha_1) > 0$, $\mathrm{Re}\,(\beta_1) > 0$, *and* $0 < y < 1$,

$$G^{2,0}_{2,2}\left(y \middle|^{\alpha_1 + \beta_1 - 1,\, \alpha_1 + \beta_1 - \frac{1}{2}}_{\alpha_1 - 1,\, \alpha_1 - \frac{1}{2}}\right) = [2\Gamma(2\beta_1)]^{-1} y^{\alpha_1 - 1}(1 - y^{\frac{1}{2}})^{2\beta_1 - 1}.$$

By specializing the parameters, one can get several such results for particular cases. Then by equating to the corresponding series expansions given in Theorems 1–4, one can also get some interesting results.

It should be remarked that there is complete symmetry in the parameter pairs $(\alpha_1, \beta_1), \ldots, (\alpha_p, \beta_p)$. Hence in the above results, one can permute these parameters among themselves.

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