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A NOTE ON RANK TWO STABLE BUNDLES OVER SURFACES

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Abstract. Let $\pi : X \to C$ be a fibration with integral fibers over a curve C and consider a polarization H on the surface X. Let E be a stable vector bundle of rank 2 on C. We prove that the pullback $\pi^*(E)$ is a H-stable bundle over X. This result allows us to relate the corresponding moduli spaces of stable bundles $\mathcal{M}_C(2, d)$ and $\mathcal{M}_{X,H}(2, df, 0)$ through an injective morphism. We study the induced morphism at the level of Brill–Noether loci to construct examples of Brill–Noether loci on fibered surfaces. Results concerning the emptiness of Brill–Noether loci follow as a consequence of a generalization of Clifford's Theorem for rank two bundles on surfaces.

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1. Introduction. Let C be a smooth irreducible complex projective curve of genus g. A fibration over C is a surjective morphism $\pi : X \to C$ from a projective non-singular surface X with connected fibers. Consider the case when $\pi : X \to C$ is a ruled surface. Let E be a stable vector bundle of rank two on C. It is shown in [**19**, Proposition 3.4] that for any ample line bundle H on X, the pullback $\pi^*(E)$ is a H-stable bundle on X. This result has been generalized by S. Misra to higher rank bundles to study stable Higgs bundles on ruled surfaces (cf. [**13**, Corollary 4.2]). In this case, there is an isomorphism between the moduli space $\mathcal{M}_C(r, d)$ of stable rank r vector bundles with degree d on C and the moduli space $\mathcal{M}_{X,H}(r, df, 0)$ of H-stable rank r vector bundles with fixed Chern classes $c_1 = df$ and $c_2 = 0$, where f denotes the class of a fiber of π on the ruled surface X (cf. [**13**, Theorem 5.1]). When $\pi : X \to C$ is a non-isotrivial elliptic fibration with $\chi(\mathcal{O}_X) > 0$, there is also an isomorphism between the moduli spaces $\mathcal{M}_C(r, d)$ and $\mathcal{M}_{X,H}(r, df, 0)$ (see [**2**] and [**21**]).

In this paper, we aim to generalize these results to fibrations with integral fibers in the case of rank two bundles. Let $\pi : X \to C$ be a fibration with integral fibers and consider an ample line bundle H over X. Let E be a stable vector bundle of rank 2 on C, we prove

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in Theorem 3.6 that the pullback $\pi^*(E)$ is a H-stable rank 2 vector bundle over X. This result allows us to relate the corresponding moduli spaces over the curve and the surface, $\mathcal{M}_C(2, d)$ and $\mathcal{M}_{X,H}(2, df, 0)$, respectively. More precisely, we prove the following:

THEOREM 3.7. *Let $\pi : X \to C$ be a fibration with integral fibers. Then, π induces an injective morphism of moduli spaces*

$$\pi^* : \mathcal{M}_C(2, d) \to \mathcal{M}_{X,H}(2, df, 0)$$
$$E \mapsto \pi^*(E).$$

According to the proof of [4, Theorem 2.3], we define the Brill–Noether locus as

$$W^k_{X,H}(2, c_1, c_2) := \{E \in \mathcal{M}_{X,H}(2, c_1, c_2) \mid h^0(E) + h^2(E) \geq k\}$$

parameterizing H-stable rank r bundles on a smooth projective surface X with fixed Chern classes c_1, c_2. In the case that the cohomology for every $E \in \mathcal{M}_{X,H}(2, c_1, c_2)$ satisfies $h^2(E) = 0$, Costa and Miró-Roig [4] computed the expected dimension of a non-empty irreducible component of $W^k_{X,H}(2, c_1, c_2)$ namely $\rho_X(2, c_1, c_2, k)$. Consider a fibration $\pi : X \to C$ as before and the Brill–Noether locus on C,

$$W^k_C(r, d) := \{E \in \mathcal{M}_C(r, d) \mid h^0(E) \geq k\}.$$

From the projection formula, the morphism given in Theorem 3.7 induces an injective morphism at the level of Brill–Noether loci

$$\pi^* : W^k_C(2, d) \to W^k_{X,H}(2, df, 0).$$

We study this morphism to understand the geography of $W^k_{X,H}(2, df, 0)$ over the surface. An interesting application of Theorem 3.7 is that it allows to use results on Brill–Noether over curves to determine properties of the locus $W^k_{X,H}(2, df, 0)$ on the surface X as follows: If X is a ruled surface, from [13], the map π^* is an isomorphism of moduli spaces for bundles of any rank $r \geq 1$, then it induces an isomorphism of Brill–Noether locus

$$W^k_C(r, d) \cong W^k_{X,H}(r, df, 0),$$

and the geometry of the locus over the surface X coincides with the one over the curve C. When π is an elliptic fibration, we show examples where the locus $W^k_{X,H}(r, df, 0)$ is non-empty and the expected dimension is negative. In the case when π is a fibration with integral fibers and the rank is $r = 1, 2$, the induced map will be injective and we prove results of non-emptiness of $W^k_{X,H}(r, df, 0)$. Furthermore, we prove a generalization of Clifford's Theorem for rank two bundles on surfaces, and as a consequence, we prove results concerning the emptiness of the loci $W^k_{X,H}(2, c_1, c_2)$.

The paper is organized as follows: Section 2 collects a number of classical results, mainly about coherent sheaves on surfaces, that will be subsequently used. Section 3 is the core of this paper; it contains the proof of Theorem 3.7. In Section 4, we recall the results of [4] about the construction of Brill–Noether locus on surfaces and we show some applications of Theorem 3.7 to the study of the geometry of Brill–Noether loci of bundles on fibered surfaces. In Section 5, we prove a generalization of Clifford's Theorem for rank two bundles over surfaces and we show examples where the Brill–Noether loci $W^k_{X,H}(2, c_1, c_2)$ are empty and the expected dimension $\rho_{X,H}(2, c_1, c_2, k)$ is negative.

Notation: We work over the field of complex numbers $\mathbb{C}$. Given a coherent sheaf $\mathcal{G}$ on a variety X, we write $h^i(\mathcal{G})$ to denote the dimension of the i-th cohomology group $H^i(X, \mathcal{G})$. The sheaf K_X will denote the canonical sheaf on X.

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2. Preliminaries. This section contains some useful results on vector bundles over surfaces that will be used in the next sections. For detailed treatment of the subject see [6] and [11].

Let X be a smooth, irreducible, complex, projective variety of dimension n and let H be an ample line bundle over X. Let $\mathcal{G}$ be a torsion free sheaf on X of rank $\mathrm{rk}(\mathcal{G})$ with Chern classes $c_i(\mathcal{G}) \in H^{2i}(X, \mathbb{Z})$. The H-slope of $\mathcal{G}$ is defined as the rational number

$$\mu_H(\mathcal{G}) = \frac{c_1(\mathcal{G}).H^{n-1}}{\mathrm{rk}(\mathcal{G})},$$

where $c_1(\mathcal{G}).H^{n-1}$ is the degree of $\mathcal{G}$ with respect to H.

DEFINITION 2.1. Let $\mathcal{G}$ be a torsion-free coherent sheaf on X. We say that $\mathcal{G}$ is H-stable (respectively, H-semistable) if for all coherent subsheaves $\mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{G})$, we have $\mu_H(\mathcal{F}) < \mu_H(\mathcal{G})$ (respectively $\leq$). We call $\mathcal{G}$ unstable if it is not semistable and strictly semistable if it is semistable but not stable.

It is well known that the stability of torsion-free sheaves over a curve is independent of the polarization. We recall the following remark for the H-semistability of vector bundles over a surface X:

REMARK 2.2. Let V be a vector bundle on a surface X. We say that V is H-(semi)stable if for all sub-bundles $W \subset V$ with $0 < \mathrm{rk}(W) < \mathrm{rk}(V)$, we have $\mu_H(W) < \mu_H(V)$ ($\leq$ respectively). Indeed, let $W \subset V$ be a proper subsheaf, then W is torsion free and there exists the following diagram

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ W^{\vee\vee} & \longrightarrow & V^{\vee\vee}. \end{array} \tag{2.1}$$

Since $W^{\vee\vee}$ is a reflexive sheaf and $c_1(W) = c_1(W^{\vee\vee})$, it follows that V is H-semistable (respectively, H-stable) if for any proper reflexive sheaf W, we have $\mu_H(W) < \mu_H(V)$ (respectively, $\leq$). Moreover, since the singular points of reflexive sheaf W have co-dimension greater than 2, this implies that W is a vector bundle.

Moduli spaces of H-stable vector bundles with fixed Chern classes c_1, c_2 on a surface X have been constructed since the 1970s by M. Maruyama (cf. [12]). We shall denote the moduli space of H-semistable vector bundles of rank r and with fixed Chern classes c_1, c_2 on X by $\mathcal{M}^{ss}_{X,H}(r, c_1, c_2)$, and by $\mathcal{M}_{X,H}(2, c_1, c_2)$ the moduli space consisting of stable bundles.

Let $\mathcal{G}$ be a torsion-free sheaf on a surface X with Chern classes c_i and rank n. The discriminant of $\mathcal{G}$ is the characteristic class

$$\Delta(\mathcal{G}) = 2nc_2 - (n-1)c_1^2.$$

The Bogomolov inequality states that if $\mathcal{G}$ is H-semistable, then $\Delta(\mathcal{G}) \geq 0$.

The following Proposition was formulated without proof in [13, Proposition 3.2] for H-stable Higgs bundles over surfaces:

PROPOSITION 2.3. *If V is a H-stable vector bundle of rank n on a smooth algebraic surface X with a fixed polarization H on X and $\Delta(V) = 0$, then the stability of the vector bundle V is independent of the chosen polarization.*

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Proof. Suppose there is a polarization H_1 such that V is not H_1-stable. Then, there exists a sub-bundle $W \subset V$ with $\mu_{H_1}(V) \leq \mu_{H_1}(W)$. For such W, we can define a rational number

$$\lambda(W) := \frac{\mu_{H_1}(W) - \mu_{H_1}(V)}{\mu_H(V) - \mu_H(W)} \geq 0,$$

and a $\mathbb{Q}$-divisor $L_W := H_1 + \lambda(W)H$. Denote as $\mu_{L_W}(W) := \mu_{H_1}(W) + \lambda(W)\mu_H(W)$. Then

$$\mu_{L_W}(W) = \mu_{L_W}(V),$$

Notice that if $\lambda(W) \leq \lambda(W_0)$ and $L_{W_0} := H_1 + \lambda(W_0)H$, then $\mu_{L_{W_0}}(W) \leq \mu_{L_{W_0}}(V)$: Indeed, we have

$$\frac{\mu_{H_1}(W) - \mu_{H_1}(V)}{\mu_H(V) - \mu_H(W)} = \lambda(W) \leq \lambda(W_0).$$

Then, $\mu_{H_1}(W) - \mu_{H_1}(V) \leq \lambda(W_0)(\mu_H(V) - \mu_H(W))$. Therefore, $\mu_{L_{W_0}}(W) \leq \mu_{L_{W_0}}(V)$. Consider

$$A := \{\lambda(W) | W \subset V \text{ is sub-bundle with } \mu_{H_1}(W) \geq \mu_{H_1}(V)\}.$$

By Grothendieck Theorem (cf. [**10**, Lemma 1.7.9]), the set A is bounded and we can consider a sub-bundle $W_0 \subset V$ such that $\lambda(W_0)$ is maximal.

We claim that: Any proper sub-bundle $W \subset V$ satisfies $\mu_{L_{W_0}}(W) \leq \mu_{L_{W_0}}(V)$.
Proof of Claim:

(i) If $\mu_{H_1}(W) < \mu_{H_1}(V)$, then $\mu_{L_{W_0}}(W) < \mu_{L_{W_0}}(V)$ since V is H-stable.

(ii) If $\mu_{H_1}(W) \geq \mu_{H_1}(V)$, since $\lambda(W_0)$ is maximal in A, then $\lambda(W) \leq \lambda(W_0)$ and $\mu_{L_{W_0}}(W) \leq \mu_{L_{W_0}}(V)$, which proves the Claim.

Consider $k > 0$ an integer such that $k\lambda(W_0) = d \in \mathbb{N}$, then $H_0 := kL_{W_0} = kH_1 + rH$ is an ample divisor over X. From the Claim follows that V is H_0-semistable because $\mu_{H_0}(F) = k\mu_{L_{W_0}}(F)$ for any vector bundle F over X.

Therefore, there is an exact sequence

$$0 \longrightarrow W_0 \longrightarrow V \longrightarrow W_1 \longrightarrow 0 \tag{2.2}$$

of torsion-free sheaves with $\mu_{H_0}(W_0) = \mu_{H_0}(V) = \mu_{H_0}(W_1)$. Since W_0 and V are H_0-semistables torsion-free sheaves, it follows that W_1 is H_0-semistable (cf. [**6**, Chapter 4, Lemma 6]).

By Bogomolov inequality, we have $\Delta(W_0) \geq 0$ and $\Delta(W_1) \geq 0$. Define

$$\mathcal{E} := nc_1(W_0) - mc_1(V),$$

where rk $(V) = n$ and rk $(W_0) = m$. Since V is stable with respect to the polarization H then $\mathcal{E}.H < 0$, in particular $\mathcal{E}$ is not numerically equivalent to zero. Notice that $\mathcal{E}.H_0 = 0$ and by Hodge Index Theorem, we have $\mathcal{E}^2 < 0$. On the other hand, from the exact sequence (2.2), we have

$$0 = \Delta(V) = \frac{n}{n-m}\Delta(W_0) + \frac{n}{m}\Delta(W_1) - \frac{\mathcal{E}^2}{m(n-m)}.$$

Since $\Delta(W_0) \geq 0$ and $\Delta(W_1) \geq 0$, hence $\mathcal{E}^2 \geq 0$ which is a contradiction. We conclude that the stability of the vector bundle V is independent of the polarization. □

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3. Main result. In this section, we prove Theorem 3.7. We begin this section by recalling the next result (see proof of Corollary 6 in p. 54 of [**14**]):

LEMMA 3.1. *(cf. [14]) Let B be a complete variety (integral separated scheme of finite type over $\mathbb{C}$) and let L be a line bundle over B. Then $L = \mathcal{O}_B$ if and only if $h^0(L) \neq 0$ and $h^0(L^\vee) \neq 0$.*

We use the previous Lemma to prove:

LEMMA 3.2. *Let $\pi : X \to C$ be a fibration with integral fibers. Let V be a line bundle such that $V|_f = \mathcal{O}_f$ for the generic fiber f. Then, there exists a line bundle L over C such that $\pi^*(L) = V$.*

Proof. We define the sets $Z := \{c \in C \mid h^0(V|_{\pi^{-1}(c)}) \geq 1\}$ and $W := \{c \in C \mid h^0(V^\vee|_{\pi^{-1}(c)}) \geq 1\}$. By upper-semicontinuity Theorem (cf. [**14**], p. 50), we have that Z and W are closed subsets of C. Since $V|_f = \mathcal{O}_f$ for the generic fiber f, it follows that $Z = W = C$. Thus, we get $V|_f = \mathcal{O}_f$ for any fiber f by Lemma 3.1. Consequently, $V = \pi^*(\pi_*(V))$, where $L = \pi_*(V)$. □

The following result characterizes semistable bundles of degree zero in terms of their sections:

LEMMA 3.3. *Let E be a semistable vector bundle of degree 0 and rank r over a smooth projective curve C. Then, $h^0(E) \leq r$ and $h^0(E) = r$ if and only if $E = \oplus_r \mathcal{O}_C$.*

Proof. If E has rank $r = 1$, the statement is clear. Let E be a vector bundle on C of degree 0 and rank ≥ 2 and suppose that the Lemma 3.3 holds for $r - 1$. Assume that $h^0(E) \neq 0$, then there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \xrightarrow{f} Q \longrightarrow 0 \tag{3.1}$$

of vector bundles. Since E is a semistable bundle of degree zero, it follows that Q is a semistable vector bundle of rank $r - 1$ and degree 0. By induction hypothesis, $h^0(Q) \leq r - 1$ and $h^0(Q) = r - 1$ if and only if $Q = \oplus_{r-1}\mathcal{O}_C$.

From the exact sequence (3.1), we get

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(C, E) \to H^0(C, Q) \to H^1(C, \mathcal{O}_C) \to \cdots$$

Therefore, $h^0(E) \leq 1 + h^0(Q) \leq r$, and this proves the first part of Lemma. Now assume that $h^0(E) = r$, then $h^0(Q) = r - 1$ and $Q = \oplus_{r-1}\mathcal{O}_C$. From the exact sequence (3.1), we get the following long exact sequence

$$0 \to \mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, \mathcal{O}_C) \to \mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, E) \to \mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, \oplus_{r-1}\mathcal{O}_C) \xrightarrow{\delta}$$
$$\mathrm{Ext}^1(\oplus_{r-1}\mathcal{O}_C, \mathcal{O}_C) \to \mathrm{Ext}^1(\oplus_{r-1}\mathcal{O}_C, E) \to \mathrm{Ext}^1(\oplus_{r-1}\mathcal{O}_C, \oplus_{r-1}\mathcal{O}_C) \to 0,$$

where

$$\dim \mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, \mathcal{O}_C) = h^0(C, \oplus_{r-1}\mathcal{O}_C) = r - 1$$
$$\dim \mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, E) = h^0(C, \oplus_{r-1}E) = r(r-1)$$
$$\dim \mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, \oplus_{r-1}\mathcal{O}_C) = h^0(C, \oplus_{(r-1)^2}\mathcal{O}_C) = (r-1)^2.$$

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Note that the morphism

$$\begin{aligned}\mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, E) &\to \mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, \oplus_{r-1}\mathcal{O}_C)\\ \varphi &\mapsto f\circ\varphi\end{aligned}$$

is surjective. It follows that for any $\phi \in \mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, \oplus_{r-1}\mathcal{O}_C)$, there exists $\varphi \in \mathrm{Hom}(\oplus_{r-1}\mathcal{O}_C, E)$ such that $f\circ\varphi = \phi$. In particular, if $\phi = Id$, the exact sequence (3.1) splits and $E = \oplus_r \mathcal{O}_C$, which completes the proof. □

The following Lemma states a relation between pullback of bundles on the curve and semistable bundles of rank two with trivial restriction:

LEMMA 3.4. *Let $\pi : X \to C$ be a fibration. Let V be a H-semistable vector bundle of rank 2 over X. Suppose that there exists a vector bundle E over C such that $\pi^*(E) = V$.*

(i) If V is H-stable, then E is a stable vector bundle.
(ii) $V|_f = \mathcal{O}_f \oplus \mathcal{O}_f$ for generic fiber f.

Proof. Let $E_1 \subset E$ be a sub-bundle. We want to prove that $\mu(E_1) < \mu(E)$. Since $\pi^*(E_1) \subset \pi^*(E) = V$ is a sub-bundle and V is H-stable, it follows that

$$\mu(E_1)H.f = \mu_H(\pi^*(E_1)) < \mu_H(\pi^*(E)) = \mu(E)H.f.$$

Since H is ample and f is a fiber, we get $\mu(E_1) < \mu(E)$. Therefore, E is a stable bundle on C, and this proves (i). By [**6**, Chapter 9, Theorem 18], the bundle $V|_f$ is semistable for the generic fiber f and by projection formula we have $E = \pi_*(V)$ since π is a fibration. The fiber dimension of $E = \pi_*(V)$ at a point $c \in C$ is given as

$$\mathrm{rk}(E)|_c = h^0(\pi^{-1}(c), V|_{\pi^{-1}(c)}) = h^0(f, V|f) = 2 = \mathrm{rk}(V),$$

where $f = \pi^{-1}(c)$. Finally, since $h^0(f, V|_f) = 2$, it follows from Lemma 3.3 that $V_{|_f} = \mathcal{O}_f \oplus \mathcal{O}_f$. This proves (ii). □

We recall the proof that the pullback of a semistable bundle under a fibration is again semistable:

REMARK 3.5. *(cf. [**13**, Theorem 3.3]) Let $\pi : X \to C$ be a fibration with a fixed polarization H on X. Let E be a semistable vector bundle on C. Then, the pullback $\pi^*(E)$ is H-semistable.*

Proof of Remark 3.5. Since H is ample, there exists a natural $n \in \mathbb{N}$ such that nH is very ample. By Bertini's Theorem, there exists a smooth curve B in the linear system $|nH|$. Consider the composition of morphisms

$$\phi : B \xrightarrow{i} X \xrightarrow{\pi} C.$$

Since $\phi = \pi \circ i : B \to C$ is a finite morphism between smooth curves and E is semistable, then $\phi^*(E)$ is a semistable vector bundle (see [**11**, Theorem 10.1.3]). Assume that $\pi^*(E)$ is not nH-semistable, then there exists a sub-bundle $W \subset \pi^*(E)$ such that

$$\mu_{nH}(W) > \mu_{nH}(\pi^*(E)).$$

Notice that $W|_B$ is a sub-bundle of $\pi^*(E)|_B := \phi^*(E)$ and the slopes are given by $\mu(W|_B) = \mu_H(W)$ and $\mu(\pi^*(E)|_B) = \mu_{nH}(\pi^*(E))$. Therefore, $\phi^*(E) = \pi^*(E)|_B$ is not a semistable vector bundle on B, which is a contradiction. It follows that $\pi^*(E)$ is a nH-semistable vector bundle. Hence, $\pi^*(E)$ is a H-semistable vector bundle. □

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Next, we prove that the pullback of rank two stable bundles is not just semistable but stable:

THEOREM 3.6. *Let $\pi : X \to C$ be a fibration with integral fibers. Then for any stable rank 2 vector bundle E over C, the pullback $\pi^*(E)$ is a H-stable bundle on X.*

Proof. From the previous result, we can assume that $\pi^*(E)$ is a strictly H-semistable vector bundle. Let $V_1 \subset \pi^*(E)$ be a line sub-bundle with $\mu_H(V_1) = \mu_H(\pi^*(E))$. Hence, there exists an exact sequence of torsion-free sheaves

$$0 \longrightarrow V_1 \longrightarrow \pi^*(E) \longrightarrow V_2 \otimes I_Z \longrightarrow 0, \tag{3.2}$$

where V_i are line bundles on X and $Z \subset X$ has co-dimension 2. Restricting the exact sequence (3.2) to a generic fiber f such that $\mathrm{Supp}(Z) \cap f = \emptyset$,

$$0 \longrightarrow V_1|_f \longrightarrow \pi^*(E)|_f \longrightarrow V_2|_f \longrightarrow 0. \tag{3.3}$$

By Lemma 3.4 part (ii), we have that $\pi^*(E)|_f = \mathcal{O}_f \oplus \mathcal{O}_f$ is semistable of degree 0 and $\deg(V_1|_f) \leq 0$.

We claim that: $\deg(V_1|_f) < 0$. Suppose that $\deg(V_1|_f) = 0$. From the exact sequence (3.3) for a generic fiber f, we have

$$\deg(V_1|_f) = \deg(V_2|_f) = 0.$$

Notice that from Lemma 3.3, $h^0(V_i|_f) \leq \mathrm{rk}(V_i) = 1$. By taking cohomology in the exact sequence (3.3),

$$2 = h^0(\pi^*(E)_f) \leq h^0(V_1|f) + h^0(V_2|f) \leq 1 + 1 = 2.$$

Therefore, $h^0(V_i|f) = 1$ and $V_i|_f = \mathcal{O}_f$ for a generic fiber f. By Lemma 3.2, it follows that there exists a line bundle L_1 over C such that $\pi^*(L_1) = V_1$. Therefore, $L_1 \subset E$ is a sub-bundle of E and

$$\mu(L_1)(H.f) = \mu_H(V_1) = \mu_H(\pi^*E) = \mu(E)(H.f).$$

Since H is ample and f is a fiber, it follows that $\mu(L_1) = \mu(E)$ which contradicts the stability of E. Thus, $\deg(V_1|_f) < 0$ which proves the claim. We have proved the following statement: for any line sub-bundle $W \subset \pi^*(E)$ with $\mu_H(W) = \mu_H(\pi^*(E))$, we have $\deg(W|_f) = c_1(W)f < 0$ for generic fiber f.

Let n be a positive integer. Since H is ample and f is nef, then $H_n = H + nf$ is ample.

We claim that: $\pi^*(E)$ is H_n-stable. Let $W \subset \pi^*(E)$ be a sub-bundle.

(i) If $\mu_H(W) = \mu_H(\pi^*(E))$, by the above statement we have $c_1(W).f < 0$ and

$$\mu_{H_n}(W) < \mu_H(\pi^*(E)) = \mu_{H_n}(\pi^*(E)).$$

(ii) If $\mu_H(W) < \mu_H(\pi^*(E))$, then

$$\mu_{H_n}(W) \leq \mu_H(W) < \mu_H(\pi^*(E)) = \mu_{H_n}(\pi^*(E)).$$

Hence, $\pi^*(E)$ is a H_n-stable bundle. Since $\Delta(\pi^*(E)) = 0$ and the stability is independent of the chosen polarization (see Proposition 2.3), it follows that $\pi^*(E)$ is a H-stable vector bundle for any polarization H on X. □

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We formulate our main result:

THEOREM 3.7. *Let $\pi : X \to C$ be a fibration with integral fibers. Then, π induces an injective morphism of moduli spaces*

$$\begin{aligned} \pi^* : \mathcal{M}_C(2, d) &\to \mathcal{M}_{X,H}(2, df, 0) \\ E &\mapsto \pi^*(E). \end{aligned} \tag{3.4}$$

Proof. The proof is similar to [13] (Theorem 5.1). Let F be a family of stable vector bundles of rank two and degree d over C, parameterized by T. That is, F is a vector bundle over $C \times T$ such that for any closed point $t \in T$, we have that $F|_{C\times\{t\}}$ is a stable vector bundle over C of degree d and rank two. By Theorem 3.6, $\tilde{F} := (\pi \times id)^*(F)$ defines a family of H-stable bundles of rank two over X parameterized by T such that for any closed point $t \in T$, we have that $c_1(\tilde{F}|_{X\times\{t\}}) = df$ and $c_2(\tilde{F}|_{X\times\{t\}}) = 0$. Thus, we get a natural morphism between the moduli spaces

$$\pi^* : \mathcal{M}_C(2, d) \to \mathcal{M}_{X,H}(2, df, 0).$$

Now we prove the injectivity of π^*: For $i = 1, 2$ consider $E_i \in \mathcal{M}(2, d)$ such that $\pi^*(E_1) \cong \pi^*(E_2)$. Since π is a fibration, it follows that

$$E_1 = \pi_*(\pi^*(E_1)) \cong \pi_*(\pi^*(E_2)) = E_2.$$

Thus, $E_1 \cong E_2$ which proves the injectivity of π^* and completes the proof of the Theorem. □

Let $\pi : X \to C$ be a fibration with integral fibers and fix a polarization H on X. In order to generalize the previous result to higher rank $r \geq 2$, we must first make sure that for any H-stable vector bundle V of rank r on X with Chern classes $c_1 = df$ and $c_2 = 0$ such that $V|_f = \bigoplus \mathcal{O}_f$, there exists a vector bundle E over C such that $V = \pi^*(E)$. Moreover, we must ask that if V is H-stable, then for all but finitely fibers f of π, $V|_f$ is semistable. Here, we conjecture that the morphism

$$\begin{aligned} \pi^* : \mathcal{M}_C(r, d) &\to \mathcal{M}_{X,H}(r, df, 0) \\ E &\mapsto \pi^*(E) \end{aligned} \tag{3.5}$$

is well defined and that its image consists of H-stable vertical bundles over X such that the restriction to the general fiber is trivial. However, this topic exceeds the scope of this paper (see [21, Section 2 and Definition 2.1] for the definition of vertical bundles and [2, Lemma 1.4] for more details).

REMARK 3.8. (i) *If $\pi : X \to C$ is a ruled surface, the morphism (3.4) is in fact an isomorphism between the respective moduli spaces of (semi)stables vector bundles for any rank $r \geq 1$ (see e.g. [13, Corollary 4.2]).*

(ii) *Let $\pi : X \to C$ be a non-isotrivial relatively minimal elliptic fibration with no multiple fibers. Then, for any rank $r \geq 1$, the morphism (3.4) is an isomorphism. (For a deeper discussion of the isomorphism, we refer the reader to [2, 7, 20] and [21].)*

COROLLARY 3.9. *Let $\pi : X \to C$ be a fibration with integral fibers. If the moduli space $M_C(2, d)$ is non-empty, then there exist H-stable torsion free sheaves of rank 2, first Chern class $c_1 = df + 2c_1(L)$ and second Chern class $c_2 = c_2 + df \cdot c_1(L) + c_1(L)^2$ for any $L \in Pic(X)$ and $c_2 \geq 0$.*

The proof of corollary makes use of the following result.

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LEMMA 3.10. *(cf. [3, Lemma 2.7]) Let L be a line bundle on a smooth surface X. Let E be a vector bundle on X, and let E' be a general elementary modification of E at a general point $p \in X$, defined as the kernel of a general surjection $\phi : E \to \mathcal{O}_p$:*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

(i) $rk(E') = rk(E)$, $c_1(E') = c_1(E)$, $c_2(E') = c_2 + 1$.
(ii) If E is H-stable, then E' is H-stable.
(iii) $H^2(X, E) \cong H^2(X, E')$.
(iv) If $h^0(X, E) > 0$, *then* $h^0(X, E') = h^0(X, E) - 1$ *and* $h^1(X, E') = h^1(X, E)$. *If* $h^0(X, E) = 0$, *then* $h^1(X, E') = h^1(X, E) + 1$. *In particular, if at most one of h^0 or h^1 is non zero for E, then at most one of h^0 or h^1 is non zero for E'.*

Proof of Corollary 3.9. Let $\pi : X \to C$ be a fibration with integral fibers. Let $V \in M_C(2, d)$ and let $E := \pi^* V$ the corresponding vector bundle in the moduli space $M_{X,H}(2, df, 0)$. let E' be a general elementary modification of E. By Lemma 3.10, it follows that E' is H-stable torsion-free sheaf of rank 2 and Chern classes $c_1(E') = df$, $c_2(E') = 1$. Repeated application of elementary modification enables us to conclude that there exist H-stable torsion-free sheaves of rank 2 and Chern classes $c_1 = df$ and c_2 for any $c_2 \geq 0$. Moreover, since $E \otimes L$ and $E' \otimes L$ are H-stable, we can conclude that there exist H-stable torsion-free sheaves of rank 2 and Chern classes $c_1 = df + 2c_1(L)$ and $c_2 = c_2 + df \cdot c_1(L) + c_1(L)^2$ for any $L \in Pic(X)$. □

4. Non-emptiness of Brill–Noether loci on fibered surfaces. Moduli spaces of stable vector bundles have been extensively studied; however, relatively little is known about their geometry in terms of the existence and structure of their subvarieties. In [4], Costa and Miró–Roig have defined and constructed the Brill–Noether locus $W^k_{X,H}(r, c_1, c_2)$ for smooth projective surfaces (in fact, these loci were constructed for smooth projective varieties) as subvarieties of $\mathcal{M}_{X,H}(r, c_1, c_2)$ satisfying cohomological properties, i.e. the support is the set of H-stable rank r vector bundles E on X with fixed Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$ for $i = 1, 2$ and $h^0(E) + h^2(E) \geq k$, i.e

$$W^k_{X,H}(r, c_1, c_2) := \{E \in \mathcal{M}_{X,H}(r, c_1, c_2) \mid h^0(E) + h^2(E) \geq k\}.$$

Moreover, if $h^2(E) = 0$ for any vector bundle $E \in \mathcal{M}_{X,H}(r, c_1, c_2)$, then each non-empty irreducible component of $W^k_{X,H}(r, c_1, c_2)$ has dimension at least the Brill–Noether number defined as

$$\rho_X(r, c_1, c_2, k) := \dim \mathcal{M}_{X,H}(r, c_1, c_2) - k(k - \chi(r, c_1, c_2)).$$

The study of these subvarieties is known as Brill–Noether theory. Basic questions concerning non-emptiness, connectedness, irreducibility, dimension, singularities, etc, have been answered when X is a curve (see for instance [1, 16] and [18]).

In this section, we use the morphism (3.4) to study properties of non-emptiness, connectedness, irreducibility, and dimension of the Brill–Noether locus $W^k_{X,H}(r, df, 0)$ over a fibered surface $\pi : X \to C$ with integral fibers. From Proposition 2.3, since stability does not depend on the polarization, we will denote by $W^k_X(2, df, 0)$ the Brill–Noether locus and by $\mathcal{M}_X(2, df, 0)$ the moduli space.

The following theorem states a relation between the locus $W^k_C(r, d)$ and $W^k_X(r, df, 0)$.

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THEOREM 4.1. *Let $r = 1, 2$ and $\pi : X \to C$ be a fibration with integral fibers. The morphism induced by the pullback*

$$\pi^* : W_C^k(r, d) \longrightarrow W_X^k(r, df, 0) \tag{4.1}$$

is injective. In particular, if $\rho_C(1, d, k+1) \geq 0$, then $W_X^k(1, df, 0) \neq \emptyset$.

Proof. Let $E \in W_C^k(r, d)$. From Theorem 3.7, it is sufficient to prove that $\pi^*(E) \in W_X^k(r, df, 0)$. Since π is a fibration, by the projection formula, it follows that $\pi_*(\pi^*(E)) = E$. Therefore, $h^0(X, \pi^*(E)) = h^0(C, E) = k$ and $\pi^*(E) \in W_X^k(r, df, 0)$ as desired. The second part follows directly from [**1**, Theorem 1.1]. □

COROLLARY 4.2. *Let $r = 1, 2$ and $0 < c_2 < k$. Let $\pi : X \to C$ be a fibration with integral fibers.*

(i) If the locus $W_C^k(r, d)$ is non-empty, then the there exist H-stable torsion-free sheaves on X of rank r with Chern classes $c_1 = df$, $c_2 > 0$ having at least $k - c_2$ sections.
(ii) Let D be an effective divisor on X. If the locus $W_C^k(r, d)$ is non-empty, then the locus $W_X^{k-c_2}(r, df + rc_1(\mathcal{O}_X(D)), c_2 + df \cdot c_1(\mathcal{O}_X(D) + c_1(\mathcal{O}_X(D)^2)$ is non-empty.

Proof.

(i) The proof follows directly from Theorem 4.1 and repeated application of Corollary 3.9 and Lemma 3.10.
(ii) The proof follows from item (i) and the exact sequence

$$0 \longrightarrow E \longrightarrow E(D) \longrightarrow E_D(D) \longrightarrow 0.$$

□

Non-emptiness of the Brill–Noether locus $W_C^k(r, d)$, $r \geq 2$ has been studied by several authors (see for instance [**8**, **16**, **17**, **18**, **15**]). However, the problem in the general case remains open. There are examples where the expected dimension $\rho_C(r, d, k) < 0$ and $W_C^k(r, d)$ is non-empty (see for instance [**8**]); and examples where $\rho_C(r, d, k) > 0$ and $W_C^k(r, d)$ is non-empty of dimension strictly greater than $\rho_C(r, d, k)$ (see for instance [**17**, Corollary 1.2]). The main interest of Theorem 4.1 is that it allows to use results of Brill–Noether over curves to determine properties of $W_X^k(r, df, 0)$ for $r = 1, 2$ as we will see in the next results.

COROLLARY 4.3. *Let $d \in \mathbb{N}$ and let $\pi : X \to C$ be fibration with integral fibers. If any $L \in Pic^{df}(X)$ satisfies $h^2(L) = 0$, then*

(i) $W_C^k(1, d) \cong W_X^k(1, df, 0)$ for any $k \in \mathbb{N}$.
(ii) If $\rho_C(1, d, k+1) \geq 1$, then $W_X^k(1, df, 0)$ is connected.
(iii) If $\rho_C(1, d, k+1) < 0$ and C is a general curve, then $W_X^k(1, df, 0) = \emptyset$.
(iv) If $\rho_C(1, d, k+1) \geq 1$ and C is a general curve, then $W_X^k(1, df, 0)$ is irreducible.

Proof. We only prove (*i*), the proof (*ii*)–(*iv*) follows as an application of (*i*) and the results well known on the classical Brill–Noether theory (see for instance [**1**, Chapter 5]). We claim that π^* is surjective. Let $L \in W_X^k(1, df, 0)$ be a line bundle over X. Since $h^2(L) = 0$, it follows that $h^0(L) \geq k \neq 0$. By Lemma 3.2, we recall that if $L|_F = \mathcal{O}_F$ for generic fiber F, then there exists a line bundle $\bar{L}$ over C such that $\pi^*(\bar{L}) = L$. Assume that $L|_F$ is not isomorphic to $\mathcal{O}_F$ for generic fiber F, then $h^0(F, L|_F) = 0$. From the exact sequence

$$0 \longrightarrow L(-F) \longrightarrow L \longrightarrow L|_F \longrightarrow 0,$$

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we get $H^0(X, L(-F)) \cong H^0(X, L)$. Therefore, any section $s \in H^0(X, L)$ vanishes at F for generic fiber F. Since the zero locus of a section $s \in H^0(X, L)$ is a divisor, it follows that $h^0(L) = 0$, which is a contradiction. Therefore, $L|_F = \mathcal{O}_F$, and there exists a line bundle $\bar{L}$ over C such that $\pi^*(\bar{L}) = L$ (see Lemma 3.2). Since π is a fibration, it follows that $h^0(\bar{L}) = h^0(L) \geq k$. Hence, by Theorem 4.1, the morphism

$$\pi^* : W_C^k(1, d) \longrightarrow W_X^k(1, df, 0)$$

is bijective. Since $W_C^k(1, d)$ and $W_X^k(1, df, 0)$ are normal varieties, it follows that π^* is an isomorphism, which is the desired conclusion. □

PROPOSITION 4.4. *Let $\pi : X \to C$ be a ruled surface. There exist an isomorphism $W_X^k(r; df, 0) \cong W_C^k(r, d)$. Moreover, in this case, $\rho_C(r, d, k) = \rho_X(r, df, 0, k)$.*

Proof. Since the morphism (3.4) is in fact an isomorphism between the respective moduli spaces of (semi)stables vector bundles for any rank $r \geq 1$ (see [**13**, Corollary 4.2]), it is sufficient to prove that $h^2(E) = 0$ for any $E \in M_X(r; df, 0)$. We recall that there is a well-defined invariant e (see [**9**, Proposition 2.8]). We denote by C_0 the section of self-intersection $-e$ and by f the class of a fiber. Let H be a line bundle numerically equivalent to $C_0 + bf$ with $b > \max\{e, e + g - 1 - \frac{er+d}{2r}\}$, then H is ample (see [**9**, Proposition 2.20]). Since

$$b > e + g - 1 - \frac{er + d}{2r},$$

it follows that $c_1(E^\vee \otimes K_X).H < 0$. Therefore, $H^2(X, E) = 0$ for any vector bundle $E \in \mathcal{M}_X(r, df, 0)$ because stability is independent of the polarization H (see Theorem 2.3) and Serre duality Theorem. □

PROPOSITION 4.5. *Let $\pi : X \to C$ be a relatively minimal elliptic fibration with no multiple fibers and $\chi = \chi(\mathcal{O}_X) > 0$. Let $r, d \in \mathbb{N}$, satisfying $d \geq r(2(g-1) + \chi)$, then*

$$W_X^k(r, df, 0) = \begin{cases} \emptyset, & \text{if } k > d + r(1-g), \\ \mathcal{M}_C(r, d), & \text{if } k \leq d + r(1-g). \end{cases}$$

In particular, if $k = d + r(1-g)$, then $W_X^k(r, df, 0) \neq \emptyset$ and $\rho_X < 0$.

Proof. Since the morphism (3.4) is in fact an isomorphism between the respective moduli spaces of (semi)stables vector bundles for any rank $r \geq 1$ (see Remark 3.8 part (ii)), it is sufficient to prove that $h^2(E) = 0$ for any $E \in M_X(r; df, 0)$. Since $h^1(\mathcal{O}_f) = 1$ for every fiber of π, base change implies that $R^1\pi_*(\mathcal{O}_X)$ is a line bundle on C. Denote by L the dual line bundle of $R^1\pi_*(\mathcal{O}_X)$ and $\deg(L) = \chi$. By [**6**, Theorem 15], the canonical line bundle for an elliptic surface is given by

$$K_X = \pi^*(K_C \otimes L).$$

If $(2g - 2 + \chi)r \leq d$, then $h^2(F) = 0$ for any $F \in M_X(r, df, 0)$ and π^* induce an isomorphism between $W_C^k(r, d)$ and $W_X^k(r; df, 0)$ for any $k \in \mathbb{N}$. Let $E \in M_C(r, d)$ be a vector bundle, we have that $h^0(E) = d + r(1-g)$ by Riemann–Roch Theorem, and $h^0(\pi^*(E)) = d + r(1-g)$. Thus,

$$W_X^k(r, df, 0) = \begin{cases} \emptyset, & \text{if } k > d + r(1-g), \\ \mathcal{M}_C(r, d), & \text{if } k \leq d + r(1-g). \end{cases}$$

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Now, the Brill–Noether number satisfies

$$\begin{aligned}\rho_X(r, df, 0, k) &:= r^2(g-1)+1-k(k-r\chi(\mathcal{O}_X)) \\ &= r^2(g-1)+1+\frac{r^2}{4}\chi^2(\mathcal{O}_X)-(k-\frac{r}{2}\chi(\mathcal{O}_X))^2.\end{aligned}$$

Thus, $\rho_X < 0$ if $k > \sqrt{r^2(g-1)+1+\frac{r^2}{4}\chi^2(\mathcal{O}_X)}+\frac{r}{2}\chi(\mathcal{O}_X)$. In particular, if $k = d + r(g-1)$ then $\rho_X < 0$. □

A similar argument proves that if $\pi : X \to C$ is a fibration with integral fibers and K_X is ample with $rK_X^2 \leq dK_X.f = 2d(g(f)-1)$ then $H^2(X, E) = 0$ for any $E \in \mathcal{M}_X(2, df, 0)$ and $\rho_X(2, df, 0, k) \leq \rho_C(2, d, k)$.

5. Emptiness of Brill–Noether loci of rank two stable vector bundles on surfaces. In this section, we give a generalization of Clifford Theorem for rank 2 vector bundles on surfaces and show the emptiness of some Brill–Noether loci.

PROPOSITION 5.1. *Let X be a smooth projective surface and let H be a very ample divisor on X such that $H.K_X \geq 0$. Let $\mathcal{F}$ be a torsion-free sheaf of rank one over X such that $0 \leq c_1(\mathcal{F}).H \leq nH^2$ with $n \in \mathbb{N}$ and $c_1(\mathcal{F}) \notin |nH|$. Then*

$$h^0(\mathcal{F}) \leq n\frac{c_1(\mathcal{F}).H}{2}+1.$$

Proof. Let C be a smooth projective curve in the linear system $|nH|$. Assume that $\mathcal{F} := L$ is a line bundle over X such that $0 \leq L.H \leq nH^2$ and $L \notin |nH|$. By Adjunction formula and the fact that $H.K_X \geq 0$, we have that $\deg(\mathcal{O}_C(L)) \leq n^2H^2 \leq C^2 + K_X.C = 2(g(C)-1)$. Therefore,

$$h^0(\mathcal{O}_C(L)) \leq n\frac{L.H}{2}+1$$

by Clifford's Theorem.

On the other hand, since H is an ample line bundle and $L.H \leq nH^2$, it follows from Nakai–Moishezon's criterion that $H^0(\mathcal{O}_X(L-nH)) = H^0(L-C) = 0$.

From the exact sequence

$$0 \to \mathcal{O}_X(L-C) \to \mathcal{O}_X(L) \to \mathcal{O}_C(L) \to 0,$$

it follows that

$$h^0(L) \leq n\frac{L.H}{2}+1.$$

which proves the theorem for line bundles. If $\mathcal{F}$ is a torsion-free sheaf of rank one, then there exists a line bundle L over X such that $\mathcal{F} = L \otimes I_Z$, where $Z \subset X$ is of co-dimension 2. Since $c_1(\mathcal{F}) = c_1(L)$ and $h^0(\mathcal{F}) \leq h^0(L)$, it follows that

$$h^0(\mathcal{F}) \leq h^0(L) \leq n\frac{c_1(\mathcal{F}).H}{2}+1.$$

□

The following theorem can be considered as a generalization of Clifford's Theorem.

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THEOREM 5.2. *Let X be a smooth projective surface and let H be a very ample divisor on X such that $H \cdot K_X \geq 0$. Let $E \in M_{X,H}(2, c_1, c_2)$ with $0 \leq \mu_H(E) < nH^2$ with $n \in \mathbb{N}$. Then*

$$h^0(E) \leq n\frac{c_1.H}{2} + n^2H^2 + 2.$$

Proof. Notice that if $h^0(E) = 0$, then the theorem follows. Assume that $h^0(E) \neq 0$. Let L_1 be a subline bundle of E of maximal slope. Since $h^0(E) \neq 0$, it follows that $0 \leq L_1.H < nH^2$. By Proposition 5.1,

$$h^0(L_1) \leq n\frac{L_1.H}{2} + 1.$$

Consider the exact sequence

$$0 \to L_1 \to E \to L_2 \otimes I_Z \to 0. \tag{5.1}$$

Since $0 \leq \mu_H(E) < nH^2$, from the exact sequence, (5.1) follows that $0 \leq \mu_H(E) < L_2.H$ and L_2 satisfies the hypothesis of Proposition 5.1. Hence,

$$h^0(E) \leq n\frac{L_1.H}{2} + 1 + 2n\frac{L_2.H}{2} + 1 \leq n\frac{c_1.H}{2} + n^2H^2 + 2.$$

□

Notice that an argument similar to the proof of Proposition 5.1 and Theorem 5.2 works when H is ample, and there exists a smooth curve $C \in |nH|$. As an application of Theorem 5.2, we obtain the following result concerning the emptiness of the Brill–Noether loci.

COROLLARY 5.3. *Let X be a smooth surface and let H be a very ample divisor on X such that $H \cdot K_X \geq 0$. Let $c_2 \gg 0$, n be integers and c_1 a divisor on X, such that $0 \leq \frac{c_1.H}{2} < nH^2$. Then,*

$$W^k_{X,H}(2, c_1, c_2) = \emptyset$$

for any $k > n\frac{c_1.H}{2} + n^2H^2 + 2$.

REMARK 5.4. *From [5, Proposition 2.4] whenever $c_2 \gg 0$ the moduli space $\mathcal{M}_{X,H}(2, c_1, c_2)$ is a non-empty generically smooth, irreducible, quasi-projective variety of the expected dimension $dim(\mathcal{M}_{X,H}(2, c_1, c_2)) = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_X)$. In particular, if $c_2 \gg 0$ and $k \geq 5$ in Corollary 5.3, when the Brill–Noether locus is empty, the expected dimension is $\rho_{X,H}(2, c_1, c_2, k) < 0$.*

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