Two ways to generate monotonic sequences: convexity and ratios

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Introduction

Here we give an account of two appealingly simple techniques to generate monotonic sequences that were developed by the author and Grahame Bennett. Sadly, Bennett died in 2016 after a lifetime of distinguished contributions in the field of inequalities. This article is dedicated to him.

The first method works with averages of values of convex functions. It was described in full in [1]; a partial version was given by Kuang in a *Gazette* Note [2]. The second method deals with ratios of partial sums. The idea has certainly been known for a long time, but the applications described here were first presented in [3], as part of a wider study.

The main application relates to the sum $S_n(p) = \sum_{r=1}^{n} r^p$. We present a companion pair of results on monotonicity of $\dfrac{S_n(p)}{n^p(n+1)}$ and $\dfrac{S_n(p)}{n(n+1)^p}$ for various values of p. At first sight, these ratios may seem a bit abstruse, but in a sense they are more natural than, for example, $\dfrac{S_n(p)}{n^{p+1}}$, because they reproduce the fact that $\dfrac{S_n(1)}{n(n+1)}$ is constant. We show how these results can be obtained by either method. Another result delivered by both methods is a strengthening of the theorem of [4] concerning $(n!)^{1/n}$.

We then present a result on sums of the form $\sum_{r=1}^{n} (2r-1)^p$ which is delivered by Method 2 but not at all easily by Method 1. We finish with a survey of various other averages of convex functions.

Method 1: *Averages of convex functions*

For a function f on the interval $[0, 1]$, we define

$$A_n(f) = \frac{1}{n-1}\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) \qquad (n \geqslant 2),$$

$$B_n(f) = \frac{1}{n+1}\sum_{r=0}^{n} f\left(\frac{r}{n}\right) \qquad (n \geqslant 1).$$

These are, respectively, the averages of the values $f\left(\frac{r}{n}\right)$ excluding and including the endpoints. For $A_n(f)$, we do not need f to be defined at 0 and 1. If f is continuous on $[0, 1]$, then both $A_n(f)$ and $B_n(f)$ tend to $\int_0^1 f(x)\,dx$ as $n \to \infty$ (we return to this point later). The point for us is that for *convex* functions, they do so in a monotonic way.

https://doi.org/10.1017/mag.2021.4 Published online by Cambridge University Press

Recall that f is 'convex' on an interval I if it lies below the straight-line chord between any two points on its graph. More exactly, for x_1, x_2 in I and $0 \leqslant \lambda \leqslant 1$, we have $f\left[(1-\lambda)x + \lambda y\right] \leqslant (1-\lambda)f(x) + \lambda f(y)$. Also, f is 'concave' if $-f$ is convex. If a function is both convex and concave, then it is linear. A sufficient condition for convexity is that $f''(x) \geqslant 0$ on the interval. So for $x > 0$, the function x^p is convex for $p \geqslant 1$ and $p \leqslant 0$, and concave for $0 \leqslant p \leqslant 1$. We can now state our results.

Theorem 1: If f is a convex function on $(0, 1)$, then $A_n(f)$ increases with n. If f is concave, then $A_n(f)$ decreases with n.

Proof: We prove the statement for convex f; the statement for concave f then follows by considering $-f$. For $1 \leqslant r \leqslant n-1$, the point $\frac{r}{n}$ lies between $\frac{r}{n+1}$ and $\frac{r+1}{n+1}$. More exactly,

$$\frac{r}{n} = \frac{n-r}{n}\frac{r}{n+1} + \frac{r}{n}\frac{r+1}{n+1}.$$

Write $f\left(\frac{r}{n+1}\right) = f_r$. Since f is convex,

$$f\left(\frac{r}{n}\right) \leqslant \frac{n-r}{n} f_r + \frac{r}{n} f_{r+1}.$$

Hence

$$\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) \leqslant \frac{n-1}{n} f_1 + \frac{1}{n} f_2 + \frac{n-2}{n} f_2 + \frac{2}{n} f_3 + \ \ldots \ + \frac{1}{n} f_{n-1} + \frac{n-1}{n} f_n$$

$$= \frac{n-1}{n} \sum_{r=1}^{n} f_r,$$

which says that $A_n(f) \leqslant A_{n+1}(f)$.

Theorem 2: If f is convex on $[0, 1]$, then $B_n(f)$ decreases with n. If f is concave, then $B_n(f)$ increases with n.

Proof: Assume f convex and take $n \geqslant 2$. This time, we use the fact that for $1 \leqslant r \leqslant n-1$,

$$\frac{r}{n} = \frac{r}{n}\frac{r-1}{n-1} + \frac{n-r}{n}\frac{r}{n-1}.$$

Write $f\left[\frac{r}{n-1}\right] = g_r$. By convexity of f, for r as above,

$$f\left(\frac{r}{n}\right) \leqslant \frac{r}{n} g_{r-1} + \frac{n-r}{n} g_r,$$

also $f\left(\frac{0}{n}\right) = g_0$ and $f\left(\frac{n}{n}\right) = g_{n-1}$. Hence

$$\sum_{r=0}^{n} f\left(\frac{r}{n}\right) \leqslant \frac{n}{n} g_0 + \frac{1}{n} g_0 + \frac{n-1}{n} g_1 + \frac{2}{n} g_1 + \ \ldots \ + \frac{1}{n} g_{n-1} + \frac{n}{n} g_{n-1}$$

$$= \frac{n+1}{n} \sum_{r=0}^{n-1} g_r,$$

which says that $B_n(f) \leqslant B_{n-1}(f)$.

https://doi.org/10.1017/mag.2021.4 Published online by Cambridge University Press

Of course, if f is linear, it follows that $A_n(f)$ and $B_n(f)$ are constant. This is easily verified directly: if $f(x) = x$, then $A_n(f) = B_n(f) = \frac{1}{2}$ for all n.

Method 2: *Ratios of partial sums*

Our second method for generating monotonic sequences is beautifully simple both to state and to prove. It is far from new: it appears as an exercise (albeit without serious applications) in [5, p. 21], published in 1908.

Theorem 3: Let (a_n) and (b_n) be sequences, with $b_n > 0$ for all n. Write $A_n = a_1 + a_2 + \ldots + a_n$, similarly B_n. If $\frac{a_n}{b_n}$ increases (or decreases) with n, so does $\frac{A_n}{B_n}$.

(Of course, this notation A_n, B_n has no connection with the previous $A_n(f)$, $B_n(f)$.)

Proof:

We prove the increasing case: the decreasing one follows by considering $-a_n$.

Write $a_n = c_n b_n$ and $A_n = K_n B_n$. Then (c_n) is increasing, and hence $A_n = \sum_{r=1}^{n} c_r b_r \leqslant c_n \sum_{r=1}^{n} b_r = c_n B_n$, so $K_n \leqslant c_n$. So also $K_n \leqslant c_{n+1}$, and we have

$$\begin{aligned} A_{n+1} &= A_n + a_{n+1} \\ &= K_n B_n + c_{n+1} b_{n+1} \\ &\geqslant K_n (B_n + b_{n+1}) \\ &= K_n B_{n+1}. \end{aligned}$$

But $A_{n+1} = K_{n+1} B_{n+1}$, so $K_{n+1} \geqslant K_n$.

The special case $b_n = 1$ says: if a_n is increasing, then so is $\frac{A_n}{n}$, the arithmetic mean of $a_1, a_2, \ldots, a_n$. We mention here that we can deduce the same for the geometric mean:

Corollary: Let (a_n) be an increasing sequence of positive numbers, and let $G_n = (a_1 a_2 \ldots a_n)^{1/n}$. Then (G_n) increases with n. In particular, $(n!)^{1/n}$ is increasing.

Proof: Since $\log a_n$ increases with n, so does $\frac{1}{n} \sum_{r=1}^{n} \log a_r = \log G_n$.

Of course, to apply Theorem 3, we have to equip ourselves with the sequence $\frac{a_n}{b_n}$. The examples to follow will illustrate what this may involve.

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Application to $S_n(p)$

Let $S_n(p) = \sum_{r=1}^{n} r^p$. We will show how both methods can be applied to this sum, with a similar outcome. Note that $S_n(0) = n$ and $S_n(1) = \frac{1}{2}n(n+1)$. We do not exclude negative values of p: for example, $S_n(-1)$ is the harmonic sum $H_n = \sum_{r=1}^{n} \frac{1}{r}$.

A basic estimation of $S_n(p)$ (which we will shortly improve) is given by comparison with the integral of x^p, as follows. Let f be any increasing function with $f(0) = 0$. Write $S_n = \sum_{r=1}^{n} f(r)$. On the interval $[r-1, r]$, we have $f(r-1) \leqslant f(x) \leqslant f(r)$; hence $f(r-1) \leqslant \int_{r-1}^{r} f(x)dx \leqslant f(r)$. Adding the right-hand inequalities for $1 \leqslant r \leqslant n$ and the left-hand inequalities for $1 \leqslant r \leqslant n+1$, we obtain

$$\int_0^n f(x)\,dx \leqslant S_n \leqslant \int_0^{n+1} f(x)\,dx.$$

For $f(x) = x^p$, where $p > 0$, this says

$$\frac{n^{p+1}}{p+1} \leqslant S_n(p) \leqslant \frac{(n+1)^{p+1}}{p+1}. \tag{1}$$

hence $\dfrac{S_n(p)}{n^{p+1}} \to \dfrac{1}{p+1}$ as $n \to \infty$. In fact, this limit also holds for $-1 < p < 0$ (we omit the details). Of course, for $p < -1$, $S_n(p)$ is simply the nth partial sum of the convergent series $\sum_{r=1}^{\infty} r^p$.

We are now ready for our Theorem on monotonicity.

Theorem 4: Let

$$c_n(p) = \frac{S_n(p)}{n(n+1)^p}.$$

Then $c_n(p)$ increases with n for $p \geqslant 1$ and $p \leqslant 0$, and decreases for $0 \leqslant p \leqslant 1$.

Also, let

$$d_n(p) = \frac{S_n(p)}{n^p(n+1)}.$$

Then $d_n(p)$ decreases with n for $p \geqslant 1$, and increases for $p \leqslant 1$.

The reversals at $p = 1$ reflect the fact that $c_n(1)$ and $d_n(1)$ have the constant value $\frac{1}{2}$. Similarly, $c_n(0)$ has the constant value 1.

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Ahead of the proof (by both methods), note that for $p \leqslant 0$, the statement for $d_n(p)$ follows from the one for $c_n(p)$. In fact

$$d_n(p) = \left(\frac{n+1}{n}\right)^{p-1} c_n(p).$$

Now $\frac{n+1}{n} = 1 + \frac{1}{n}$, which is decreasing. Since $p - 1 < 0$, $\left(\frac{n+1}{n}\right)^{p-1}$ is increasing.

Recall that x^p is convex for $p \geqslant 1$ and $p \leqslant 0$, and concave for $0 \leqslant p \leqslant 1$. With this remark, we can read off Theorem 4 from Theorems 1 and 2 without further work.

Proof by Method 1:

The statements for $c_n(p)$ are given by Theorem 1 with $f(x) = x^p$, since

$$A_{n+1}(f) = \frac{1}{n}\sum_{r=1}^{n} \frac{r^p}{(n+1)^p} = c_n(p).$$

By the preceding remark, we only need to consider $d_n(p)$ for $p \geqslant 0$. The statements are given by Theorem 2, since

$$B_n(f) = \frac{1}{n+1}\sum_{r=0}^{n} \frac{r^p}{n^p} = d_n(p).$$

The proof by Method 2 needs a little more work. We include it for comparison, and because it paves the way for the proof of another result to follow.

Proof by Method 2: We apply Theorem 3 to $\dfrac{1}{c_n(p)}$: this equals $\dfrac{A_n}{B_n}$, where

$$a_n = n(n+1)^p - (n-1)n^p, \qquad b_n = n^p.$$

So for $p \neq 0$,

$$\frac{a_n}{b_n} = n\left(1 + \frac{1}{n}\right)^p - n + 1.$$

Now

$$\frac{d}{dx}\left(1 + \frac{x}{n}\right)^p = \frac{p}{n}\left(1 + \frac{x}{n}\right)^{p-1},$$

and hence

$$\frac{a_n}{b_n} = n\left[\left(1 + \frac{x}{n}\right)^p\right]_0^1 + 1 = p\int_0^1 \left(1 + \frac{x}{n}\right)^{p-1} dx + 1.$$

Now $1 + \frac{x}{n}$ decreases with n, so $\left(1 + \frac{x}{n}\right)^{p-1}$, and hence its integral, decreases if $p \geqslant 1$ and increases if $p < 1$. For $\frac{a_n}{b_n}$, this is reversed by the factor p when $p < 0$. The result follows.

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For $d_n(p)$, with $p > 0$, we have $\dfrac{1}{d_n(p)} = \dfrac{A_n}{B_n}$, where

$$a_n = n^p(n+1) - (n-1)^p n, \qquad b_n = n^p.$$

So, in similar style,

$$\frac{a_n}{b_n} = n + 1 - n\left(1 - \frac{1}{n}\right)^p = 1 + p\int_0^1 \left(1 - \frac{x}{n}\right)^{p-1} dx.$$

This increases with n if $p \geqslant 1$ and decreases if $0 < p < 1$.

Example 1: We can illustrate the case $p = 2$ by giving exact expressions. Since $S_n(2) = \frac{1}{6}n(n+1)(2n+1)$, we have

$$6c_n(2) = 2 - \frac{1}{n+1}, \qquad 6d_n(2) = 2 + \frac{1}{n},$$

showing that $c_n(2)$ is indeed increasing and $d_n(2)$ decreasing.

Example 2: The case $p = -1$ for $c_n(p)$ says that $\left(1 + \frac{1}{n}\right)H_n$ increases with n. On writing it out, one finds that this actually follows from the inequality $H_n \leqslant n$.

We mention a number of consequences of Theorem 4.

If $p > 0$, it follows from (1) that $c_n(p)$ and $d_n(p)$ tend to $\frac{1}{p+1}$ as $n \to \infty$. Since the terms of an increasing sequence are not greater than the limit, we can deduce the following refinement of (1):

Corollary 1: For $p \geqslant 1$, we have

$$\frac{1}{p+1}n^p(n+1) \leqslant S_n(p) \leqslant \frac{1}{p+1}n(n+1)^p.$$

The inequalities reverse when $0 < p \leqslant 1$.

At the cost of some loss of accuracy, we can deduce simpler statements where the denominators are n^{p+1} and $(n+1)^{p+1}$.

Corollary 2: Let $u_n(p) = \dfrac{S_n(p)}{n^{p+1}}$. Then $u_n(p)$ decreases if $p \geqslant 0$ and increases if $p \leqslant 0$.

Proof: We have $u_n(p) = \left(1 + \frac{1}{n}\right)d_n(p)$, hence is decreasing for $p \geqslant 1$. Also, $u_n(p) = \left(1 + \frac{1}{n}\right)^p c_n(p)$, hence is decreasing for $0 \leqslant p \leqslant 1$ and increasing for $p < 0$.

Corollary 3: Let $v_n(p) = \dfrac{S_n(p)}{(n+1)^{p+1}}$. Then $v_n(p)$ is increasing for all p.

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Proof: For $p \geqslant 1$ and $p < 0$, this follows from $v_n(p) = \left(\frac{n}{n+1}\right)c_n(p)$, since $\frac{n}{n+1}$ is increasing. For $0 < p \leqslant 1$, it follows from $v_n(p) = \left(\frac{n}{n+1}\right)^p d_n(p)$.

Note that, unlike $c_n(p)$ and $d_n(p)$, these ratios fail to be constant when $p = 1$.

Application to n!

We now describe another result delivered by both methods.

Theorem 5: The expression $\frac{1}{n+1}(n!)^{1/n}$ decreases with n.

It was shown in [4] that $\frac{1}{n}(n!)^{1/n}$ decreases; this statement is weaker than ours, because their expression equates to ours multiplied by the decreasing factor $1 + \frac{1}{n}$.

Proof 1: We apply Theorem 1 to the concave function $\log x$ (recall that for this Theorem there is no need for f to be defined at 0). Then $A_{n+1}(f)$ decreases with n. The result follows, since

$$A_{n+1}(f) = \frac{1}{n}\sum_{r=1}^{n}\left(\log r - \log(n+1)\right) = \frac{1}{n}\log(n!) - \log(n+1).$$

Proof 2: Let $a_n = n\left[\log(n+1) - \log n\right]$. Then $a_n = \int_0^1 \frac{n}{n+x}dx$. For $x \geqslant 0$, $\frac{n}{n+x}$ increases with n, since $\frac{n+x}{n} = 1 + \frac{x}{n}$ decreases. So a_n increases. By Theorem 3, so does $\frac{A_n}{n}$. But

$$\frac{A_n}{n} = \frac{1}{n}\sum_{r=1}^{n} r\left[\log(r+1) - \log r\right] = \log(n+1) - \frac{1}{n}\sum_{r=1}^{n}\log r.$$

Note: There is another application of Proof 2: take $a_n = \log(n+1) - \log n$ and $b_n = \frac{1}{n}$, to conclude that $\frac{\log(n+1)}{H_n}$ increases with n.

Some further applications of Method 2

Method 1 gives an instant proof for sequences that are of one of the forms described, but it is limited to such sequences. Method 2 is effective in some areas beyond the reach of Method 1. One such area concerns odd-number sums. Let

$$U_n(p) = \sum_{r=1}^{n}(2r-1)^p.$$

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An immediate application of Theorem 3 is that $\frac{U_n(p)}{S_n(p)}$ increases with n for $p > 0$, since this is of the form $\frac{A_n}{B_n}$, where $\frac{a_n}{b_n} = \left(2 - \frac{1}{n}\right)^p$. More seriously, by a suitable modification of the proof by Method 2, we can derive a pleasantly simple analogue of Theorem 4 for $U_n(p)$. This proof follows [3, p. 559].

Theorem 6: The expression $\frac{U_n(p)}{n^{p+1}}$ increases with n if $p \geqslant 1$ or $p \leqslant 0$, and decreases if $0 \leqslant p \leqslant 1$.

Remark: The reversal at $p = 1$ reflects the fact that $U_n(1) = n^2$.

Lemma: Let $g(t) = (1 + t)^p + (1 - t)^p$. If $p \geqslant 1$ or $p \leqslant 0$, then $g(t)$ increases with t for $0 \leqslant t < 1$. If $0 \leqslant p \leqslant 1$, then $g(t)$ decreases.

Proof: We have $g'(t) = p\left[(1 + t)^{p-1} - (1 - t)^{p-1}\right]$. Hence $g'(t) \geqslant 0$ on $[0, 1)$ if $p \geqslant 1$ or $p \leqslant 0$, and $g'(t) \leqslant 0$ if $0 \leqslant p \leqslant 1$.

Proof of Theorem 6: If $p \leqslant -1$, then $\frac{1}{n^{p+1}} = n^q$ where $q \geqslant 0$, so increases with n, hence the statement is trivial. So assume that $p > -1$. We have $\frac{n^{p+1}}{U_n(p)} = \frac{A_n}{B_n}$, where

$$a_n = n^{p+1} - (n - 1)^{p+1}, \qquad b_n = (2n - 1)^p.$$

Then

$$\frac{a_n}{b_n} = (2n - 1)\left[\left(\frac{n}{2n - 1}\right)^{p+1} - \left(\frac{n - 1}{2n - 1}\right)^{p+1}\right]$$

$$= \frac{2n - 1}{2^{p+1}}\left[\left(1 + \frac{1}{2n - 1}\right)^{p+1} - \left(1 - \frac{1}{2n - 1}\right)^{p+1}\right].$$

Now

$$(2n - 1)\frac{d}{dx}\left(1 + \frac{x}{2n - 1}\right)^{p+1} = (p + 1)\left(1 + \frac{x}{2n - 1}\right)^p,$$

so

$$\frac{a_n}{b_n} = \frac{p + 1}{2^{p+1}} I_n,$$

where

$$I_n = \int_{-1}^{1}\left(1 + \frac{x}{2n-1}\right)^p dx = \int_0^1\left[\left(1 + \frac{x}{2n-1}\right)^p + \left(1 - \frac{x}{2n-1}\right)^p\right]dx.$$

By the Lemma, I_n decreases with n if $p \geqslant 1$ or $p \leqslant 0$, and increases with n if $0 \leqslant p \leqslant 1$. The statement follows, by Theorem 3.

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Clearly, this sum is not a case of $A_n(f)$ or $B_n(f)$. In the next section we will see what happens when this problem is addressed by function averages.

A slight variation of the proof of Theorem 6 (which sufficiently keen readers may care to write out) delivers the following further result of this kind for $S_n(p)$. The expression

$$\frac{S_n(p)}{(2n+1)^{p+1}-1}$$

increases with n if $p \geqslant 1$ or $-1 < p \leqslant 0$, and decreases if $0 \leqslant p \leqslant 1$. (Note that again the expression is constant when $p = 1$.)

Some other averages of convex functions

First we investigate averages including one endpoint. Write

$$L_n(f) = \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right), \qquad R_n(f) = \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right).$$

Simple formulae link these sums with $A_n(f)$ and $B_n(f)$, for example

$$nR_n(f) = \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = (n+1)B_n(f) - f(0),$$

hence $R_n(f) = \left(1+\frac{1}{n}\right)B_n(f) - \frac{1}{n}f(0)$. Note also that if $g(x) = f(1-x)$, then $L_n(g) = R_n(f)$.

These unsymmetrical averages are arguably less natural than $A_n(f)$ and $B_n(f)$. They fail to be constant for $f(x) = x$: in fact, then $L_n(f) = \frac{1}{2}\left(1-\frac{1}{n}\right)$ and $R_n(f) = \frac{1}{2}\left(1+\frac{1}{n}\right)$. So $R_n(f)$ is decreasing for $f(x) = x$, but increasing for $f(x) = -x$: there is no result to the effect that $R_n(f)$ is either increasing or decreasing for all convex f. However, restricted results on monotonicity do follow easily from Theorems 1 and 2.

Theorem 7: If f is convex and increasing, then $R_n(f)$ decreases with n and $L_n(f)$ increases. If f is convex and decreasing, then $R_n(f)$ increases and $L_n(f)$ decreases.

Proof: The statements for $L_n(f)$ follow from those for $R_n(f)$ by applying them to $g(x) = f(1-x)$. Suppose that f is convex and increasing. Let $h(x) = f(x) - f(0)$. Then $R_n(h) = R_n(f) - f(0)$, $h(0) = 0$ and $h(x) \geqslant 0$ on $[0, 1]$. So $R_n(h) = \left(1+\frac{1}{n}\right)B_n(h)$ and $B_n(h) \geqslant 0$. By Theorem 2, $B_n(h)$ is decreasing. Hence $R_n(h)$ is decreasing.

Similarly, if f is decreasing, let $k(x) = f(x) - f(1)$. Then $R_n(k) = \left(1-\frac{1}{n}\right)A_n(k)$, which is increasing, by Theorem 1.

This result was presented in [2]. While it serves to satisfy curiosity about averages with one endpoint, it is clear that it is weaker than Theorems 1 and 2.

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Applied to $f(x) = x^p$, Theorem 7 reproduces Corollaries 2 and (for $p > 0$) 3 of Theorem 3, since $u_n(p) = R_n(f)$ and $v_n(p) = L_{n+1}(f)$.

We can now clarify the relation with $\int_0^1 f(x)\,dx$. If f is continuous on [0, 1], then $L_n(f)$ and $R_n(f)$ are Riemann sums for this integral (in the sense that a value of $f(x)$ is chosen from each successive interval). So they both converge to the integral as $n \to \infty$, and it follows that the same is true for $A_n(f)$ and $B_n(f)$, as stated earlier.

We now revisit Theorem 6. The sum is of the form

$$M_n(f) = \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{2r-1}{2n}\right).$$

What can we say about this kind of average? It passes the test of being constant for $f(x) = x$. It describes the approximation to $\int_0^1 f(x)\,dx$ given by taking the tangent to the curve at the midpoint of each interval $\left[\frac{r-1}{n}, \frac{r}{n}\right]$. A convex function f lies above its tangents, so $M_n(f) \leqslant \int_0^1 f(x)\,dx$.

A dual approximation to the integral is given by the trapezium rule: let

$$T_n(f) = \frac{1}{2n}f(0) + \frac{1}{n}\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) + \frac{1}{2n}f(1).$$

From the definition of convexity, it is clear that $T_n(f) \geqslant \int_0^1 f(x)\,dx$ for convex functions.

In the light of these facts, it seems plausible that for convex f, $M_n(f)$ increases with n and $T_n(f)$ decreases. However, this is not true, as the following example shows.

Example 3: Let $f(x) = \left|x - \frac{1}{2}\right|$. Then $M_2(f) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}$, while $M_3(f) = \frac{1}{3}\left(\frac{1}{3} + 0 + \frac{1}{3}\right) = \frac{2}{9}$. The reader can easily verify that $T_2(f) = \frac{1}{4}$ and $T_3(f) = \frac{5}{18}$.

The true state of affairs was established in [1]: the suspected results hold if f is convex and also f is either convex or concave. This, of course, delivers Theorem 6. However, the proofs given in [1] are long and intricate. A more pleasant proof of both results is presented in a forthcoming article [6].

A natural generalisation of $A_n(f)$ and $B_n(f)$ is a *weighted* average of the form $B_n(W,f) = \sum_{r=0}^{n} w_{n,r} f\left(\frac{r}{n}\right)$, where $w_{n,r} \geqslant 0$ and $\sum_{r=0}^{n} w_{n,r} = 1$ for all n. This is explored in [7]. For example, it is shown that $B_n(W, f)$ decreases with n for convex f when W is the *Euler matrix* defined by $w_{n,r} = \binom{n}{r} x^r (1-x)^{n-r}$ for a chosen x in (0, 1). As some readers will recognise, $B_n(W, f)$ is then the value at x of the nth *Bernstein polynomial* for f.

https://doi.org/10.1017/mag.2021.4 Published online by Cambridge University Press

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10.1017/mag.2021.4

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From the Editor-in-Chief : MA 150

Very sharp-eyed readers may have noticed that the logo on the front of this issue of *The Mathematical Gazette* has been slightly enhanced. That is because The Mathematical Association is celebrating the 150th Anniversary of its foundation. This took place on 17th January 1871 at a meeting arranged by leading schoolteachers of that time and as their creation is still going strong their initiative has proved to be well worthwhile.

The foundation meeting was held at University College, London and those who were there agreed to the setting up of the 'Association for the Improvement of Geometrical Teaching'. In its early years, the work of the Association was achieved by means of Reports. The first issue of the *Gazette* was dated April 1894 and despite two World Wars has appeared regularly ever since. In 1897, the organisation became The Mathematical Association.

In 1971, the MA launched *Mathematics in School* which had as its focus teachers in the secondary school sector and, in 1977, *Primary Mathematics* for teachers of younger pupils. So both of these are in very significant Jubilee Years and are to be congratulated. The *Gazette* is now into Volume 105 and Issue 562 but, though of great merit, these numbers do not appear to be particularly special. On your behalf, I offer thanks to all who have worked on these magazines and to those who are still doing so.

10.1017/mag.2021.45

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https://doi.org/10.1017/mag.2021.4 Published online by Cambridge University Press