The solid is therefore a sphere, volume $\frac{4}{3}\pi t^3 = \frac{4}{3}\pi\left(R^2 - r^2\right)^{3/2}$. In other words, if the bored sphere sits on a table, its volume is that of a complete sphere of the same height. Hence the title of the puzzle based on this observation, the 'napkin-ring problem' [7, pp. 54-55]. The questioner asks: What is the volume of a napkin-ring of height h? Martin Gardner's shrewd puzzler reasons that, since he has not been given the diameter of the hole, he is free to shrink it to zero, producing a sphere of diameter h.

I am pleased to see that Wikipedia now honours Mamikon by devoting the entry 'Visual calculus' to his method.

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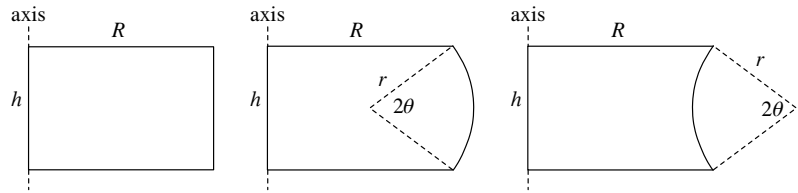
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An area conundrum

Three solids of revolution are formed by rotating the areas in Figure 1 about the axes indicated. These form the basis of an interesting classroom discussion: which of solids (a), (b) and (c) has the greatest surface area? This is equivalent to asking: which of the solids has the greatest curved surface area?



(a) cylinder (b) outer curved edge (c) inner curved edge

FIGURE 1

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Intuitively, it seems clear that (a) and (c) have a smaller curved surface area than (b), but the interesting comparison is of (a) with (c). The respective formulae for S_a, the curved surface area of (a), etc. are derived below where, for S_b and S_c we use Pappus' theorem that the curved surface area generated is the product of the arc length and the distance that the centre of mass of the arc travels in one revolution about the axis. (Alternatively, we could use the standard integration formula for finding surface areas of revolution.)

For the cylinder in (a), we have

$$S_a = 2\pi Rh = 4\pi rR \sin\theta, \text{ since } h = 2r \sin\theta.$$

For S_b, the arc of length $2r\theta$ has centre of mass distance

$$R - r\cos\theta + \frac{r\sin\theta}{\theta}$$

from the axis, so that

$$\begin{aligned} S_b &= 2r\theta \cdot 2\pi\left(R - r\cos\theta + \frac{r\sin\theta}{\theta}\right) \\ &= 4\pi rR\theta + 4\pi r^2(\sin\theta - \theta\cos\theta) \\ &= 4\pi rR\theta + 4\pi r^2\cos\theta(\tan\theta - \theta). \end{aligned}$$

It follows that $S_b > S_a$ because $\theta > \sin\theta$ and $\tan\theta > \theta$; this is also clear from Pappus' theorem, because the arc revolved is longer than h and its centre of mass is further from the axis than R.

A similar analysis gives

$$\begin{aligned} S_b &= 2r\theta \cdot 2\pi\left(R + r\cos\theta - \frac{r\sin\theta}{\theta}\right) \\ &= 4\pi rR\theta + 4\pi r^2(\theta\cos\theta - \sin\theta) \\ &= 4\pi rR\theta + 4\pi r^2\cos\theta(\theta - \tan\theta). \end{aligned}$$

It follows that $S_b > S_c$ always holds (as expected from Pappus), but the surprise comes from comparing S_a and S_c. For each value of θ, $0 < \theta < \frac{1}{2}\pi$, $S_a = S_c$ if

$$\frac{R}{r} = \frac{\sin\theta - \theta\cos\theta}{\theta - \sin\theta} = \frac{\cos\theta(\tan\theta - \theta)}{\theta - \sin\theta}. \tag{1}$$

For larger values of $\frac{R}{r}$, we have $S_a < S_c$; for smaller values of $\frac{R}{r}$ (but with $\frac{R}{r} \geqslant 1$ for sense), we have $S_a > S_c$.

The range of values of $\frac{R}{r}$ given by (1) is quite small because, if $f(\theta) = \frac{\sin\theta - \theta\cos\theta}{\theta - \sin\theta}$ then $\lim_{\theta\to 0} f(\theta) = 2$ and $f(\frac{\pi}{2}) = \frac{2}{\pi - 2} \approx 1.752$. In fact $f(\theta)$ is a decreasing function and we finish with a slightly unusual

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proof of this. We have

$$f(\theta) = \frac{\sin\theta - \theta\cos\theta}{\theta - \sin\theta} = \frac{\int_0^\theta x\sin x\,dx}{\int_0^\theta 1 - \cos x\,dx} \tag{2}$$

But $\dfrac{1-\cos x}{x\sin x} = \dfrac{\tan\frac{1}{2}x}{x}$ so, since $\dfrac{\tan x}{x}$ is increasing, $\dfrac{1-\cos x}{x\sin x} < \dfrac{1-\cos\theta}{\theta\sin\theta}$ for $0 < x < \theta$ so that $\theta\sin\theta\int_0^\theta 1-\cos x\,dx < (1-\cos\theta)\int_0^\theta x\sin x\,dx$. By the quotient rule on (2), this establishes that $f'(\theta) < 0$ for $0 < \theta < \frac{1}{2}\pi$ (indeed, for $0 < \theta < \pi$).

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Trigonometric identities from regular polygons

Introduction

This piece concerns trigonometric identities derived from regular n-gons by projecting their sides onto an inradius, length r, or circumradius, length R, to which the sum of the projections is equal. We find we can generalise such equations according to the form of n. This yields (at least) five useful cases and either one or two identities for each. Where there are two, one has the inradius as subject, the other, the circumradius. As shown in Figure 1,

$$r = \frac{1}{2}\cot\left(\frac{\pi}{n}\right) \quad \text{and} \quad R = \frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right),$$

and the two are related by the equation $4R^2 = 4r^2 + 1$.

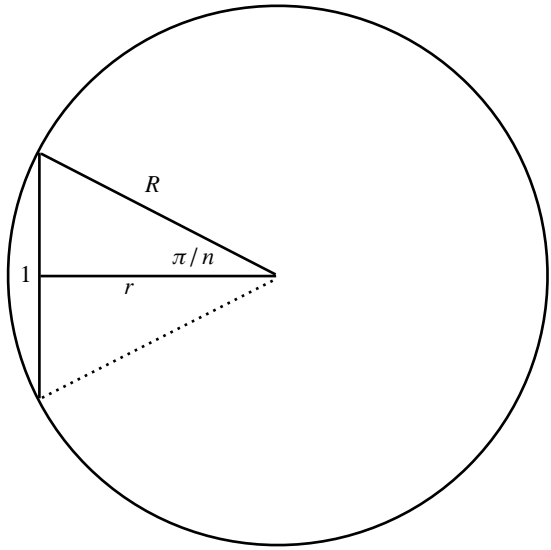


FIGURE 1

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