*Probability in the Engineering and Informational Sciences*, **33**, 2019, 205–219.
doi:10.1017/S0269964818000128

# SHARP BOUNDS FOR SURVIVAL PROBABILITY WHEN AGEING IS NOT MONOTONE

RUHUL ALI KHAN AND MURARI MITRA
*Department of Mathematics,*
*Indian Institute of Engineering Science and Technology, Shibpur,*
*P.O. - Botanic Garden, Howrah-711103, West Bengal, India*
*E-mail: ruhulali.khan@gmail.com*
*and*
*Department of Mathematics,*
*Indian Institute of Engineering Science and Technology, Shibpur,*
*P.O. - Botanic Garden, Howrah-711103, West Bengal, India*
*E-mails: murarimitra@yahoo.com, mmitra@math.iiests.ac.in*

We exploit a novel bounding argument to obtain sharp bounds for survival functions belonging to the Increasing initially then Decreasing Mean Residual Life (IDMRL) class introduced by Guess, Hollander and Proschan (1986) [8]. The bounds obtained are in terms of the mean, change point and pinnacle of the mean residual life function. The bounds for the monotonic ageing classes Decreasing Mean Residual Life (DMRL) and Increasing Mean Residual Life (IMRL) are obtained as special cases. Discussions on the bounds as well as two concrete illustrative examples are included.

**Keywords:** life distribution and survival function and sharp bounds and MRL function and IDMRL class and change point and pinnacle

## 1. INTRODUCTION

In reliability theory, the main motivation behind considering nonparametric classes of life distributions (such as Increasing Failure Rate (IFR) or DMRL or New Better than Used in Expectation (NBUE)) is the possibility of capturing natural ageing phenomena through certain intuitively appealing notions which are formalized by the properties of the distribution. The advantage of this approach is its wide scope of applicability since no definite parametric form of the distribution is being assumed thereby making the model less restrictive and much more robust. Thus reliability bounds in the context of various nonparametric families assume great relevance since they enable us to estimate survival probabilities even in the absence of precise knowledge about the distribution.

The topic has generated considerable interest among reliability theorists. The problem is to determine the sharpest possible upper and lower bounds on the survival function $\bar{F}(t)$ for a given $t$, assuming that the lifetime distribution is a member of a class (say, IFR or NBUE) with certain specifications (e.g., specified mean or quantile or higher moment). Results in this area abound in the literature. Upper and lower bounds for IFR and Increasing Failure

© Cambridge University Press 2018 0269-9648/18 $25.00 

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

Rate Average (IFRA) classes with a specified mean are available in the classic text by Barlow and Proschan (1981) [3], lower bounds on $\bar{F}(t)$ for the NBU and NBUE families can be found in Marshall and Proschan (1972) [16] while Haines and Singpurwalla (1974) [10] provide an upper bound for the New Worse than Used in Expectation (NWUE) class. Korzeniowski and Opawski (1976)[13] derived an upper bound of the reliability function for the NBU class. Klefsjö (1982) [12] has obtained both the upper and lower bounds of $\bar{F}(t)$ for the Harmonic New Better than Used in Expectation (HNBUE) class and its dual. Sengupta (1994) [22] has developed a unified approach to derive the upper and lower bounds (in terms of the mean) for IFR, Decreasing Failure Rate (DFR), IFRA, Decreasing Failure Rate Average (DFRA), NBU and NWU distributions. Sengupta and Das (2016) [23] have obtained sharp reliability bounds for DMRL and IMRL classes with a specified mean.

The ageing pattern exhibited in all the families discussed above is *monotonic*, i.e., the direction of ageing, so to say, remains *same* throughout the entire lifespan of the unit. However, it is worth noting that in many practical situations, the ageing profile happens to be *non-monotonic* where ageing is initially beneficial (the so-called *burn-in phase*) but after a certain period, the effect of age is adverse (the *wear-out phase*).

To model such ageing scenarios, Guess et al. (1986) [8], in a seminal paper, introduced for the first time the concept of the 'Increasing initially then Decreasing Mean Residual Life' (IDMRL) family of life distributions.

Definition 1: A life distribution $F$ with a finite first moment is called an *"Increasing then Decreasing Mean Residual Life"* distribution if there exists a *turning point* (or *change point*) $t_0 \geq 0$ such that

$$\begin{aligned} m_F(x) &\leq m_F(y) \quad \text{for } 0 \leq x \leq y < t_0, \\ m_F(x) &\geq m_F(y) \quad \text{for } t_0 \leq x \leq y, \end{aligned} \tag{1}$$

where

$$m_F(t) := E(X - t | X > t) = \begin{cases} \dfrac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(u)du & \text{if } \bar{F}(t) > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In this case, we write $F$ is IDMRL($t_0$) and we say that $t_0$ is a change point of the distribution in the IDMRL sense.

Examples of this type of non-monotonic ageing are work hardening of metals and cutting tools, employment time with a given company, infant mortality among human beings and lengths of political coalitions. The DMRL class of life distributions is a special case of the IDMRL family when $t_0 = 0$ and if $t_0 \to \infty$ then the limiting distribution becomes an IMRL distribution. The IDMRL class has partial overlap with the 'Bathtub Failure Rate' (BFR) class of life distributions [see Glaser (1980) [6], Rajarshi and Rajarshi (1988) [21]] and is contained in the 'New Worse then Better than Used in Expectation' (NWBUE) class [see Mitra and Basu (1994) [18], Lai and Xie (2006) [14]]. (Definitions, properties and interrelations between different ageing classes may be found in Barlow and Proschan (1981) [3], Gertsbakh (1989) [5], Lai and Xie (2006) [14] and Gupta, Zeng and Wu (2010) [9]). The estimation of the change point $t_0$ has been addressed by Mitra and Basu (1995)[19]. Anis (2012) [2] shows that the IDMRL class of life distributions is closed under weak convergence, investigates some of its closure properties and develops some related shock model theory. Tests of exponentiality against IDMRL alternatives have also received widespread attention with many researchers making significant contributions [see [1,8,11,15,20]].

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

A large volume of literature exists on reliability bounds for monotonic ageing classes but the corresponding problem for non–monotonic ageing families such as BFR, IDMRL and NWBUE is yet to be addressed. In this work, we focus on the derivation of upper and lower bounds on the reliability function in the context of IDMRL distributions. We exploit an idea in Sengupta (1994) [22] and our approach is to construct a tractable, parametric subclass over which the optimization would suffice. The construction of such a subclass is, as we shall see, far from obvious and involves considerable ingenuity.

In Section 2 we present certain preliminary results which are germane to our discussion and proceed to obtain an upper bound for the reliability function of an IDMRL distribution. Section 3 deals with the derivation of the corresponding lower bound. In Section 4, the separation between the bounds is studied graphically and the qualitative findings are reported. We also consider two specific examples where we plot the actual survival functions together with the upper and lower bounds for illustration.

## 2. AN UPPER BOUND ON IDMRL SURVIVAL FUNCTIONS

The reliability function $\bar{F}(t)$ can be expressed in terms of the corresponding MRL function by the relation

$$\bar{F}(t) = \frac{\mu}{m_F(t)} \exp\left[-\int_0^t \frac{du}{m_F(u)}\right]. \tag{3}$$

Cox (1962) [4] assigns as an exercise the demonstration that MRL determines the reliability. Meilijson (1972) [17] gives a simple and elegant proof of (3). It is clear from (2) and (3) that there is a one–one correspondence between survival functions and corresponding MRL functions. We now introduce the following important definition whose relevance will become clear immediately afterwards.

DEFINITION 2: The maximum value (if it exists) of a MRL function is called its *"pinnacle"*.

It is clear from the above definition that for an IDMRL distribution the *pinnacle* equals $m_F(t_0)$ where $t_0$ is a change point of the distribution in the IDMRL sense. Given two IDMRL distributions having the same mean $\mu$ and the same change point $t_0$, there may be a widespread difference between their MRL functions if there is a drastic gap between the maxima attained by the MRL functions. Consequently, since there is a one–one correspondence between survival functions and MRL functions, it stands to reason that a bound for the survival function would, of necessity, involve not only the change point of the distribution but also the pinnacle of the distribution.

LEMMA 1: *If $m_F(t)$ is the MRL function of a life distribution $F$, then $m'_F(t) \geq -1$.*

PROOF: This lemma is a consequence of the fact that $m_F(x) + x$ is a *non–decreasing* function [see Guess and Proschan (1985) [7]]. ■

Let $\mathcal{A}$ denote the class of all IDMRL reliability functions having mean $\mu$, change point $t_0$ and pinnacle $m_F(t_0)$, i.e.,

$$\mathcal{A} = \left\{ \bar{F} : m_F(t) \uparrow t \in [0, t_0)\,,\ m_F(t) \downarrow t \in (t_0, \infty)\,, \right. \\ \left. \int_0^\infty \bar{F}(u)du = \mu, \sup_t m_F(t) = m_F(t_0) \right\}. \tag{4}$$

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

Now we consider a parametric family of distributions with survival function defined by

$$\bar{F}_{c,b}(t) = \begin{cases} 1 & \text{for } t = 0 \\ \frac{\mu}{c} e^{-\frac{t}{c}} & \text{for } 0 < t < t_0 \\ \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{c} - \frac{t - t_0}{m_F(t_0)}} & \text{for } t_0 \le t \le b \\ \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{c} - \frac{b - t_0}{m_F(t_0)}} & \text{for } b < t \le b + m_F(t_0) \\ 0 & \text{for } t > b + m_F(t_0) \end{cases} \tag{5}$$

where $c \in [\mu, m_F(t_0)]$ and $b \ge t_0$.
Routine but tedious calculations yield the corresponding MRL function as

$$m_{c,b}(t) = \begin{cases} \mu & \text{for } t = 0 \\ c & \text{for } 0 < t < t_0 \\ m_F(t_0) & \text{for } t_0 \le t \le b \\ b + m_F(t_0) - t & \text{for } b < t \le b + m_F(t_0) \\ 0 & \text{for } t > b + m_F(t_0) \end{cases} \tag{6}$$

Observe that $\bar{F}_{c,b}(t)$ is a member of $\mathcal{A}$ $\forall$ $c \in [\mu, m_F(t_0)]$ and $\forall$ $b \ge t_0$. Now for a fixed $t \ge t_0$ choose $b \in [t_0 \vee (t - m_F(t_0)), t]$. Notice that this choice of $b$ is in the permissible range, i.e., $b \ge t_0$. Now we have the following lemma:

LEMMA 2: *Let $\bar{F}(t)$ be any member of $\mathcal{A}$ and $m_F(t)$ be the corresponding MRL function. Also let $\bar{F}_{c_0,b_0}(t)$ be as defined in (5) and $m_{c_0,b_0}(t)$ be the corresponding MRL function, where $b_0 = m_F(t) + t - m_F(t_0)$ and $c_0 = m_F(t)1_{[0,t_0)}(t) + m_F(t_0)1_{[t_0,\infty)}(t)$. Then for fixed $t \ge 0$,*
*(i) $m_{c_0,b_0}(t) = m_F(t)$ and (ii) $m_{c_0,b_0}(u) \ge m_F(u)$ $\forall u \in [0, t]$.*

PROOF: Now fix any $t \ge 0$,
**Case I:** Let $t \in [0, t_0)$.

Here $m_{c_0,b_0}(0) = \mu = m_F(0)$. It is easy to see that $c_0$ is a legitimate choice of $c$. For $t \in (0, t_0)$ using (6) and by the choice of $c_0$, we have

$$m_{c_0,b_0}(t) = c_0 = m_F(t).$$

$\forall$ $u \in (0, t]$, $m_{c_0,b_0}(u) = c_0 = m_F(t) \ge m_F(u)$ as $m_F(t)$ is non-decreasing on $[0, t_0)$. Thus

$$m_{c_0,b_0}(u) \ge m_F(u) \quad \forall u \in [0, t].$$

**Case II:** Let $t \in [t_0, \infty)$.

Note that $b_0 = m_F(t) + t - m_F(t_0) \le t$, since $m_F(t)$ is non-decreasing on $(t_0, \infty)$ i.e., $m_F(t) - m_F(t_0) \le 0$ and $b_0 = m_F(t) + t - m_F(t_0) \ge t - m_F(t_0)$ as $m_F(t) \ge 0$. As $m(x) + x$ is a non-decreasing function (see Guess and Proschan (1985) [7]), we have for $t > t_0$, $m_F(t_0) + t_0 \le m_F(t) + t$ which implies that $t_0 \le m_F(t) + t - m_F(t_0) = b_0$. So $b_0 \in [t_0 \vee (t - m_F(t_0)), t]$, thereby establishing that $b_0$ is a legitimate choice of $b$.

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

Now, $b_0 = m_F(t) + t - m_F(t_0) \implies b_0 + m_F(t_0) = m_F(t) + t \geq t$, as$m_F(t) \geq 0$. Hence, $b_0 \leq t \leq b_0 + m_F(t_0)$. So for fixed $t \in (t_0, \infty)$ the choice of $b_0$ is such that $t$ lies between $b_0$ and $b_0 + m_F(t_0)$. Thus,

$$m_{c_0,b_0}(t) = b_0 + m_F(t_0) - t = m_F(t) + t - m_F(t_0) + m_F(t_0) - t = m_F(t).$$

Now, $\forall\ u \in [0, t]$ we want to show that $m_{c_0,\,b_0}(u) \geq m_F(u)$. Now, $m_{c_0,\,b_0}(0) = \mu = m_F(0)$ and $\forall\, u \in (0,\, t_0)$, $m_{c_0,\,b_0}(u) = c_0 = m_F(t_0) \geq m_F(u)$ as $m_F(t_0)$ is the pinnacle and $\forall\, u \in [t_0,\, b_0]$, $m_{c_0,\,b_0}(u) = m_F(t_0) \geq m_F(u)$ since $m_F(t_0)$ is the pinnacle and now $\forall\, u \in [b_0,\, t]$,

$$\begin{aligned} m_{c_0,b_0}(u) &= b_0 + m_F(t_0) - u \\ &= m_F(t) + t - u \text{ [using the choice of } b_0] \\ &= m_F(u) + \int_u^t m'(v)dv + t - u \\ &\geq m_F(u) + \int_u^t (-1)dv + t - u \\ &= m_F(u). \end{aligned}$$

So, we have shown that for fixed $t \geq 0$, $m_{c_0,\,b_0}(t) = m_F(t)$ and $m_{c_0,\,b_0}(u) \geq m_F(u)\ \forall\, u \in [0,\, t]$. ■

The following theorem shows that the problem of maximizing $\bar{F}(t)$ over $\mathcal{A}$ is equivalent to the corresponding restricted maximization problem over the subclass of survival functions of the form $\bar{F}_{c,b}(t)$.

THEOREM 1: *Let $\mathcal{A}$ and $\bar{F}_{c,b}(t)$ be as defined in (4) and (5) respectively. Then for each fixed $t \geq 0$,*

$$\sup_{\bar{F}(t)\in\mathcal{A}} \bar{F}(t) \quad = \sup_{\substack{c\in[\mu, m_F(t_0)] \\ b\in[t_0 \vee (t - m_F(t_0)),\, t]}} \bar{F}_{c,b}(t).$$

PROOF: Let $\bar{F}(t)$ be any member of $\mathcal{A}$ and $m_F(t)$ be the corresponding MRL function. We shall show that for fixed $t \geq 0$, $\bar{F}(t) \leq \bar{F}_{c_0,\,b_0}(t)$ for $c_0 = m_F(t)1_{[0,t_0)}(t) + m_F(t_0)1_{[t_0,\,\infty)}(t)$ and $b_0 = m_F(t) + t - m_F(t_0)$. Now, by Lemma 2, we have

$$\begin{aligned} \bar{F}(t) &= \frac{\mu}{m_F(t)} \exp\left[-\int_0^t \frac{du}{m_F(u)}\right] \\ &= \frac{\mu}{m_{c_0,b_0}(t)} \exp\left[-\int_0^t \frac{du}{m_F(u)}\right] \\ &\leq \frac{\mu}{m_{c_0,b_0}(t)} \exp\left[-\int_0^t \frac{du}{m_{c_0,b_0}(u)}\right] \\ &= \bar{F}_{c_0,b_0}(t). \end{aligned}$$

Thus we have shown that for every $\bar{F} \in \mathcal{A}$, there exists $c_0 \in [\mu,\, m_F(t_0)]$ and $b_0 \in [t_0 \vee (t - m_F(t_0)),\, t]$ such that $\bar{F}(t) \leq \bar{F}_{c_0,\,b_0}(t)$. Therefore we conclude that,

$$\sup_{\bar{F}(t)\in\mathcal{A}} \bar{F}(t) \leq \sup_{\substack{c\in[\mu, m_F(t_0)] \\ b\in[t_0 \vee (t - m_F(t_0)),\, t]}} \bar{F}_{c,b}(t).$$

Since $\{\bar{F}_{c_0,\,b_0}(t)\} \subseteq \mathcal{A}$, we can say that the reverse inequality also holds. This completes the proof. ■

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

THEOREM 2: *Let $\bar{F}$ be an IDMRL survival function having change point $t_0$ and fixed mean $\mu$ and fixed pinnacle $m_F(t_0)$. Then*

$$\sup_{\bar{F}(t)\in\mathcal{A}} \bar{F}(t) = \begin{cases} e^{-\frac{t}{\mu}} & \text{for } 0 \le t \le \mu \le min(t_0, m_F(t_0)) \\ \frac{\mu}{t}e^{-1} & \text{for } 0 < \mu \le t \le min(t_0, m_F(t_0)) \\ e^{-\frac{t}{\mu}} & \text{for } 0 \le t \le t_0 < \mu \le m_F(t_0) \\ \frac{\mu}{m_F(t_0)} e^{-\frac{t}{m_F(t_0)}} & \text{for } 0 < \mu \le m_F(t_0) < t < t_0 \\ \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{m_F(t_0)}} & \text{for } 0 \le t_0 < t < t_0 + m_F(t_0) \\ \frac{\mu}{m_F(t_0)} e^{1-\frac{t}{m_F(t_0)}} & \text{for } t \ge t_0 + m_F(t_0) \end{cases} \tag{7}$$

PROOF: By virtue of Theorem 1,

$$\sup_{\bar{F}(t)\in\mathcal{A}} \bar{F}(t) = \sup_{\substack{c\in[\mu, m_F(t_0)] \\ b\in[t_0 \vee (t-m_F(t_0)), t]}} \bar{F}_{c,b}(t).$$

Consequently, it is sufficient to calculate the latter supremum. Let $f(c) = \frac{\mu}{c}e^{-\frac{t}{c}}$ ($\mu$, $t$ fixed). Then $f'(c) = \frac{\mu}{c^2}e^{-\frac{t}{c}}(\frac{t}{c} - 1)$ and so $f(c)$ is *increasing* when $c \le t$ and *decreasing* when $c \ge t$. Recall that the permissible range of $c$ is between $\mu$ and $m_F(t_0)$ i.e. $\mu \le c \le m_F(t_0)$.

Thus, three cases may arise for each $t \in (0, t_0]$.

**Case a: $\mu \le t_0 \le m_F(t_0)$.**

**Subcase a(i): $0 \le t \le \mu \le t_0 \le m_F(t_0)$.**

Here, clearly $c \ge t$ as $c \ge \mu$. When $c \ge t$, $\frac{\mu}{c}e^{-\frac{t}{c}}$ decreasing in $c$, so the maximum is at the least possible value of $c$ i.e., at $c = \mu$. So, $\sup_{c\in[\mu, m_F(t_0)]} \frac{\mu}{c}e^{-\frac{t}{c}} = e^{-\frac{t}{\mu}}$.

**Subcase a(ii): $0 < \mu \le t \le t_0 \le m_F(t_0)$.**

Here $\mu \le c < t$ or $c \ge t$. In any case, sup is at $c = t$. So, $\sup_{c\in[\mu, m_F(t_0)]} \frac{\mu}{c}e^{-\frac{t}{c}} = \frac{\mu}{t}e^{-1}$.

**Case b: $t_0 \le \mu \le m_F(t_0)$.**

So, $t < t_0 \le \mu \le c \le m_F(t_0)$. Here also $c \ge t$ and hence, arguing as in Subcase a(i), we have $\sup_{c\in[\mu, m_F(t_0)]} \frac{\mu}{c}e^{-\frac{t}{c}} = e^{-\frac{t}{\mu}}$.

**Case c: $\mu \le m_F(t_0) \le t_0$.**

**Subcase c(i): $0 < t \le \mu \le m_F(t_0) \le t_0$.**

Here, $c \ge t$ and so once again arguing as in Subcase a(i), we get $\sup_{c\in[\mu, m_F(t_0)]} \frac{\mu}{c}e^{-\frac{t}{c}} = e^{-\frac{t}{\mu}}$.

**Subcase c(ii): $0 < \mu < t \le m_F(t_0) \le t_0$.**

Then $c \le t$ or $c > t$ with $c \in [\mu, m_F(t_0)]$. Now arguing as in Subcase a(ii), we get $\sup_{c\in[\mu, m_F(t_0)]} \frac{\mu}{c}e^{-\frac{t}{c}} = \frac{\mu}{t}e^{-1}$.

**Subcase c(iii): $0 < \mu \le m_F(t_0) \le t \le t_0$.**

Here, $c \le t$. So, $\frac{\mu}{c}e^{-\frac{t}{c}}$ increasing in $c$, so the supremum is attained at $c = m_F(t_0)$. So, $\sup_{c\in[\mu, m_F(t_0)]} \frac{\mu}{c}e^{-\frac{t}{c}} = \frac{\mu}{m_F(t_0)}e^{-\frac{t}{m_F(t_0)}}$.

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

For $t \in (t_0, \infty)$, the case $t_0 < t \leq b$ does not arise as $b \in [t_0 \vee (t - m_F(t_0)), t]$. Now

$$\bar{F}_{c,b}(t) = = \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{c} - \frac{b-t_0}{m_F(t_0)}}$$
$$= \frac{\mu}{m_F(t_0)} e^{t_0\left(\frac{1}{m_F(t_0)} - \frac{1}{c}\right)} . e^{-\frac{b}{m_F(t_0)}} \tag{8}$$

which is maximum when both of $e^{t_0\left(\frac{1}{m_F(t_0)} - \frac{1}{c}\right)}$ and $e^{-\frac{b}{m_F(t_0)}}$ attain their respective maxima. The maximum value of $e^{t_0\left(\frac{1}{m_F(t_0)} - \frac{1}{c}\right)}$ over $c$ is 1 for the choice of $c = m_F(t_0)$. Now $e^{-\frac{b}{m_F(t_0)}}$ is maximum when $b$ is minimum. Here, two cases arise.

**Case I: $t - m_F(t_0) < t_0$ i.e. $t \leq t_0 + m_F(t_0)$.**

Then $b \in [t_0, t]$. In this case, the choice is $b = t_0$. So, $\sup_{b \in [t_0, t]} e^{-\frac{b}{m_F(t_0)}} = e^{-\frac{t_0}{m_F(t_0)}}$.

Hence, for $t_0 \leq t \leq t_0 + m_F(t_0)$

$$\sup_{\substack{c \in [\mu, m_F(t_0)] \\ b \in [t_0 \vee (t - m_F(t_0)), t]}} \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{c} - \frac{b-t_0}{m_F(t_0)}} = \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{m_F(t_0)}}.$$

**Case II: $t - m_F(t_0) \geq t_0$ i.e. $t \geq t_0 + m_F(t_0)$.**

Then $b \in [t - m_F(t_0), t]$. In this case, the choice is $b = t - m_F(t_0)$. So,

$$\sup_{b \in [t - m_F(t_0), t]} e^{-\frac{b}{m_F(t_0)}} = e^{1 - \frac{t}{m_F(t_0)}}.$$

Hence, $t_0 < t \leq t_0 + \mu_F(t_0)$,

$$\sup_{\substack{c \in [\mu, m_F(t_0)] \\ b \in [t_0 \vee (t - m_F(t_0)), t]}} \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{c} - \frac{b-t_0}{m_F(t_0)}} = \frac{\mu}{m_F(t_0)} e^{1 - \frac{t}{m_F(t_0)}}. \qquad \blacksquare$$

*Remark 1*: For the special case $t_0 = 0$, the IDMRL family reduces to DMRL. Putting $t_0 = 0$ in (7), we get $m_F(t_0) = m_F(0) = \mu$ and here (7) reduces to

$$\sup_{a \in [0 \vee (t-\mu), t]} \bar{F}(t) = \begin{cases} 1 & \text{for } t \leq \mu, \\ e^{1 - \frac{t}{\mu}} & \text{for } t > \mu. \end{cases}$$

which is just the upper bound for DMRL distributions obtained in Sengupta and Das (2016) [23].

*Remark 2*: For the special case $t_0 \to \infty$ the IDMRL family reduces to IMRL. Then (7) reduces to

$$\sup_{c \geq \mu} \bar{F}(t) = \begin{cases} e^{-\frac{t}{\mu}} & \text{for } t \leq \mu, \\ \frac{\mu}{t} e^{-1} & \text{for } t > \mu. \end{cases}$$

which is once again the upper bound for IMRL distributions obtained in Sengupta and Das (2016) [23].

*Remark 3*: Now, we have $\mu \leq m_F(t) \leq m_F(t_0)$ for $0 \leq t \leq t_0$ and $\forall\ u \in [0, t_0]$, $m_F(u) \leq m_F(t_0)$. So, $-\int_0^t \frac{1}{m_F(u)} du \leq -\frac{t}{m_F(t_0)}$ for $0 \leq t < t_0$.

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

Hence, for $0 \le t \le t_0$,

$$\begin{aligned}\bar{F}(t) &= \frac{\mu}{m_F(t)} e^{-\int_0^t \frac{du}{m_F(u)}} \\ &\le \frac{\mu}{\mu} e^{-\int_0^t \frac{du}{m_F(u)}} \text{ as } \mu \le m_F(t) \text{ for } 0 \le t \le t_0 \\ &= e^{-\int_0^t \frac{du}{m_F(u)}} \\ &\le e^{-\frac{t}{m_F(t_0)}}.\end{aligned}$$

This furnishes an upper bound on $\bar{F}(t)$ for $0 \le t \le t_0$ but the upper bounds obtained in (7) are much sharper as is to be expected in the presence of more precise knowledge.

## 3. A LOWER BOUND ON IDMRL SURVIVAL FUNCTION

Now, we consider a parametric family of life distributions with survival functions defined by

$$\bar{F}_a(t) = \begin{cases} e^{-\frac{t}{\mu}} & \text{for } 0 \le t < t_0 \\ \dfrac{\mu}{m_F(t_0)} e^{-\frac{t_0}{\mu}} & \text{for } t_0 \le t < t_0 + a \\ \dfrac{\mu}{m_F(t_0)} e^{-\frac{t_0}{\mu} - \frac{(t-t_0)-a}{m_F(t_0)-a}} & \text{for } t \ge t_0 + a \end{cases} \tag{9}$$

where $a \in [0,\, m_F(t_0)]$.
Routine but tedious calculations yield the associated MRL function as

$$m_a(t) = \begin{cases} \mu & \text{for } 0 \le t < t_0 \\ m_F(t_0) + t_0 - t & \text{for } t_0 \le t < t_0 + a \\ m_F(t_0) - a & \text{for } t \ge t_0 + a \end{cases} \tag{10}$$

Therefore, $\bar{F}_a(t)$ is a member of $\mathcal{A}$ for all $a \in [0,\, m_F(t_0)]$. Now, consider a fixed $t \ge t_0$ and choose $a \in [0,\, (t - t_0) \wedge m_F(t_0)]$. Note that this choice of $a$ is in the permissible range. We now have the following lemma:

LEMMA 3: *Let $\bar{F}(t)$ be any member of $\mathcal{A}$ and $m_F(t)$ be the corresponding MRL function. Also let $\bar{F}_{a_0}(t)$ be as defined in (9) and $m_{a_0}(t)$ be the corresponding MRL function, where $a_0 = m_F(t_0) - m_F(t)$.*
*Then for fixed $t \ge t_0$,*
*(i) $m_{a_0}(t) = m_F(t)$ and (ii) $m_{a_0}(u) \le m_F(u)\ \forall\, u \in [0,\, t]$.*

PROOF: (i) Now, fix any $t \ge t_0$.
Observe that $a_0 = m_F(t_0) - m_F(t) \ge 0$. and $a_0 = m_F(t_0) - m_F(t) \le m_F(t_0)$
Now,

$$\begin{aligned} m_F(t) &= m_F(t_0) + \int_{t_0}^{t} m'_F(v)dv \\ &\ge m_F(t_0) + \int_{t_0}^{t} (-1)dv \\ &= m_F(t_0) - t + t_0. \end{aligned}$$

Hence, $t - t_0 \ge m_F(t_0) - m_F(t) = a_0$. So $a_0 \in [0,\, (t - t_0) \wedge m_F(t_0)]$, thereby establishing that $a_0$ is a legitimate choice of $a$.

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

Now, from $t - t_0 \geq a_0$, we get $t \geq t_0 + a_0$. So for a fixed value of $t > t_0$, $a_0$ also takes a fixed value in the permissible range such that $t$ will always be greater than or equal to $a_0 + t_0$. So, as $t \geq a_0 + t_0$, from (10), we have,

$$\begin{aligned} m_{a_0}(t) &= m_F(t_0) - a_0 \\ &= m_F(t_0) - m_F(t_0) + m_F(t) \\ &= m_F(t). \end{aligned}$$

(ii) Now, $\forall\, u \in [0,\, t_0]$, $m_{a_0}(u) = \mu = m_F(0) \leq m_F(u)$ as $m_F(u)$ is non-decreasing on $[0,\, t_0]$. Again $\forall\, u \in [t_0,\, t_0 + a_0]$,

$$\begin{aligned} m_F(u) &= m_F(t_0) + \int_{t_0}^{u} m'_F(v)dv \\ &\geq m_F(t_0) + \int_{t_0}^{u} (-1)dv \\ &= m_F(t_0) + t_0 - u \\ &= m_{a_0}(u). \end{aligned}$$

Now, $\forall\, u \in [t_0 + a_0,\, t]$,

$$\begin{aligned} m_{a_0}(u) &= m_F(t_0) - a_0 \\ &= m_F(t_0) - m_F(t_0) + m_F(t) \\ &= m_F(t) \\ &\leq m_F(u) \text{ as } m_F(u) \text{ is non-increasing on } [t_0, \infty). \end{aligned}$$

So, we have shown that for fixed $t \geq t_0$, $m_{a_0}(t) = m_F(t)$ and $m_{a_0}(u) \leq m_F(u)\ \forall u \in [0,\, t]$. ■

The following theorem shows that the problem of minimizing $\bar{F}(t)$ over $\mathcal{A}$ is equivalent to the corresponding restricted minimization problem over the subclass of survival functions of the form $\bar{F}_a(t)$.

THEOREM 3: *Let $\mathcal{A}$ and $\bar{F}_a(t)$ be as defined in (4) and (9) respectively. Then for each fixed $t \geq t_0$,*

$$\inf_{\bar{F}(t) \in \mathcal{A}} \bar{F}(t) = \inf_{a \in [0, (t - t_0) \wedge m_F(t_0)]} \bar{F}_a(t).$$

PROOF: Let $\bar{F}(t)$ be any member of $\mathcal{A}$ and $m_F(t)$ be the corresponding MRL function. We shall show that for fixed $t \geq t_0$, $\bar{F}(t) \geq \bar{F}_a(t)$ for $a_0 = m_F(t_0) - m_F(t)$. Let $m_{a_0}(t)$ be the corresponding MRL function of $\bar{F}_{a_0}(t)$. Now by Lemma 3,

$$\begin{aligned} \bar{F}(t) &= \frac{\mu}{m_F(t)} \exp\left[-\int_0^t \frac{du}{m_F(u)}\right] \\ &= \frac{\mu}{m_{a_0}(t)} \exp\left[-\int_0^t \frac{du}{m_F(u)}\right] \\ &\geq \frac{\mu}{m_{a_0}(t)} \exp\left[-\int_0^t \frac{du}{m_{a_0}(u)}\right] \\ &= \bar{F}_{a_0}(t). \end{aligned}$$

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

Thus we have shown that for a fixed $t \geq t_0$ and $\bar{F} \in \mathcal{A}$, $\exists a_0 \in [0, (t-t_0) \wedge m_F(t_0)]$ such that $\bar{F}(t) \geq \bar{F}_{a_0}(t)$. Then for each $t \geq t_0$,

$$\inf_{\bar{F}(t)\in\mathcal{A}} \bar{F}(t) \geq \inf_{a\in[0,(t-t_0)\wedge m_F(t_0)]} \bar{F}_a(t).$$

Since $\{\bar{F}_{a_0}(t)\} \subseteq \mathcal{A}$, we can say that the reverse inequality also holds. This completes the proof. ■

THEOREM 4: *Let $F$ be an IDMRL reliability function having change point $t_0$ and fixed mean $\mu$ and fixed pinnacle $m_F(t_0)$. Then for each $t \geq 0$,*

$$\inf_{\bar{F}(t)\in\mathcal{A}} \bar{F}(t)) = \begin{cases} \frac{\mu}{m_F(t_0)} e^{-\frac{t}{\mu}} & \text{for } 0 \leq t < t_0 \\ \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{\mu} - \frac{(t-t_0)}{m_F(t_0)}} & \text{for } t_0 \leq t < t_0 + m_F(t_0) \\ 0 & \text{fort} \geq t_0 + m_F(t_0) \end{cases} \tag{11}$$

PROOF: Now, for $0 \leq t < t_0$, $\mu \leq m_F(t) \leq m_F(t_0)$. So, $\frac{\mu}{m_F(t)} \geq \frac{\mu}{m_F(t_0}$ for $t \leq t_0$.

Again, $\mu \leq m_F(u)\ \forall 0 \leq u < t_0$, and so $e^{-\int_0^t \frac{1}{m_F(u)}} \geq e^{-\frac{t}{\mu}}\ \forall\ 0 \leq t < t_0$. Hence, for $0 \leq t < t_0$,

$$\bar{F}(t) = \frac{\mu}{m_F(t)} e^{-\int_0^t \frac{du}{m_F(u)}} \geq \frac{\mu}{m_F(t_0)} e^{-\frac{t}{\mu}}. \tag{12}$$

Now, by the virtue of Theorem 3, for each fixed $t \geq t_0$,

$$\inf_{\bar{F}(t)\in\mathcal{A}} \bar{F}(t) = \inf_{a\in[0,(t-t_0)\wedge m_F(t_0)]} \bar{F}_a(t).$$

Consequently, it suffices to calculate the latter infimum. Now $\bar{F}_a(t) = \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{\mu}}$ for $t_0 \leq t < t_0 + a$ does not arise as $a \in [0, (t-t_0) \wedge m_F(t_0)]$.

Note that $\frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{\mu} - \frac{(t-t_0)-a}{m_F(t_0)-a}}$ will be minimized over $a \in [0, (t-t_0) \wedge m_F(t_0)]$ when $\frac{(t-t_0)-a}{m_F(t_0)-a}$ is maximized over $a \in [0, (t-t_0) \wedge m_F(t_0)]$.

For $a_1, a_2 \in [0, (t-t_0) \wedge m_F(t_0)]$, let

$$\frac{(t-t_0)-a_1}{m_F(t_0)-a_1} \leq \frac{(t-t_0)-a_2}{m_F(t_0)-a_2}$$

$$\text{i.e., } a_2[m_F(t_0)+t_0-t] \leq a_1[m_F(t_0)+t_0-t]. \tag{13}$$

**Case 1: $t - t_0 \leq m_F(t_0)$** i.e., **$t \leq t_0 + m_F(t_0)$**.

Then from (13), we get that $\frac{(t-t_0)-a}{m_F(t_0)-a}$ is a *decreasing* function of $a$ and hence will be maximum when $a$ is minimum when $a = 0$ since $a \in [0, (t-t_0) \wedge m_F(t_0)] = [0, (t-t_0)]$.

Hence, for $t_0 \leq t < t_0 + m_F(t_0)$,

$$\inf_{a_0\in[0,(t-t_0)\wedge m_F(t_0)]} \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{\mu} - \frac{(t-t_0)-a}{m_F(t_0)-a}} = \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{\mu} - \frac{(t-t_0)}{m_F(t_0)}}.$$

**Case 2: $t - t_0 \geq m_F(t_0)$** i.e., **$t \geq t_0 + m_F(t_0)$**.

Then from (13), we get that $\frac{(t-t_0)-a}{m_F(t_0)-a}$ is a *increasing* function of $a$. Note that in this

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

case $[0,\ (t-t_0) \wedge m_F(t_0)] = [0,\ m_F(t_0))]$, so that $\frac{(t-t_0)-a}{m_F(t_0)-a} \to +\infty$ as $a \to m_F(t_0)$. Hence, for $t \geq t_0 + m_F(t_0)$,

$$\inf_{a_0 \in [0,(t-t_0)\wedge m_F(t_0)]} \frac{\mu}{m_F(t_0)} e^{-\frac{t_0}{\mu} - \frac{(t-t_0)-a}{m_F(t_0)-a}} = 0. \qquad \blacksquare$$

*Remark 4*: For the special case, $t_0 = 0$, the IDMRL family reduces to DMRL. Putting $t_0 = 0$ in (11), we get $m_F(t_0) = m_F(0) = \mu$ and here (11) reduces to

$$\inf_{a \in [0 \vee (t-\mu), t]} \bar{F}(t) = \begin{cases} e^{-\frac{t}{\mu}} & \text{for } t \leq \mu, \\ 0 & \text{for } t > \mu. \end{cases}$$

which is just the lower bound for DMRL distributions obtained in Sengupta and Das (2016) [23].

*Remark 5*: The lower bound for an IMRL survival function with mean $\mu$ is 0. This is because the lower bound for DFR distributions is 0 and the DFR class is contained in the IMRL class [see Klefsjö (1982) [12]].

## 4. ILLUSTRATIVE EXAMPLES

We study the separation between the upper and lower bounds graphically. We set $x := t/\mu$, $x_0 := t_0/\mu$ and $m := m_F(t_0)/\mu$ in the expressions for our bounds. The survival probability bounds are then plotted against $x$ in Fig. 1–4 for certain choices of $x_0$ and $m$. A quick look at the figures reveals that the separation between the bounds reduce as the change

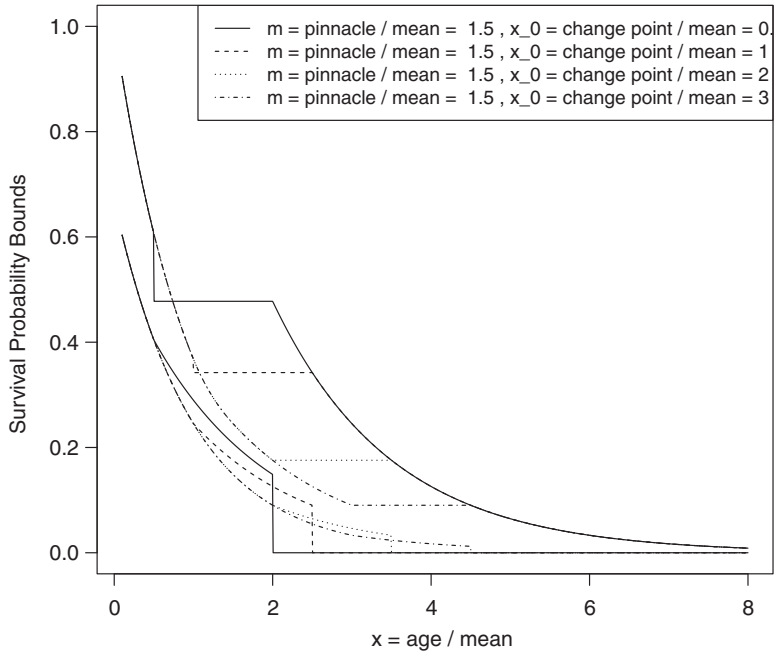



FIGURE 1. Plots of survival probability bounds for $m = 1.5$.

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

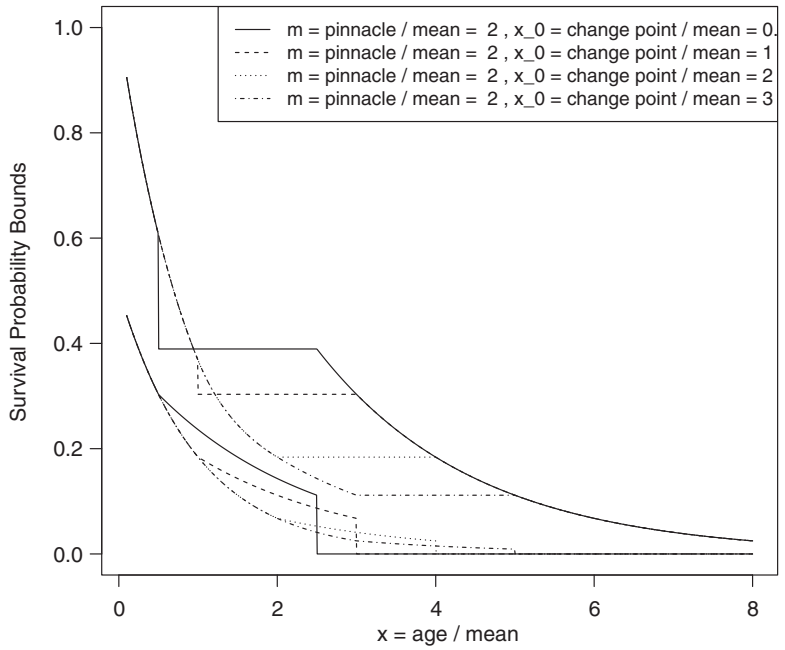



FIGURE **2.** Plots of survival probability bounds for $m = 2$.

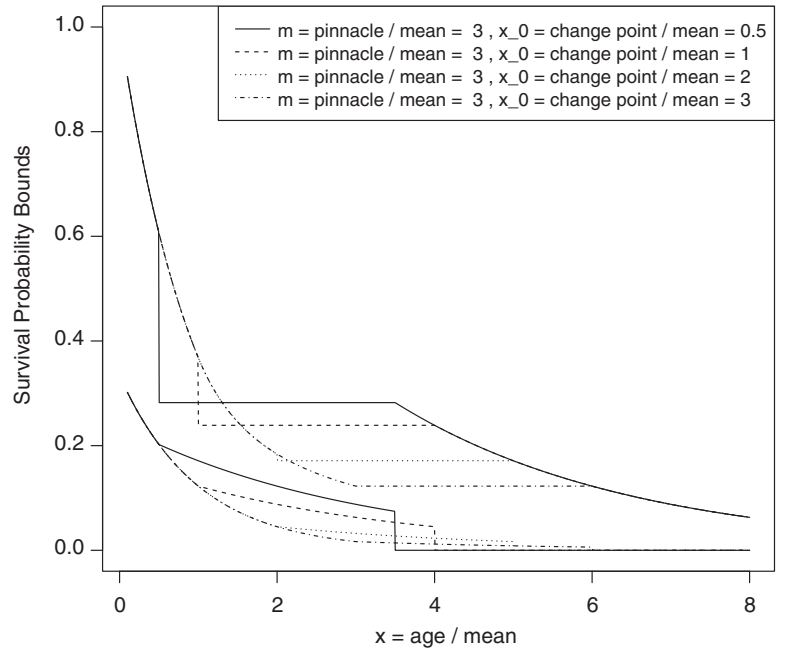



FIGURE **3.** Plots of survival probability bounds for $m = 3$.

point shifts to the right. The graphs also seem to indicate that the gap between bounds increase as the pinnacle reduces.

We next look at two concrete examples of IDMRL distributions and in each case, display plots of the actual survival function along with the lower and upper bounds in the same graph. We consider the two distributions mentioned in Example 4.1. of Anis (2012) [2].

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

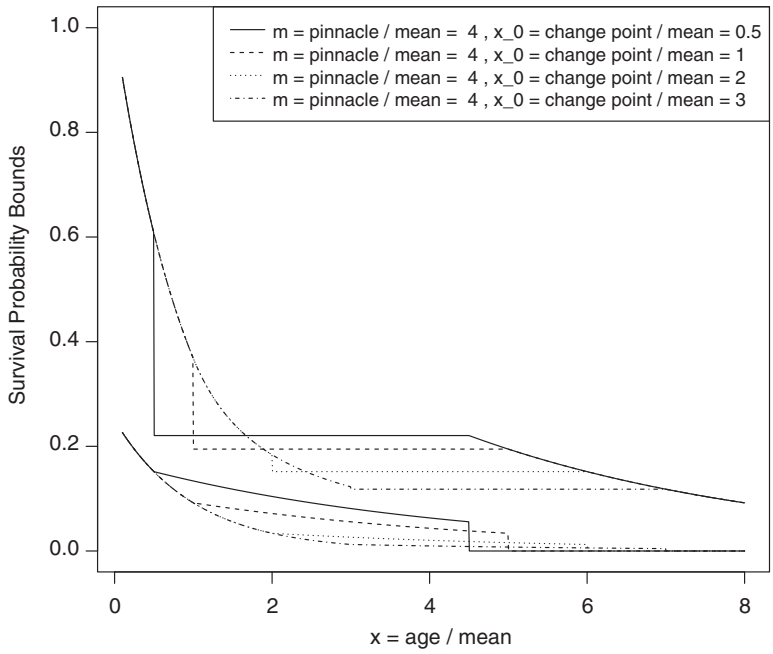



FIGURE **4.** Plots of survival probability bounds for $m = 4$.

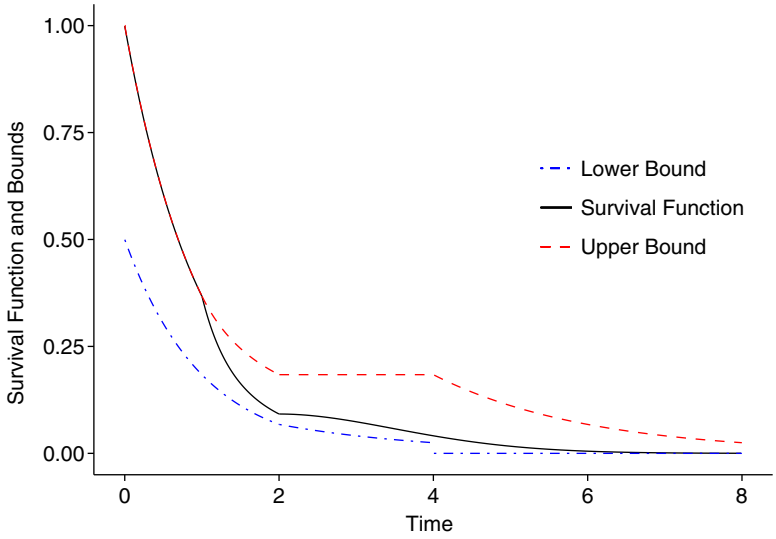



FIGURE **5.** Plots of survival function and bounds for Example 1.

*Example 1*: The survival function given by

$$\bar{F}(t) = \begin{cases} \exp(-t) & \text{for } 0 \le t < 1 \\ \dfrac{1}{t^2}\exp(-1) & \text{for } 1 \le t < 2 \\ \dfrac{t}{8}\exp\left[-\left(\dfrac{t^2}{8}+\dfrac{1}{2}\right)\right] & \text{for } t \ge 2 \end{cases} \tag{14}$$

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

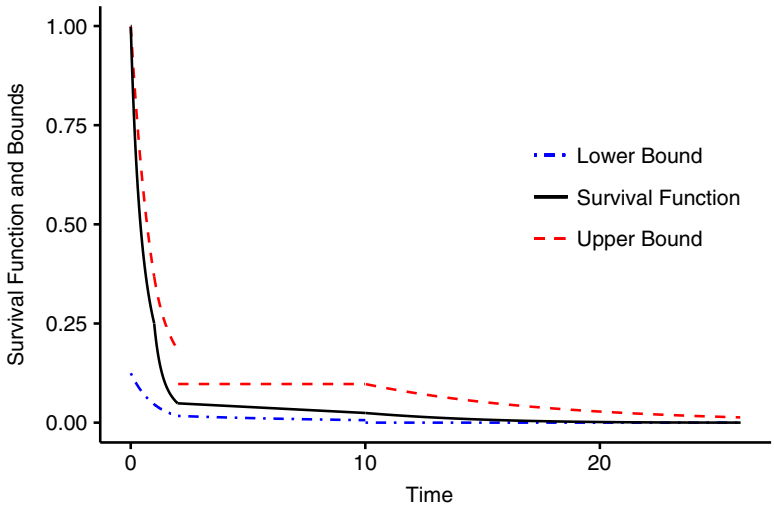



FIGURE **6.** Plots of survival function and bounds for Example 2.

is easily seen to be IDMRL having mean $\mu = 1$, change point $t_0 = 2$ and pinnacle $m_F(t_0) = 2$. Figure 5 shows the graphs of the true survival function and the corresponding upper and lower bounds as obtained in Theorems 2 and 4, respectively. It is interesting to note that in this example the upper bound coincides with the true survival function in the interval $[0, 1]$.

*Example 2*: Consider the survival function

$$\bar{G}(t) = \begin{cases} \dfrac{1}{(1+t)^2} & \text{for } 0 \le t < 1 \\ \dfrac{1}{4t^2}\exp\left[\dfrac{1}{2}\left(\dfrac{1}{t}-1\right)\right] & \text{for } 1 \le t < 2 \\ \dfrac{(18-t)\exp(-1/4)}{2^8} & \text{for } 2 \le t < 10 \\ \dfrac{\exp(-1/4)}{2^{13}}\left(6-\dfrac{t}{5}\right)^4 & \text{for } 10 \le t < 25 \\ \dfrac{\exp(3/4)}{2^{13}}\exp\left(t-25-\exp\left(t-25\right)\right) & \text{for } t \ge 25. \end{cases} \tag{15}$$

Note that $G$ is IDMRL having mean $\mu = 1$, change point $t_0 = 2$ and pinnacle $m_F(t_0) = 8$. A plot analogous to Figure 5 for this example is given in Figure 6.

## ACKNOWLEDGMENT

We are grateful to the anonymous referee for his valuable comments which have substantially improved the presentation of the paper.

### References

1. Aly, E.E.A.A. (1990). Tests for monotonicity properties of the mean residual life function. *Scandinavian Journal of Statistics* 17: 189–200.

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press

2. Anis, M.Z. (2012). On some properties of the IDMRL class of life distributions. *Journal of Statistical Planning and Inference* 142: 3047–3055.
3. Barlow, Richard E. & Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing: Probability Models*. To Begin With, MD, USA: Silver Spring.
4. Cox, D.R. (1962). *Renewal Theory*. London: Methuen.
5. Gertsbakh, I.B. (1989). *Statistical Reliability Theory*. New York: Marcel Dekker.
6. Glaser, R.E. (1980). Bathtub and related failure rate characterizations. *Journal of the American Statistical Association* 75: 667–672.
7. Guess, F. & Proschan, F. (1985) Mean Residual Life: Theory and Applications. FSU Statistics Report M4702, AFOSR Technical Report No. 85–178.
8. Guess, F., Hollander, M. & Proschan, F. (1986). Testing exponentiality versus a trend change in mean residual life. *The Annals of Statistics* 14(4): 1388–1398.
9. Gupta, A.K., Zeng, W.B. & Wu, Y. (2010). *Probability and Statistical Models: Foundations for Problems in Reliability and Financial Mathematics*. New York: Springer.
10. Haines, A.L. & Singpurwalla, N.D. (1974). Some contributions to the stochastic characterization of wear. In Proschan, F., Serfling, R.J. (eds.), *Reliability and Biometry. Society for Industrial and Applied Mathematics*. Philadelphia, pp. 47–80.
11. Hawkins, D.L., Kochar, S. & Loader, C. (1992). Testing exponentiality against IDMRL distributions with unknown change point. *The Annals of Statistics* 20: 280–290.
12. Klefsjö, B. (1982). The HNBUE and HNWUE classes of life distributions. *Naval Research Logistics Quarterly* 29(2): 331–345.
13. Korzeniowski, A. & Opawski, A. (1976). Bounds for reliability in the NBU, NWU, NBUE and NWUE classes. *Zastosowanià Matematyki Applicationes Mathematicae* 15: 1–5.
14. Lai, C.D. & Xie, M. (2006). *Stochastic Ageing and Dependence for Reliability*. New York: Springer.
15. Lim, J.H. & Park, D.H. (1998). A family of tests for trend change in mean residual life. *Communications in Statistics–Theory and Methods* 27: 1163–1179.
16. Marshall, A.W. & Proschan, F. (1972). Classes of distributions applicable in replacement with renewal theory implications. In Cam, Le (eds.), *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*. Vol. 11, pp. 395–415.
17. Meilijson, I. (1972). Limiting properties for the mean residual lifetime function. *The Annals of Statistics* 1: 354–357.
18. Mitra, M. & Basu, S.K. (1994). On a nonparametric family of life distributions and its dual. *Journal of Statistical Planning and Inference* 39: 385–397.
19. Mitra, M. & Basu, S.K. (1995). Change point estimation in non-monotonic aging models. *Annals of the Institute of Statistical Mathematics* 47: 483–491.
20. Na, M.H. & Lee, S. (2003). A family of IDMRL tests with unknown turning point. *Statistics* 37: 457–462.
21. Rajarshi, M.B. & Rajarshi, S.M. (1988). Bathtub distributions: A review. *Communications in Statistics–Theory and Methods* 17: 2597–2622.
22. Sengupta, D. (1994). Another look at the moment bounds on reliability. *Journal of Applied Probability* 31: 777–787.
23. Sengupta, D. & Das, S. (2016). Sharp bounds on DMRL and IMRL classes of life distributions with specified mean. *Statistics and Probability Letters* 119: 101–107.

https://doi.org/10.1017/S0269964818000128 Published online by Cambridge University Press