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DIRICHLET IMPROVABILITY FOR *S*-NUMBERS

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Abstract

We study the problem of improving Dirichlet's theorem of metric Diophantine approximation in the most general and multiplicative form in the *S*-adic setting. Our approach is based on translation of the problem related to Dirichlet improvability into a dynamical one, and the main technique of our proof is the *S*-adic version of quantitative nondivergence estimate due to Kleinbock and Tomanov. The main result of this paper can be regarded as the number field version of earlier works of Kleinbock and Weiss ['Dirichlet's theorem on Diophantine approximation and homogeneous flows', *J. Mod. Dyn.* **4** (2008), 43–62], and of the second named author and Ghosh ['Dirichlet's theorem in function fields', *Canad. J. Math.* **69**(3) (2017), 532–547]. Also, this in turn generalizes a result of Datta and Radhika ['Singular Vectors on Manifolds over Number Fields', *Monatsh. Math.* **200**(3) (2023), 545–568] on singularity of vectors to any number field K and S containing all Archimedian places.

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1. Introduction

In this paper, we study the improvement of Dirichlet's theorem of Diophantine approximation in the *S*-adic setting. The branch of number theory called 'Diophantine approximation' deals with approximation of real numbers by rationals and their higher dimensional analogues, and Dirichlet's theorem is one of its foundational results. It is indeed quite natural to ask how much improvement the above-mentioned theorem of Dirichlet undergoes for a given real number, vector, matrix and so forth. Recall that given $A \in M_{m\times n}(\mathbb{R})$ and $0 < \varepsilon < 1$, we say that Dirichlet's theorem can be ε-improved for A, and write $A \in DI_{\varepsilon}(m,n)$ if for any $t \gg 1$, there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ satisfying

$$\|A\mathbf{q} - \mathbf{p}\| < \varepsilon e^{-t/m} \quad \text{and} \quad \|\mathbf{q}\| < \varepsilon e^{t/n},$$

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where $\|\cdot\|$ denotes the sup norm on $\mathbb{R}^n$. Also, we say that $A \in M_{m\times n}(\mathbb{R})$ is *singular* if $A \in DI_\varepsilon(m,n)$ for any $\varepsilon > 0$. In the pioneering works [14, 15], Davenport and Schmidt showed that the irrational numbers that are Dirichlet improvable are precisely those that are badly approximable [15]. Furthermore, they have also established the first metrical result in complete generality, which says that $DI_\varepsilon(m,n)$ has zero Lebesgue measure for any $\varepsilon \in (0,1)$.

In 1964, V. Sprindžuk developed a new avenue of investigation, usually referred to as 'metric Diophantine approximation on manifolds', while resolving a long standing conjecture of K. Mahler that says almost every point on the curve $(x, x^2, \ldots, x^n)$ is not 'very well approximable' by rationals with respect to the induced one-dimensional Lebesgue measure. The central problem of this subject is to investigate the extent the properties of a generic point in $\mathbb{R}^n$ with respect to Lebesgue measure or some nice measures are inherited by embedded submanifolds. Later, V. Sprindžuk himself conjectured a more general form of that of Mahler. Partial progress on this conjecture had been made, yet the original conjecture seemed to remain beyond one's reach. In 1998, Kleinbock and Margulis proved the Sprindžuk conjecture in their landmark work [32] using techniques from homogeneous dynamics. Undoubtedly, it was a stroke of genius for them to observe a Dani-type correspondence between *very well approximability* of a vector and cusp excursion behaviour of the trajectory of a certain unimodular lattice. At the core of their argument, there is a sharp quantitative estimate, known as the 'quantitative nondivergence' estimate, on nondivergence of trajectories under multidimensional diagonal flows that shows trajectories with such a nature are rare. The reader is also referred to [6, 24, 28, 30] for other generalizations of the quantitative nondivergence estimate that have number theoretic implications.

Davenport and Schmidt first studied the question of Dirichlet improvability for points lying in manifolds; in fact, they showed in [15] that the set of all $x \in \mathbb{R}$, for which $(x, x^2) \in DI_\varepsilon(1,2)$, has zero Lebesgue measure for any $\varepsilon < 4^{-1/3}$. Further developments in this direction were made by Baker [3, 4], Bugeaud [7], and Dodson, Rynne and Vickers [17, 18]. Kleinbock and Weiss [29] significantly generalize earlier works of Davenport, Schmidt, Baker and Bugeaud on Dirichlet improvability. One of the main results of [29] is the existence of a constant $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $\mathbf{f}_*\nu(DI_\varepsilon(1,n)) = 0$, for continuous, *good* and *nonplanar* maps $\mathbf{f}$ and Radon *Federer* measure ν. Note that the above-mentioned ε_0 is quite far from 1, and getting the same result as above for all $\varepsilon < 1$ is open and seems to be quite challenging. For analytic curves not contained in any affine hyperplane, this has been resolved by Shah in [43]. See also [44, 45] for some more results related to this.

Eventually, there was a drive to generalize classical results in Diophantine approximation to the setting of an arbitrary number field. The adelic successive minima theorem has been established independently by McFeat [36], and Bombieri and Vaaler [6]. See also [8, 26] and the references therein for S-adic versions of other foundational results of geometry of numbers. There seem to exist multiple versions, although with very little variance, of Dirichlet's theorem over number fields in the literature [8, 23, 40–42]. In a recent work [1], Alam and Ghosh obtained several quantitative improve-

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ments to Dirichlet's theorem in number fields generalizing a weighted spiralling of approximates. The metric theory on manifolds in this context germinated some time ago. The S-arithmetic version of Sprindžuk's conjecture was proved by Kleinbock and Tomanov in [28] modifying the technique of [32]. Subsequently, using the results of [28], Mohammadi and Golsefidy settled the convergence case of an S-arithmetic Khintchine-type theorem for S-adic nondegenerate analytic manifolds in [38] and [39]. An inhomogeneous theory on manifolds of the same, both convergence and divergence cases, was developed in the S-arithmetic setting in [11] by Datta and Ghosh. For other developments, see [2, 12, 19, 31].

Since, together with the Euclidean and S-adic ones, function field counterparts complete the general theory of Diophantine approximation over local fields, it is worth mentioning some of developments that have taken place of late in the setting of function fields. The reader should note that there are many interesting parallels, that is, many results of usual Euclidean Diophantine approximation also hold over function fields; however, there are certain striking exceptions as well. For example, there is no analogue of Roth's theorem for function fields over a finite field. The geometry of numbers in the context of function fields was developed by Mahler [34] in 1940 and, using results analogous to that of the Euclidean case, one can obtain a Dirichlet type theorem in positive characteristic. Sprindžuk established the positive characteristic analogue of Mahler's conjecture in [46]. We refer the reader to [16, 33] for general surveys and to, for example, [9, 10, 20, 22] for some of the recent developments.

It is quite expected to be inquisitive about Dirichlet improvability over number fields. Notwithstanding that, in this setting, there have been multiple versions of Dirichlet's theorem in the literature as mentioned earlier. The problem of improvability of the same does not seem to have been pursued much to the best of the authors' knowledge, although a more restrictive notion, namely *singular vectors*, for totally real number fields K and S, being the collection of all Archimedian places, has been coined and discussed at length in the very recent paper [13] by Datta and Radhika. We take up this as the theme of this project. The main result of this paper, namely Theorem 3.13, similar to those of [29, Theorem 1.5] and [20, Theorem 3.7], deals with the Dirichlet improvability of a generic point, with respect to some nice class of measures, lying in the image of a good and nonplanar map in the S-adic setting. Thus, Theorem 3.13 of this paper can be regarded as the number field version of [29, Theorem 1.5] and [20, Theorem 3.7].

Further to the above, the notion of singularity of vectors is now legitimately extended to any arbitrary number field K and S containing all Archimedian places, and that, too, in any direction (see Definition 3.2). This is indeed equivalent to that of [13] (see Lemma 3.4), when K is totally real and S consists of all Archimedian places, from whence our Theorem 3.13 can be regarded as a generalization of [13, Theorem 2.3].

We first establish a multiplicative version (see Theorem 2.1) of Dirichlet's theorem for S-adic matrices. Although it can be proved in the exactly similar manner to that

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of [8, Lemma 5.1], it is clearly the most general version of Dirichlet's theorem over number fields. To provide a brief overview of our result to the reader, here, we first state a simplified version of the above-mentioned Theorem 2.1 as follows. To begin with, we suppose K is a number field of degree d over $\mathbb{Q}$. The completion of K with respect to the place v is denoted by K_v. Let S be a finite collection of pairwise nonequivalent valuations of K containing all Archimedean places. By K_S, we denote the direct product of all K_v for $v \in S$. Let $O_S = \{x \in K : |x|_v \le 1 \text{ for all } v \notin S\}$ denote the *ring of S-integers* of K, where $|\cdot|_v$ denotes the normalized absolute value in K_v. We define the field constant

$$\text{const}_K = \left(\frac{2}{\pi}\right)^s |D_K|^{1/2},$$

where s is the number of complex places of K and D_K is the discriminant of K.

THEOREM 1.1. *For each $v \in S$, let $\mathbf{y}^{(v)} = (y_1^{(v)}, \dots, y_n^{(v)}) \in K_v^n$. Also, let $\varepsilon_v,\ \delta_v \in K_v \setminus \{0\}$ be given for each $v \in S$ with $|\varepsilon_v|_v < 1$ and $|\delta_v|_v \ge 1$ for each $v \in S$, and*

$$\prod_{v \in S} |\varepsilon_v|_v |\delta_v|_v^n = (\text{const}_K)^{n+1}.$$

Then, there exist $\vec{\mathbf{q}} = (q_1, \dots, q_n) \in O_S^n \setminus \{0\}$ and $p \in O_S$ satisfying

$$|q_1 y_1^{(v)} + q_2 y_2^{(v)} + \cdots + q_n y_n^{(v)} - p|_v \le |\varepsilon_v|_v \quad \text{and} \quad \max_{1 \le j \le n} |q_j|_v \le |\delta_v|_v,$$

for all $v \in S$.

For a proof, we refer to [8, Lemma 5.1]. The notion of Dirichlet improvability can be defined as follows.

DEFINITION 1.2. Given a vector $\mathbf{y} = (\mathbf{y}^{(v)})_v \in K_S^n$, where $\mathbf{y}^{(v)} = (y_1^{(v)}, \dots, y_n^{(v)}) \in K_v^n$ for each $v \in S$ and $0 < \varepsilon < 1$, we say that Dirichlet's theorem can be ε-improved for $\mathbf{y}$ or use the notation $\mathbf{y} \in DI_\varepsilon(1, n, K_S)$ if, given any $\varepsilon_v,\ \delta_v \in K_v \setminus \{0\}$ for each $v \in S$ with $|\varepsilon_v|_v < 1$ and $|\delta_v|_v \ge M$ for each $v \in S$, where $M \ge 1$ is sufficiently large and

$$\prod_{v \in S} |\varepsilon_v|_v |\delta_v|_v^n = (\text{const}_K)^{n+1},$$

there exist $\vec{\mathbf{q}} = (q_1, \dots, q_n) \in O_S^n \setminus \{0\}$ and $p \in O_S$ satisfying

$$|q_1 y_1^{(v)} + q_2 y_2^{(v)} + \cdots + q_n y_n^{(v)} - p|_v \le \varepsilon |\varepsilon_v|_v \quad \text{and} \quad \max_{1 \le j \le n} |q_j|_v \le \varepsilon |\delta_v|_v,$$

for all $v \in S$.

In the next couple of sections, we prove a multiplicative version of Dirichlet's theorem over number fields (Theorem 2.1) and define a more general version of Dirichlet improvability (Definition 3.1). We state a special case of our Theorem 3.13.

THEOREM 1.3. *Let $X = \prod_{v \in S} K_v^{l_v}$, where $l_v \in \mathbb{N}$ and $U = \prod_{v \in S} U_v \subseteq X$, be open and let $\mathbf{f} = (\mathbf{f}^{(v)})_v : U \to K_S^n$ be a map, where $\mathbf{f}^{(v)} = (f_1^{(v)}, \dots, f_n^{(v)})$ are continuous maps*

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from U_v to K_v^n such that $f_1^{(v)}, \dots, f_n^{(v)}$ are polynomials and $1, f_1^{(v)}, \dots, f_n^{(v)}$ are linearly independent over K_v for each $v \in S$. Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $\mathbf{f}(\mathbf{x})$ is not Dirichlet ε-improvable for λ-almost every $\mathbf{x} \in U$, where $\lambda = \prod_{v \in S} \lambda_v$ and each λ_v is a Haar measure on $K_v^{l_v}$.

Note that one can obtain an explicit estimate of the constant ε_0 appearing in the above theorem; indeed, it is not difficult to compute its value in terms of known quantities for $n = 2, 3$ and nice maps, and so forth. So, in that sense, our proof can be regarded as 'effective'. However, optimality for that value of ε_0, computed as above, cannot be expected; although at the same time, the set up of Theorem 3.13 is very much general. Our proof is based on the S-arithmetic version of the quantitative nondivergence estimate (see [27, Section 6.6]).

2. Dirichlet's theorem over number fields

In this section, we first state Dirichlet's theorem over number fields, which is a slight modification of [8, Lemma 5.1]. Before stating Dirichlet's theorem, we recall some notation and terminology from [8].

We take K to be an algebraic number field of degree d over $\mathbb{Q}$. The collection of all nontrivial places of K is denoted by P_K. For any $v \in P_K$, K_v denotes the completion of K with respect to the place v. We know that for any $v \in P_K$, K_v is isomorphic to a finite extension of either $\mathbb{R}$ or $\mathbb{Q}_p$ for some prime p. For any given $v \in P_K$, if v is an Archimedean place, we say that v lies over infinity, denoted by $v|\infty$. If v is a non-Archimedean place, then v is the extension of some p-adic valuation. In this case, we say v lies over the finite prime p, denoted by $v|p$:

- $d_v = [K_v : \mathbb{Q}_v]$ denotes the local degree of K_v for each place $v \in P_K$;
- $\|\cdot\|_v$ denotes the absolute value on K_v for each place $v \in P_K$.

For each place $v \in P_K$, we normalize the absolute value $\|\cdot\|_v$ as follows:

(1) if $v|p$, then $\|p\|_v = p^{-1}$;
(2) if $v|\infty$, then for $x \in K_v$, $\|x\|_v = |x|$, where $|\cdot|$ is the Euclidean absolute value on $\mathbb{R}$ or $\mathbb{C}$.

So, $\|\cdot\|_v$ extends the usual p-adic absolute value if $v|p$ and the Euclidean absolute value if $v|\infty$. Our second normalized absolute value $|\cdot|_v$ on K_v is defined by

$$|x|_v = \|x\|_v^{d_v}.$$

Due to this normalization, we have the product formula:

$$\prod_{v \in P_K} |x|_v = 1 \tag{2-1}$$

for all $x \in K$, $x \neq 0$.

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- For $\vec{x} = (x_1, x_2, \ldots, x_n) \in K_v^n$, we extend our absolute values as follows:

$$|\vec{x}|_v = \max_{1 \le i \le n} \{|x_i|_v\},$$
$$\|\vec{x}\|_v = \max_{1 \le i \le n} \{\|x_i\|_v\}.$$

- Let v be a finite place of K. We denote the maximal compact (open) subring of K_v by $\mathcal{O}_v$, that is,

$$\mathcal{O}_v = \{x \in K_v : |x|_v \le 1\}.$$

- A subset R_v in K_v^n is a K_v-lattice if it is a compact open $\mathcal{O}_v$-module in K_v^n. It is easy to observe that $\mathcal{O}_v^n$ is a K_v-lattice in K_v^n.
- Let S be a finite collection of places of K containing all Archimedean places. We define the ring of S-integers as

$$\mathcal{O}_S = \{x \in K : x \in \mathcal{O}_v \text{ for all } v \notin S\}.$$

- The adele ring of K is denoted by $K_{\mathbb{A}}$. Here, K is diagonally embedded inside $K_{\mathbb{A}}$, so that we may view $K \subseteq K_{\mathbb{A}}$ by the natural diagonal map $\rho : K \to K_{\mathbb{A}}$ defined by

$$\rho(\alpha) = (\alpha, \alpha, \alpha, \ldots)$$

  for $\alpha \in K$.
- We say that a nonempty subset $R_v \subseteq K_v^n$ is a regular set if it has one of the following forms:

  (1) if $v|\infty$, then R_v is a bounded, convex, closed, symmetric subset of K_v^n with nonzero volume;
  (2) if $v \nmid \infty$, then R_v is a K_v-lattice in K_v^n.

- We say that a subset $\mathcal{R}$ of $K_{\mathbb{A}}^n$ is *admissible* if $\mathcal{R} = \prod_{v \in P_K} R_v$, where R_v is a regular set in K_v^n for each $v \in P_K$ and $R_v = \mathcal{O}_v^n$ for almost all places v.
- We define the field constant for the number field K as

$$\text{const}_K = \left(\frac{2}{\pi}\right)^s |D_K|^{1/2},$$

  where D_K is the discriminant of K and s is the number of complex places of K.

We are now ready to state and prove a version of Dirichlet's theorem over number fields. The following theorem is a slight modification of [8, Lemma 5.1], in the sense that we are considering a multiplicative form of approximation. Although the proof follows from [8, Lemma 5.1], we include it here for completeness.

THEOREM 2.1. *Let A_v be an $m \times n$ matrix over K_v for each $v \in S$. Also, let*

$$\varepsilon_v^{(1)}, \ldots, \varepsilon_v^{(m)}, \delta_v^{(1)}, \ldots, \delta_v^{(n)} \in K_v \setminus \{0\}$$

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be given for each $v \in S$ *so that* $|\varepsilon_v^{(i)}|_v < 1$, *for all* $i = 1, \dots, m$ *and* $|\delta_v^{(j)}|_v \geq 1$, *for all* $j = 1, \dots, n$ *for each* $v \in S$, *and*

$$\prod_{v \in S} |\varepsilon_v^{(1)}|_v \cdots |\varepsilon_v^{(m)}|_v |\delta_v^{(1)}|_v \cdots |\delta_v^{(n)}|_v = (\mathrm{const}_K)^{m+n}.$$

Then, there exists $\vec{x} = (x_1, \dots, x_n) \in \mathcal{O}_S^n \setminus \{\vec{0}\}$ *and* $\vec{y} = (y_1, \dots, y_m) \in \mathcal{O}_S^m$ *satisfying*

$$\begin{cases} \|A_v^{(i)}\vec{x} - y_i\|_v \leq \|\varepsilon_v^{(i)}\|_v & \text{for } i = 1, 2, \dots, m, \\ \|x_j\|_v \leq \|\delta_v^{(j)}\|_v & \text{for } j = 1, 2, \dots, n, \end{cases}$$

for all $v \in S$, *where* $A_v^{(i)}$ *is the* i*th row of* A_v *for* $i = 1, 2, \dots, m$.

PROOF. The proof of this theorem goes along the lines of the original proof by Burger with slight modifications. For each place $v \in P_K$, we define an $(m+n) \times (m+n)$ matrix B_v over K_v by

$$B_v = \begin{pmatrix} {\delta_v}^{-1} I_n & \mathbf{0} \\ \varepsilon_v^{-1} A_v & {\varepsilon_v}^{-1} I_m \end{pmatrix}$$

for all $v \in S$, where $\varepsilon_v^{-1} A_v$ is the $m \times n$ matrix over K_v, whose ith row is ${\varepsilon_v^{(i)}}^{-1} A_v^{(i)}$ for $i = 1, \dots, m$; ${\delta_v}^{-1} I_n = \mathrm{diag}({\delta_v^{(1)}}^{-1}, {\delta_v^{(2)}}^{-1}, \dots, {\delta_v^{(n)}}^{-1})$ and we also have ${\varepsilon_v}^{-1} I_m = \mathrm{diag}({\varepsilon_v^{(1)}}^{-1}, {\varepsilon_v^{(2)}}^{-1}, \dots, {\varepsilon_v^{(m)}}^{-1})$. Also, define $B_v = I_{m+n}$ for all $v \notin S$.

Define $R_v \subseteq K_v^{m+n}$ by

$$R_v = \{\vec{z} \in K_v^{m+n} : \|B_v \vec{z}\|_v \leq 1\}$$

and let $\mathcal{R} = \prod_v R_v$. It is easy to see that $\mathcal{R} \subseteq K_{\mathbb{A}}^{m+n}$ and $\mathcal{R}$ is admissible. Observe that

$$\begin{aligned} V(\mathcal{R}) &= 2^{d(m+n)} \left(\frac{\pi}{2}\right)^{s(m+n)} |D_K|^{-(m+n)/2} \left(\prod_v |\det B_v|_v\right)^{-1} \\ &= 2^{d(m+n)} \left(\left(\frac{\pi}{2}\right)^s |D_K|^{-1/2}\right)^{m+n} \left(\prod_v |\varepsilon_v^{(1)}|_v \cdots |\varepsilon_v^{(m)}|_v |\delta_v^{(1)}|_v \cdots |\delta_v^{(n)}|_v\right) \\ &= 2^{d(m+n)}. \end{aligned}$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{(m+n)}$ be the successive minima of $\mathcal{R} \subseteq K_{\mathbb{A}}^{m+n}$ with respect to K^{m+n}. Then, by the adelic successive minima theorem [6, Theorem 3], we know that

$$(\lambda_1 \lambda_2 \cdots \lambda_{m+n})^d V(\mathcal{R}) \leq 2^{d(m+n)}.$$

Thus, we have $\lambda_1 \leq 1$, since λ_n are increasing and $V(\mathcal{R}) = 2^{d(m+n)}$. Hence, there exists a point $\begin{pmatrix} \vec{x} \\ -\vec{y} \end{pmatrix} \in K^{m+n} \setminus \{0\} \cap \mathcal{R}$. By our definition of R_v for $v \notin S$,

$$\begin{pmatrix} \vec{x} \\ -\vec{y} \end{pmatrix} \in \mathcal{O}_S^{m+n}.$$

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Also, for $v \in S$,

$$\|A_v^{(i)}\vec{x} - y_i\|_v \le \|\varepsilon_v^{(i)}\|_v$$

and

$$\|x_j\|_v \le \|\delta_v^{(j)}\|_v$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. In addition, it is easy to see that $\vec{x} \neq \vec{0}$ due to the product formula (2-1). □

REMARK 2.2. It is natural to ask about the *optimal constant* in Dirichlet's theorem. We illustrate this through two basic examples.

EXAMPLE 2.3. Let $m = n = 1$, $K = \mathbb{Q}$ and $S = \{\infty\}$. In this case, Theorem 2.1 asserts that for any $x \in \mathbb{R}$ and any $Q \ge 1$, there exist integers $p, q \in \mathbb{Z}$ with $1 \le q \le Q$ such that

$$|qx + p| < \frac{\text{const}_{\mathbb{Q}}}{Q}.$$

Here, $\text{const}_{\mathbb{Q}} = 1$ and this value is optimal in the following sense.

There exists $x \in \mathbb{R}$ such that, for every $c < 1$, the system

$$|qx + p| < \frac{c}{Q}, \quad 1 \le q \le Q,$$

has no solution in integers p, q for infinitely many Q.

EXAMPLE 2.4. Let $m = n = 1, K = \mathbb{Q}(i)$, where i is a root of $x^2 + 1 = 0$, and take $S = \{\infty\}$. Then, Theorem 2.1 gives: for any $\alpha \in \mathbb{C}$ and any $Q \ge 1$, there exist $p, q \in \mathbb{Z}[i]$ with $1 \le |q| \le Q$ such that

$$|q\alpha + p| < \frac{\text{const}_{\mathbb{Q}(i)}}{Q}.$$

In this case,

$$\text{const}_{\mathbb{Q}(i)} = \frac{4}{\pi}.$$

However, unlike the real case, this constant is *not optimal*. In fact, one can show that the optimal constant is strictly smaller, namely [37, Satz LXII, Section 8, Ch. 6]

$$\frac{\sqrt{2}}{3 - \sqrt{3}}.$$

More precisely, there exists $\alpha \in \mathbb{C}$ such that for every $c < (\sqrt{2}/(3 - \sqrt{3}))$, the system

$$|q\alpha + p| < \frac{c}{Q}, \quad 1 \le |q| \le Q,$$

has no solution in $p, q \in \mathbb{Z}[i]$ for infinitely many Q.

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REMARK 2.5. These examples highlight that the constant appearing in Dirichlet's theorem (Theorem 2.1) obtained from Minkowski's theorem is not always optimal. In the case where $K = \mathbb{Q}$, the Minkowski constant is optimal, whereas for $K = \mathbb{Q}(i)$, it can be improved. It is clear that the critical value depends sensitively on the underlying number field K as well as the choice of places in S. However, at present, it is not clear how to determine this exponent in full generality for arbitrary number fields.

Our view is that the problem of identifying the critical exponent must be approached through specific cases rather than in complete generality. A fundamental difficulty is that the constant arising in Dirichlet's theorem from Minkowski's convex body theorem is, in general, not optimal. The determination of the optimal constant depends on the geometry of lattices and is often closely connected with optimal lattice covering problems in the corresponding Euclidean space. These covering problems are well understood only in low dimensions and remain largely open in higher dimensions. See [25] for more details. In [25], Kleinbock and Rao considered the problem of Dirichlet improvability in $\mathbb{R}^2$ for norms different than the sup norm, and showed that the optimal constant in this specific scenario is related to critical lattices and critical loci.

To summarize, we do not currently have a method to determine the optimal constant for Theorem 2.1 over general number fields.

We would, however, like to emphasize that our results on Dirichlet improvability still have meaningful applications. In particular, they yield new results for singular vectors, extending earlier work of Datta and Radhika.

3. Dirichlet improvability and the main theorem

Before introducing the notion of 'Dirichlet improvability', we first recall some notation from [27].

- Throughout this article, K represents a number field of degree d.
- We denote a finite set of places of K containing all the Archimedean places as S. Precisely, let $S = \{v_1, \dots, v_\ell\}$, where $v_i \in P_K$ for $i = 1, \dots, \ell$, such that S contains all Archimedean places.
- We denote the normalized absolute value on K_v by $|\cdot|_v$, which is the same as the earlier defined normalization $|\cdot|_v$ on K_v.
- K_S is the direct product of all the completions K_v, $v \in S$, that is,

$$K_S = \prod_{v \in S} K_v.$$

  K is embedded inside K_S via the diagonal embedding. The set of all Archimedean places of K is denoted by $S_a \subseteq S$ and we denote $S \setminus S_a$ by S_f. Let S_c and S_r respectively denote the set of all complex places and the set of all real places of K.
- The elements of K_S are denoted as $x = (x^{(v)})_{v \in S}$, or simply $x = (x^{(v)})_v$, where $x^{(v)} \in K_v$. For $x \in K_S$, we define the content $c(x)$ of x as follows:

$$c(x) = \prod_{v \in S} |x^{(v)}|_v.$$

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- As usual, the ring of integers of K is denoted by $\mathcal{O}_K$.
- The ring of S-integers of K, that is, $\mathcal{O}_S$, is also diagonally embedded inside K_S. We use this identification for convenience.
- We denote the elements of K_S^m by bold characters as $\mathbf{x} = (\mathbf{x}^{(v)})_v \in K_S^m$, where $\mathbf{x}^{(v)} = (x_1^{(v)}, \ldots, x_m^{(v)}) \in K_v^m$.
- We denote by $\|\cdot\|_{v,2}$ the square of the usual Hermitian norm on K_v^m if v is complex, the usual Euclidean norm on K_v^m if v is real and the sup norm defined by

$$\|(x_1^{(v)}, \ldots, x_m^{(v)})\|_{v,2} = \max_i |x_i^{(v)}|_v$$

if v is non-Archimedean.

Note that if v is non-Archimedean and $\vec{x} = (x_1, \ldots, x_m) \in K_v^m$, then $\|\vec{x}\|_{v,2} = |\vec{x}|_v$.

- We endow both K_S and K_S^m with the product metric on them, respectively. More precisely, for any $x = (x^{(v)})_v \in K_S$ and $\mathbf{x} = (\mathbf{x}^{(v)})_v \in K_S^m$, the norms are given by

$$\|x\| := \sup_{v \in S} |x^{(v)}|_v$$

and

$$\|\mathbf{x}\| := \sup_{v \in S} \|\mathbf{x}^{(v)}\|_{v,2},$$

respectively.
- For $\mathbf{x} = (\mathbf{x}^{(v)})_v \in K_S^m$, we define the content $c(\mathbf{x})$ of $\mathbf{x}$ by

$$c(\mathbf{x}) = \prod_{v \in S} \|\mathbf{x}^{(v)}\|_{v,2}.$$

- For the sake of convenience, we keep the notation $\|\cdot\|_{v,2}$ and $c(\cdot)$ to denote the norms and the content on the exterior powers $\bigwedge^r(K_v^m)$ and $\bigwedge^r(K_S^m)$, respectively.
- Given a field F, the set of all $m \times n$ matrices over F is denoted by $M_{m\times n}(F)$.
- We define $M_{m\times n}(K_S)$ as

$$M_{m\times n}(K_S) := \prod_{v \in S} M_{m\times n}(K_v).$$

Elements of $M_{m\times n}(K_S)$ are denoted by $g = (g^{(v)})_v$, where $g^{(v)} \in M_{m\times n}(K_v)$. Similarly,

$$\mathrm{GL}(m, K_S) := \prod_{v \in S} \mathrm{GL}(m, K_v)$$

and

$$\mathrm{GL}^1(m, K_S) := \left\{ (g^{(v)})_{v\in S} \in \mathrm{GL}(m, K_S) : \prod_{v \in S} \left|\det(g^{(v)})\right|_v = 1 \right\}.$$

- $$\mathfrak{a}^+ := \{\mathbf{t} := (t_{v_1}^{(1)}, t_{v_1}^{(2)}, \ldots, t_{v_1}^{(m+n)}, \ldots, t_{v_\ell}^{(1)}, t_{v_\ell}^{(2)}, \ldots, t_{v_\ell}^{(m+n)}) \in \mathbb{R}_+^{m+n} \times \cdots \times \mathbb{R}_+^{m+n} : t_{v_r}^{(i)} < 1,\ t_{v_r}^{(j)} \geq 1 \quad \text{for } i = 1, \ldots, m;\ j = m+1, \ldots, m+n;\ r = 1, \ldots, \ell \text{ and } \prod_{v\in S} t_v^{(1)} \cdots t_v^{(m+n)} = (\mathrm{const}_K)^{m+n}\},$$

where $\mathbb{R}^+$ is the set of positive real numbers.

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- Let $\mathcal{I}$ be a subset of $\mathfrak{a}^+$ defined by
$$\mathcal{I} := \{\mathbf{t} \in \mathfrak{a}^+ : t_{v_r}^{(1)} = \cdots = t_{v_r}^{(m)}, t_{v_r}^{(m+1)} = \cdots = t_{v_r}^{(m+n)} \text{ for } r = 1, \ldots, \ell\}.$$
- Let S contain precisely all Archimedian places of K. Given $\mathbf{t} \in \mathcal{I}$, let $t_{v_r} := t_{v_r}^{(1)} = \cdots = t_{v_r}^{(m)}$ and $t'_{v_r} := t_{v_r}^{(m+1)} = \cdots = t_{v_r}^{(m+n)}$ for $r = 1, \ldots, \ell$. Keeping this in mind, let $\mathcal{I}_0$ be the subset of $\mathcal{I}$ defined as follows:
$$\mathcal{I}_0 := \{\mathbf{t} \in \mathcal{I} : t_{v_r} = t_{v_s} \text{ and } t'_{v_r} = t'_{v_s} \text{ for } r, s = 1, \ldots, \ell\}.$$
Here $\mathcal{I}_0$ is called the *central ray* of $\mathfrak{a}^+$.
- For $\mathbf{t} \in \mathfrak{a}^+$, define $\|\mathbf{t}\|_\infty = \max\limits_{\substack{i=1,\ldots,m+n \\ r=1,\ldots,\ell}} t_{v_r}^{(i)}$.
- For $i = 1, \ldots, \ell$, let $p_i : \mathfrak{a}^+ \to \mathbb{R}_+^{m+n}$ be the projection of $\mathfrak{a}^+$ on the ith factor $\mathbb{R}_+^{m+n}$ defined by
$$p_i(\mathbf{t}) = (t_{v_i}^{(1)}, t_{v_i}^{(2)}, \ldots, t_{v_i}^{(m+n)}) \quad \text{for all } \mathbf{t} \in \mathfrak{a}^+.$$
- We say that a subset $\mathcal{T}$ of $\mathfrak{a}^+$ is S-unbounded if $p_i|_{\mathcal{T}}$ is unbounded for each $i = 1, \ldots, \ell$.

We define Dirichlet improvability in this context. Let $\mathfrak{a}^+$ be as above and $\mathcal{T}$ be an S-unbounded subset of $\mathfrak{a}^+$.

DEFINITION 3.1. Given $A = (A_v)_v \in M_{m\times n}(K_S)$, where A_v is an $m \times n$ matrix over K_v for each $v \in S$, and $0 < \varepsilon < 1$, we say that Dirichlet's theorem can be ε-improved for A along $\mathcal{T}$ or use the notation $A \in DI_\varepsilon(m, n, K_S, \mathcal{T})$ if there is $t_0 > 0$ such that given any $\varepsilon_v^{(1)}, \varepsilon_v^{(2)}, \ldots, \varepsilon_v^{(m)}, \delta_v^{(1)}, \delta_v^{(2)}, \ldots, \delta_v^{(n)} \in K_v \setminus \{0\}$ for each $v \in S$ with
$$\begin{aligned}\mathbf{t} := (&|\varepsilon_{v_1}^{(1)}|_{v_1}, \ldots, |\varepsilon_{v_1}^{(m)}|_{v_1}, |\delta_{v_1}^{(1)}|_{v_1}, \ldots, |\delta_{v_1}^{(n)}|_{v_1}, \ldots, |\varepsilon_{v_\ell}^{(1)}|_{v_\ell}, \ldots, |\varepsilon_{v_\ell}^{(m)}|_{v_\ell},\\ &|\delta_{v_\ell}^{(1)}|_{v_\ell}, \ldots, |\delta_{v_\ell}^{(n)}|_{v_\ell}) \in \mathcal{T}\end{aligned} \tag{3-1}$$
and $\|p_i|_{\mathcal{T}}(\mathbf{t})\|_\infty > t_0$ for each $i = 1, \ldots, \ell$, there exists $\vec{x} = (x_1, \ldots, x_n) \in O_S^n \setminus \{\vec{0}\}$ and $\vec{y} = (y_1, \ldots, y_m) \in O_S^m$ satisfying
$$\begin{cases} \|A_v^{(i)}\vec{x} - y_i\|_v \le \varepsilon\|\varepsilon_v^{(i)}\|_v & \text{for } i = 1, 2, \ldots, m, \\ \|x_j\|_v \le \varepsilon\|\delta_v^{(j)}\|_v & \text{for } j = 1, 2, \ldots, n, \end{cases} \tag{3-2}$$
for all $v \in S$, where $A_v^{(i)}$ is the ith row of A_v for $i = 1, 2, \ldots, m$. In particular, a vector $\mathbf{x} = (\mathbf{x}^{(v)})_v \in K_S^n$, where $\mathbf{x}^{(v)} = (x_1^{(v)}, \ldots, x_n^{(v)}) \in K_v^n$ is said to be *Dirichlet ε-improvable along $\mathcal{T}$* if the corresponding matrix $A = ([x_1^{(v)} \cdots x_n^{(v)}])_{v\in S} \in DI_\varepsilon(1, n, K_S, \mathcal{T})$. When $\mathcal{T} = \mathcal{I}$, then simply we say A is Dirichlet ε-improvable.

Singular vectors can be defined in the S-adic context in a similar way to that of the Euclidean context.

DEFINITION 3.2. Let $m, n \in \mathbb{N}$ and $\mathcal{T}$ be an S-unbounded subset of $\mathfrak{a}^+$. We say $A = (A_v)_v \in M_{m\times n}(K_S)$, where $A_v \in M_{m\times n}(K_v)$, for each $v \in S$, is *singular* along $\mathcal{T}$ if for all

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$\varepsilon \in (0,1)$, A is Dirichlet ε-improvable along $\mathcal{T}$. In particular, a vector $\mathbf{x} = (\mathbf{x}^{(v)}) \in K_S^n$, where $\mathbf{x}^{(v)} = (x_1^{(v)}, \dots, x_n^{(v)}) \in K_v^n$, for each $v \in S$, is said to be singular along $\mathcal{T}$ if $A = ([x_1^{(v)} \cdots x_n^{(v)}])_{v\in S}$ is singular along $\mathcal{T}$. When S consists of precisely all Archimedian places of K and $\mathcal{T} = \mathcal{I}_0$, then simply we say A is *singular*. We denote the set of all vectors of K_S^n that are singular along $\mathcal{T}$ by $Sing(K_S, n, \mathcal{T})$.

When K is totally real and S = the collection of all Archimedian places of K, the notion of singular vectors has been introduced in [13].

DEFINITION 3.3. [13, Definition 2.2] Let K be a totally real number field and S the collection of all Archimedian places of K. We say $\mathbf{x} = (x_1, \dots, x_n) \in K_S^n$ is *singular* if for every $c > 0$, for all sufficiently large $Q > 0$, there exists $\mathbf{0} \neq \mathbf{q} \in O_K^n$, $q_0 \in O_K$ satisfying the following system:

$$\begin{cases} \|\mathbf{q} \cdot \mathbf{x} + q_0\| < \dfrac{c}{Q^n}, \\ \|\mathbf{q}\| \leq Q\,. \end{cases} \tag{3-3}$$

The set of all singular vectors in K_S^n, in this sense, is denoted by $Sing_S^n$.

Indeed, when S is the set of all Archimedian places of K, Definition 3.3 is equivalent to our definition. The following proposition establishes that.

LEMMA 3.4. *We have $Sing(K_S, n, \mathcal{I}_0) = Sing_S^n$ if S contains precisely all normalized Archimedean places of K.*

PROOF. By a standard algebraic argument using fractional ideals and so forth, one can easily see that, in this case, $O_S = O_K$. Let $\mathbf{x} \in Sing_S^n$. We show that $\mathbf{x} \in Sing(K_S, n, \mathcal{I}_0)$. For that, let us take $0 < \varepsilon < 1$. Suppose that $\varepsilon_v,\ \delta_v \in K_v \setminus \{0\}$, for each $v \in S$ with $|\varepsilon_v|_v = |\varepsilon_{v'}|_{v'} = \varepsilon_S$, $|\delta_v|_v = |\delta_{v'}|_{v'} = \delta_S$ for all $v, v' \in S$ and $\varepsilon_S < 1$, $\delta_S \gg 1$ with

$$\varepsilon_S^{\ell} \delta_S^{n\ell} = (\text{const}_K)^{n+1}.$$

We choose $c > 0$ and Q sufficiently large so that

$$\begin{cases} \dfrac{c}{Q^n} < \varepsilon\varepsilon_S, \\ Q \leq \varepsilon\delta_S\,, \end{cases} \tag{3-4}$$

holds. Since $\mathbf{x}$ is singular, for this chosen c and sufficiently large Q, there exists $\mathbf{0} \neq \mathbf{q} \in O_K^n$, $q_0 \in O_K$ such that (3-3) holds. Observe that if $\mathbf{x} = (x_1, \dots, x_n) \in K_S^n$ and $x_i = (x_i^{(v)}) \in K_S$ for $i = 1, \dots, n$, then (3-3) can be rewritten as

$$|q_1 x_1^{(v)} + q_2 x_2^{(v)} + \cdots + q_n x_n^{(v)} + q_0|_v \leq \frac{c}{Q^n} \quad \text{and} \quad \max_{1\leq j\leq n} |q_j|_v \leq Q \tag{3-5}$$

for all $v \in S$.

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Combining (3-4) and (3-5), we get that $\mathbf{x} \in K_S^n$ is Dirichlet ε-improvable along $\mathcal{I}_0$. Therefore,

$$Sing_S^n \subseteq Sing(K_S, n, \mathcal{I}_0).$$

The other containment $Sing(K_S, n, \mathcal{I}_0) \subseteq Sing_S^n$ is obvious. □

REMARK 3.5. In the one-dimensional case, singular points are precisely the elements of K [13, Theorem 2.6]. In fact, when $K = \mathbb{Q}$, one can adopt the strategy of the proof of [21, Theorem 2.4] and easily see that, if $0 < \varepsilon < \frac{1}{2}$, the real numbers for which Dirichlet's theorem can be improved by ε are precisely the rationals. The authors are grateful to Sanju Velani for drawing the attention of the second-named author to this fact in an informal discussion.

Unless specified otherwise, we henceforth let $\mathbf{t}$ stand for (3-1). Before going to our main result, let us recall some terminology from [24, 27, 28].

DEFINITION 3.6. Let X be a metric space. We say that X is *Besicovitch* if for any bounded subset A of X and for any family $\mathcal{B}$ of nonempty open balls in X such that every $x \in A$ is the centre of some open ball of $\mathcal{B}$, there exists a constant N_X (called the Besicovitch constant of X) and there is a finite or countable subfamily $\{B_i\}$ of $\mathcal{B}$ such that

$$1_A \leq \sum_i 1_{B_i} \leq N_X\,,$$

that is, $A \subset \bigcup_i B_i$, and the multiplicity of that subcovering is at most N_X.

Throughout this article, N_X always represents the Besicovitch constant of the metric space X.

EXAMPLE 3.7.

(1) By Besicovitch's covering theorem [35, Theorem 2.7], the Euclidean spaces $\mathbb{R}^n$ are Besicovitch spaces.
(2) The space of our interest $X = \prod_{v\in S} K_v^{l_v}$, where $l_v \in \mathbb{N}$, is also Besicovitch [27, Section 1.6].

Let X be a Besicovitch metric space and $(F, |\cdot|)$ a valued field. For a subset B of X and a given function $f : X \to F$, we set

$$\|f\|_B := \sup_{x\in B} |f(x)|.$$

If $f : X \to F$ is a measurable function, μ is a Radon measure on X and B is a subset of X with $\mu(B) > 0$, we define

$$\|f\|_{\mu,B} := \|f\|_{B\cap \operatorname{supp}(\mu)}.$$

DEFINITION 3.8. Given $C, \alpha > 0$, open $U \subset X$ and a Radon measure μ on X, $f : X \to F$ is said to be (C, α)-*good* on U with respect to μ if for every ball $B \subset U$ with centre

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in supp (μ) and any $\varepsilon > 0$,

$$\mu(\{x \in B : |f(x)| < \varepsilon\}) \le C\Big(\frac{\varepsilon}{\|f\|_{\mu,B}}\Big)^{\alpha} \mu(B)\,.$$

The properties listed below follow immediately from Definition 3.8.

LEMMA 3.9. *Let $X, U, \mu, F, f, C, \alpha$, be as given above. Then:*

(a) *f is (C,α)-good on U with respect to $\mu \iff$ so is $|f|$;*
(b) *f is (C,α)-good on U with respect to $\mu \implies$ so is cf for all $c \in F$;*
(c) *for all $i \in I$, f_i are (C,α)-good on U with respect to $\mu \implies$ so is $\sup_{i\in I} |f_i|$;*
(d) *f is (C,α)-good on U with respect to μ, and $c_1 \le |f(x)/g(x)| \le c_2$ for all $x \in U \implies g$ is $(C(c_2/c_1)^{\alpha},\alpha)$-good on U with respect to μ;*
(e) *let $C_2 > 1$ and $\alpha_2 > 0$; then, f is (C_1,α_1)-good on U with respect to μ and $C_1 \le C_2, \alpha_2 \le \alpha_1 \implies f$ is (C_2,α_2)-good on U with respect to μ.*

We recall an important lemma from [27, Section 2.3], which is useful later in our discussion.

LEMMA 3.10. *For each $i = 1,\dots,m$, let X_i be a metric space, μ_i be a measure on X_i, $U_i \subseteq X_i$ open and $f_i : X_i \to \mathbb{R}$ be a function that is (C_i,α_i)-good on U_i with respect to μ_i. Then, the function $f : \prod_{i=1}^m X_i \to \mathbb{R}$ defined by $f(x_1,\dots,x_m) = f(x_1)f(x_2)\cdots f(x_m)$ is (C,α)-good on $U_1 \times \cdots \times U_m$ with respect to $\mu_1 \times \cdots \times \mu_m$, with C and α explicitly computed in terms of C_i, α_i for $i = 1,\dots,m$.*

Given a Radon measure μ on X, an open subset U of X with $\mu(U) > 0$ and a map $\mathbf{f} = (f_1, f_2, \dots, f_m) : U \to F^m$, we say that the pair $(\mathbf{f},\mu)$ is (C,α)-good on U if any F-linear combination of $1, f_1, \dots, f_m$ is (C,α)-good on U with respect to μ.

DEFINITION 3.11. Let $\mathbf{f} = (f_1, f_2, \dots, f_m)$ be a map from U to F^m, where $m \in \mathbb{N}$. We say that $(\mathbf{f},\mu)$ is *nonplanar* on U if for any ball $B \subset U$ centred in $\mathrm{supp}(\mu)$, the restrictions of $1, f_1, f_2, \dots, f_m$ to $B \cap$ supp (μ) are linearly independent over F; in other words, if $\mathbf{f}(B \cap \text{supp }(\mu))$ is not contained in any proper affine subspace of F^m.

If $B = B(x,r)$, where $x \in X$ and $r > 0$, is a ball in X and $c > 0$, we use the notation cB to denote the ball $B(x,cr)$.

DEFINITION 3.12. Let μ be a Radon measure on X and $D > 0$. Here, μ is said to be *D-Federer* on an open subset U of X if for all balls B centred at supp (μ) with $3B \subset U$, one has

$$\frac{\mu(3B)}{\mu(B)} \le D\,.$$

We finish this section by stating the main result of this paper, which says that for sufficiently small $\varepsilon > 0$, μ-almost every vector lying in the image of a good and nonplanar map is not Dirichlet ε-improvable along $\mathcal{T}$ for a large class of measures μ in this S-adic setup.

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THEOREM 3.13. *Let $X = \prod_{v\in S} X_v$, where $X_v = K_v^{l_v}$ with $l_v \in \mathbb{N}$ and $v \in S$ is a Besicovitch metric space, μ_v be a measure on $X_v\,(v \in S)$ so that $\mu = \prod_{v\in S}\mu_v$ is D-Federer measure on X, and $U = \prod_{v\in S} U_v \subseteq X$ be open with $\mu(U) > 0$. Let $\mathbf{f} = (\mathbf{f}^{(v)})_v : U \to K_S^n$ be a map, where $\mathbf{f}^{(v)}$ are continuous maps from $U_v \subseteq K_v^{l_v}$ to K_v^n such that $(\mathbf{f}^{(v)}, \mu_v)$ is nonplanar on U_v for all $v \in S$ and for each $v \in S$, $(\mathbf{f}^{(v)}, \mu_v)$ is (C_v, α_v)-good on U_v. Then, there exists $\varepsilon_0 = \varepsilon_0(n, C_{v_1}, \dots, C_{v_\ell}, \alpha_{v_1}, \dots, \alpha_{v_\ell}, D, N_X, K, S) > 0$ such that for any $\varepsilon < \varepsilon_0$,*

$$\mathbf{f}_*\mu(DI_\varepsilon(1, n, K_S, \mathcal{T})) = 0 \quad \text{for any } S\text{-unbounded } \mathcal{T} \subseteq \mathfrak{a}^+ .$$

COROLLARY 3.14. *Let the hypothesis be the same as that of Theorem 3.13. Then,*

$$\mathbf{f}_*\mu(Sing(K_S, n, \mathcal{T})) = 0 \quad \text{for any } S\text{-unbounded } \mathcal{T} \subseteq \mathfrak{a}^+.$$

In particular, $\mathbf{f}_*\mu(Sing(K_S, n, \mathcal{I}_0)) = 0$.

In view of Lemma 3.4, one sees that Corollary 3.14 generalizes [13, Theorem 2.3] to the case $X = \prod_{v\in S} K_v^{l_v}$ with $l_v \in \mathbb{N}$ for $v \in S$.

THEOREM 3.15 [13, Theorem 2.3]. *Suppose $X = \prod_{v\in S} X_v$ is a Besicovitch space and $\mu = \prod_{v\in S}\mu_v$ is a Federer measure on X, and let $\mathbf{f} : X \to K_S^n$, $\mathbf{f}(x) = (\mathbf{f}_v(x_v))_{v\in S}$ be a continuous map such that $(\mathbf{f}, \mu)$ is nonplanar for μ-almost every point of X and for each $v \in S$, $(\mathbf{f}_v, \mu_v)$ is good for μ_v-almost every point of X_v. Then, $\mathbf{f}_*\mu(Sing_S^n) = 0$.*

The key idea to prove Theorem 3.13 is the so-called 'quantitative nondivergence' technique, a generalization of nondivergence of unipotent flows on homogeneous spaces. To apply this technique, first we need to translate our problem related to Dirichlet ε-improvability into certain properties of flows on some homogeneous spaces, which is the main theme of our next section.

4. The correspondence

Define

$$\Omega_{S,m} := \{gO_S^m : g \in \mathrm{GL}(m, K_S)\}$$

and

$$\Omega_{S,m}^1 := \{\Lambda \in \Omega_{S,m} : \mathrm{cov}(\Lambda) = (\sqrt{D_K})^m\}.$$

Let $G := \mathrm{GL}^1(m, K_S)$ and $\Gamma := \mathrm{GL}(m, O_S)$, and let $\pi : G \to G/\Gamma$ be the quotient map. Then, G acts on G/Γ by left translation via the rule $g\pi(h) = \pi(gh)$ for $g, h \in G$. We define a map $\tau : M_{m\times n}(K_S) \longrightarrow \mathrm{GL}^1(m+n, K_S)$ given by

$$\tau((A_v)_v) = (g_{A_v})_v,$$

where

$$g_{A_v} := \begin{bmatrix} I_m & A_v \\ 0 & I_n \end{bmatrix}.$$

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We put $\overline{\tau} := \pi \circ \tau$. The homogeneous space $\mathrm{GL}^1(m, K_S)/\mathrm{GL}(m, O_S)$ is naturally identified with $\Omega^1_{S,m}$, so that we have $\overline{\tau}((A_v)_v) = (g_{A_v})_v O_S^{m+n}$. Define

$$\Omega_\varepsilon := \{\Lambda \in \Omega_{S,m+n} : \delta(\Lambda) \geq \varepsilon\},$$

where $\delta : \Omega_{S,m+n} \longrightarrow \mathbb{R}_+$ is defined by

$$\delta(\Lambda) := \min\{c(\mathbf{x}) : \mathbf{x} \in \Lambda \setminus \{0\}\}$$

for all $\Lambda \in \Omega_{S,m+n}$.

Again, we let $\varepsilon_v^{(1)}, \varepsilon_v^{(2)}, \ldots, \varepsilon_v^{(m)}, \delta_v^{(1)}, \delta_v^{(2)}, \ldots, \delta_v^{(n)} \in K_v \setminus \{0\}$ for each $v \in S$ and define $g_{\varepsilon_v,\delta_v}$ by

$$g_{\varepsilon_v,\delta_v} := \mathrm{diag}((\varepsilon_v^{(1)})^{-1}, \ldots, (\varepsilon_v^{(m)})^{-1}, (\delta_v^{(1)})^{-1}, \ldots, (\delta_v^{(n)})^{-1}).$$

Then, clearly $(g_{\varepsilon_v,\delta_v})_v \in \mathrm{GL}(m+n, K_S)$.

The following proposition, which may be regarded as an analogue of *'Dani correspondence'* over number fields, helps us to convert our problem regarding Dirichlet improvability to a problem of homogeneous dynamics.

PROPOSITION 4.1. *Let $0 < \varepsilon < 1$, and suppose that $A = (A_v)_v \in DI_\varepsilon(m, n, K_S, \mathcal{T})$. For each $v \in S$, choose $\varepsilon_v^{(1)}, \varepsilon_v^{(2)}, \ldots, \varepsilon_v^{(m)}, \delta_v^{(1)}, \delta_v^{(2)}, \ldots, \delta_v^{(n)} \in K_v \setminus \{0\}$ with $\mathbf{t} \in \mathcal{T}$ given by (3-1) and $\|p_i|_{\mathcal{T}}(\mathbf{t})\|_\infty \gg 1$. Then*

$$(g_{\varepsilon_v,\delta_v})_v (g_{A_v})_v O_S^{m+n} \notin \Omega_{(m+n)^{|S_r|/2+|S_c|}\varepsilon}.$$

PROOF. Let $A = (A_v)_v \in DI_\varepsilon(m, n, K_S, \mathcal{T})$. Then, given any $\varepsilon_v^{(1)}, \varepsilon_v^{(2)}, \ldots, \varepsilon_v^{(m)}, \delta_v^{(1)}, \delta_v^{(2)}, \ldots, \delta_v^{(n)}$ in $K_v \setminus \{0\}$ for $v \in S$, with $\mathbf{t} \in \mathcal{T}$, where $\mathbf{t}$ is given by (3-1), and $\|p_i|_{\mathcal{T}}(\mathbf{t})\|_\infty >> 1$ for each $i = 1, \ldots, \ell$, there exists $\vec{x} = (x_1, \ldots, x_n) \in O_S^n \setminus \{0\}$ and $\vec{y} = (y_1, \ldots, y_m) \in O_S^m$ such that (3-2) holds. Observe that

$$\begin{aligned}(g_{\varepsilon_v,\delta_v})_v (g_{A_v})_v O_S^{m+n} = \{&(\varepsilon_v^{(1)^{-1}}(A_v^{(1)}\vec{b} - a_1), \ldots, \varepsilon_v^{(m)^{-1}}(A_v^{(m)}\vec{b} - a_m),\\ &\delta_v^{(1)^{-1}} b_1, \ldots, \delta_v^{(n)^{-1}} b_n)_v : \mathbf{z} = (\mathbf{z}^{(v)})_v \in O_S^{m+n}\},\end{aligned}$$

where $\mathbf{z}^{(v)} = (\vec{a}, \vec{b}) \in K^{m+n}$ for each $v \in S$, $\vec{a} = (a_1, \ldots, a_m) \in O_S^m$ and $\vec{b} = (b_1, \ldots, b_n) \in O_S^n$.

Consider $\vec{x} \in O_S^n \setminus \{0\}$ and $\vec{y} \in O_S^m$ satisfying (3-2) as elements of $K_S^n \setminus \{0\}$ and K_S^m, respectively, and denote them as $\mathbf{x} = (\mathbf{x}^{(v)})_v \in O_S^n \setminus \{0\}$ and $\mathbf{y} = (\mathbf{y}^{(v)})_v \in O_S^m$, respectively. Then, $(\mathbf{x}, \mathbf{y}) \in O_S^{m+n}$ and for this point, let $\mathbf{w} = (\mathbf{w}^{(v)})_v$ be the corresponding point on the lattice $(g_{\varepsilon_v,\delta_v})_v (g_{A_v})_v O_S^{m+n}$. Using (3-1) and (3-2),

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$$\begin{aligned} c(\mathbf{w}) = \prod_{v\in S} \|(\mathbf{w}^{(v)})\|_{v,2} &= \Big(\prod_{v\in S_r} \|\mathbf{w}^{(v)}\|_{v,2}\Big)\Big(\prod_{v\in S_c} \|\mathbf{w}^{(v)}\|_{v,2}\Big)\Big(\prod_{v\in S_f} \|\mathbf{w}^{(v)}\|_{v,2}\Big) \\ &< \Big(\prod_{v\in S_r} \sqrt{(m+n)\varepsilon^2}\Big)\Big(\prod_{v\in S_c} (m+n)\varepsilon^2\Big)\Big(\prod_{v\in S_f} \varepsilon\Big) \\ &= ((m+n)^{|S_r|/2}\varepsilon^{|S_r|})((m+n)^{|S_c|}\varepsilon^{2|S_c|})(\varepsilon^{|S_f|}) \\ &= (m+n)^{|S_r|/2+|S_c|}\varepsilon^{|S_r|+2|S_c|+|S_f|} \\ &< (m+n)^{|S_r|/2+|S_c|}\varepsilon, \end{aligned}$$

since $d_v \geq 1$, $|S_r| + 2|S_c| + |S_f| \geq |S| \geq 1$ and $0 < \varepsilon < 1$, where for any finite set A, $|A|$ denotes the number of elements of A.

Therefore, $\delta(\Lambda) < (m+n)^{|S_r|/2+|S_c|}\varepsilon$, where $\Lambda = (g_{\varepsilon_v,\delta_v})_v (g_{A_v})_v \mathcal{O}_S^{m+n}$ and so we finally get our desired result. □

Using the above proposition, we can say that given a Besicovitch metric space X, a Federer measure μ on X, an open subset U of X with $\mu(U) > 0$ and a map $F : U \longrightarrow M_{m\times n}(K_S)$ to prove $F_*\mu(DI_\varepsilon(m,n,K_S,\mathcal{T})) = \mu(F^{-1}(DI_\varepsilon(m,n,K_S,\mathcal{T}))) = 0$, it is enough to show that

$$\mu(F^{-1}(\{(A_v)_v \in M_{m\times n}(K_S) : (g_{\varepsilon_v,\delta_v})_v (g_{A_v})_v \mathcal{O}_S^{m+n} \notin \Omega_{(m+n)^{|S_r|/2+|S_c|}\varepsilon}\})) = 0 \tag{4-1}$$

for all $\varepsilon_v^{(1)}, \varepsilon_v^{(2)}, \ldots, \varepsilon_v^{(m)}, \delta_v^{(1)}, \delta_v^{(2)}, \ldots, \delta_v^{(n)} \in K_v \setminus \{0\}$ (for each $v \in S$) with $\mathbf{t} \in \mathcal{T}$ and $\|p_i|_{\mathcal{T}}(\mathbf{t})\|_\infty >> 1$ for each $i = 1, \ldots, \ell$, where $\mathbf{t}$ is given by (3-1).

We suppose that for any ball $B \subseteq U$ centred in $\operatorname{supp}(\mu)$, there exists a $c \in (0,1)$ such that

$$\begin{aligned} &\mu(B \cap F^{-1}(\{(A_v)_v \in M_{m\times n}(K_S) : (g_{\varepsilon_v,\delta_v})_v \overline{\tau}((A_v)_v) \notin \Omega_{(m+n)^{|S_r|/2+|S_c|}\varepsilon}\})) \\ &\quad = \mu(\{\mathbf{x} \in B : (g_{\varepsilon_v,\delta_v})_v \overline{\tau}(F(\mathbf{x})) \notin \Omega_{(m+n)^{|S_r|/2+|S_c|}\varepsilon}\}) \leq c\mu(B) \end{aligned} \tag{4-2}$$

holds for all $\varepsilon_v^{(i)}$ and $\delta_v^{(j)}$ satisfying the aforementioned conditions.

If (4-2) holds true, then for any ball $B \subseteq U$ centred in $\operatorname{supp}(\mu)$,

$$\begin{aligned} &\frac{\mu(B \cap F^{-1}(\{(A_v)_v \in M_{m\times n}(K_S) : (g_{\varepsilon_v,\delta_v})_v \overline{\tau}((A_v)_v) \notin \Omega_{(m+n)^{|S_r|/2+|S_c|}\varepsilon}\}))}{\mu(B)} \\ &\quad \leq \frac{c\mu(B)}{\mu(B)} = c < 1 \end{aligned} \tag{4-3}$$

for all $\varepsilon_v^{(1)}, \varepsilon_v^{(2)}, \ldots, \varepsilon_v^{(m)}, \delta_v^{(1)}, \delta_v^{(2)}, \ldots, \delta_v^{(n)} \in K_v \setminus \{0\}$ (for each $v \in S$) with $\mathbf{t} \in \mathcal{T}$ and $\|p_i|_{\mathcal{T}}(\mathbf{t})\|_\infty >> 1$ for each $i = 1, \ldots, \ell$.

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Since (4-3) holds for any ball $B \subseteq U$, it follows that no point $\mathbf{x} \in U \cap \text{supp}(\mu)$ is a point of density of the sets

$$F^{-1}(\{(A_v)_v \in M_{m\times n}(K_S) : (g_{\varepsilon_v,\delta_v})_v \overline{\tau}((A_v)_v) \notin \Omega_{(m+n)^{|S_r|/2+|S_c|}\varepsilon}\})$$

for all $\varepsilon_v^{(1)}, \varepsilon_v^{(2)}, \ldots, \varepsilon_v^{(m)}, \delta_v^{(1)}, \delta_v^{(2)}, \ldots, \delta_v^{(n)} \in K_v \setminus \{0\}$ (for each $v \in S$) with $\mathbf{t} \in \mathcal{T}$ and $\|p_i|_{\mathcal{T}}(\mathbf{t})\|_\infty >> 1$ for each $i = 1, \ldots, \ell$, which gives us (4-1), because of the following version of the Lebesgue density theorem for our concerned metric space $X = \prod_{v\in S} K_v^{l_v}$, where $l_v \in \mathbb{N}$, endowed with the product metric.

LEMMA 4.2. *Let $X = \prod_{v\in S} K_v^{l_v}$, where $l_v \in \mathbb{N}$, be endowed with the product metric. If μ is a Radon measure on X and Ω is any measurable subset of X, then almost every point $\mathbf{x} \in \Omega \cap$ supp (μ) is a point of density of Ω, that is,*

$$\lim_{\substack{\mu(B)\to 0 \\ \mathbf{x}\in B}} \frac{\mu(B\cap\Omega)}{\mu(B)} = 1\,.$$

The proof of Lemma 4.2 goes along the same lines as the Lebesgue density theorem for Euclidean spaces with some appropriate modifications. Hence, we leave it to the reader.

As the space of our interest X is locally compact, Hausdorff and second countable, to prove Theorem 3.13, it is enough to prove the theorem locally, that is, after choosing a suitable $\varepsilon_0 > 0$, it is enough to show that for all $\mathbf{w} \in U \cap$ supp (μ), there exists a ball $B' \subseteq U$ containing $\mathbf{w}$ such that

$$\mu(B' \cap \mathbf{f}^{-1}(DI_\varepsilon(1, n, K_S, \mathcal{T}))) = \mu(\{\mathbf{x} \in B' : \mathbf{f}(\mathbf{x}) \in DI_\varepsilon(1, n, K_S, \mathcal{T})\}) = 0$$

for all $\varepsilon < \varepsilon_0$. From the above discussion, it follows that Theorem 3.13 is an immediate consequence of the following proposition.

PROPOSITION 4.3. *Fix $n \in \mathbb{N}$, and for each $v \in S$, let $l_v \in \mathbb{N}$. Set $X_v := K_v^{l_v}$ and $X := \prod_{v\in S} X_v$. For each $v \in S$, suppose that μ_v is a measure on X_v, and let $\mu := \prod_{v\in S} \mu_v$ be a Radon measure on X. Let $\mathbf{x} = (x_v)_v \in$ supp (μ) and $r > 0$. For each $v \in S$, define $B_v := B(x_v, r)$, $\tilde{B}_v := B(x_v, 3^{n+1}r)$. Set*

$$B := \prod_{v\in S} B_v \quad \text{and} \quad \tilde{B} := \prod_{v\in S} \tilde{B}_v = 3^{n+1}B.$$

Suppose that μ is D-Federer on $\tilde{B}$. For every $v \in S$, let $\mathbf{f}^{(v)} = (f_1^{(v)}, \ldots, f_n^{(v)}) : \widetilde{B}_v \longrightarrow K_v^n$ be continuous, and define $\mathbf{f} := (\mathbf{f}^{(v)})_v : \widetilde{B} \longrightarrow K_S^n$. Suppose that, for every $v \in S$, there exist $C_v, \alpha_v > 0$ such that:

(i) *any K_v-linear combination of $1, f_1^{(v)}, \ldots, f_n^{(v)}$ is (C_v, α_v) good on $\tilde{B}_v$ with respect to μ_v for each $v \in S$; and*
(ii) *the restrictions of $1, f_1^{(v)}, \ldots, f_n^{(v)}$ to $B(x_v, r) \cap \text{supp}(\mu_v)$ are linearly independent over K_v for each $v \in S$.*

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Then there exist constants $\alpha, \rho, \widetilde{C} > 0$ with the following property. For each $v \in S$, choose $\varepsilon_v^{(1)}, \dots, \varepsilon_v^{(m)}, \delta_v^{(1)}, \dots, \delta_v^{(n)} \in K_v \setminus \{0\}$. Let $\mathbf{t} \in \mathcal{T}$ be given by (3-1), and suppose that $\|p_i|_{\mathcal{T}}(\mathbf{t})\|_\infty \gg 1$ for each $i = 1, \dots, \ell$. Then for any $0 < \varepsilon \le \rho/\sqrt{D_K}$,

$$\mu(\{\mathbf{x} \in B \,:\, (g_{\varepsilon_v,\delta_v})_v \overline{\tau}(\mathbf{f}(\mathbf{x})) \notin \Omega_{(n+1)^{|S_r|/2+|S_c|}\varepsilon}\}) \le \widetilde{C}\varepsilon^\alpha \mu(B), \tag{4-4}$$

where the constants ρ and $\widetilde{C}$ are given explicitly by (5-4) and (5-5), respectively.

Theorem 3.13 follows immediately from Proposition 4.3. We choose our $0 < \varepsilon_0 \le (\rho/\sqrt{D_K})$ so that $\widetilde{C}\varepsilon_0^\alpha < 1$. In view of the expression for $\widetilde{C}$, (5-5), it is easy to see that ε_0 depends only on $(n, C_{v_1}, \dots, C_{v_\ell}, \alpha_{v_1}, \dots, \alpha_{v_\ell}, D, N_X, K, S)$. Consider $\mathbf{x} \in U \cap \operatorname{supp}(\mu)$ and choose a ball B' containing $\mathbf{x}$ such that $B' \subseteq \tilde{B}' := 3^{n+1}B' \subseteq U$. Next, choose any ball $B \subseteq B'$ with centre in $\operatorname{supp}(\mu)$ and consider $\tilde{B} := 3^{n+1}B$. Since by the hypothesis of Theorem 3.13, for each $v \in S$, $(\mathbf{f}^{(v)}, \mu_v)$ is nonplanar and (C_v, α_v)-good on U_v with respect to μ_v, Conditions (i) and (ii) of Proposition 4.3 hold immediately. We set $c = \widetilde{C}\varepsilon_0^\alpha < 1$ and observe that (4-2) holds whenever $0 < \varepsilon < \varepsilon_0$, in view of Proposition 4.3. This completes the proof of Theorem 3.13, assuming Proposition 4.3.

We need to prove Proposition 4.3. We do it using the so-called 'quantitative nondivergence' technique and our next section is all about that.

5. Quantitative nondivergence and the proof of Proposition 4.3

We first recall the S-arithmetic quantitative nondivergence theorem from [27].

5.1. Quantitative nondivergence. We denote the set of all primitive submodules of $\mathcal{O}_S^m$ by $\mathfrak{P}(\mathcal{O}_S, m)$. We recall the following theorem from [27, Section 6.3].

THEOREM 5.1. *Let X be a Besicovitch metric space and μ be a D-Federer measure on X. For $m \in \mathbb{N}$, let a ball $B \subseteq X$ and a continuous map $h : \tilde{B} \longrightarrow \mathrm{GL}(m, K_S)$ be given, where $\tilde{B} := 3^m B$. Suppose that for some $C, \alpha > 0$ and $0 < \rho < 1$, one has:*

(C1) *for every $\Delta \in \mathfrak{P}(\mathcal{O}_S, m)$, the function $\operatorname{cov}(h(\cdot)\Delta)$ is (C, α)-good on $\tilde{B}$ with respect to μ;*

(C2) *for every $\Delta \in \mathfrak{P}(\mathcal{O}_S, m)$, $\|\operatorname{cov}(h(\cdot))\Delta\|_{\mu,B} \ge \rho$.*

Then, for any positive $\varepsilon \le \rho/\sqrt{D_K}$,

$$\mu(\{x \in B \,:\, \delta(h(x)\mathcal{O}_S^m) \,<\, \varepsilon\}) \le mC(N_X D^2)^m \Big(\frac{\varepsilon\sqrt{D_K}}{\rho}\Big)^\alpha \mu(B)\,. \tag{5-1}$$

For more details and the proof of Theorem 5.1, see [27].

5.2. The proof of Proposition 4.3. From the definition of Ω_ε, it follows that

$$(g_{\varepsilon_v,\delta_v})_v \overline{\tau}(\mathbf{f}(\mathbf{x})) \notin \Omega_{(n+1)^{|S_r|/2+|S_c|}\varepsilon} \iff \delta((g_{\varepsilon_v,\delta_v})_v (g_{\mathbf{f}^{(v)}(x_v)})_v \mathcal{O}_S^{n+1}) < (n+1)^{|S_r|/2+|S_c|}\varepsilon\,.$$

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We use Theorem 5.1 in the setting

$$X = \prod_{v\in S} X_v = \prod_{v\in S} K_v^{l_v}, m = n+1;$$
$$\mu, B, C_v, \alpha_v \text{ and } D \text{ as in Proposition 4.3;}$$
$$h(\mathbf{x}) = (g_{\varepsilon_v,\delta_v})_v (g_{\mathbf{f}^{(v)}(x_v)})_v \text{ for all } \mathbf{x} = (x_v)_v \in \tilde{B},$$

where

$$g_{\varepsilon_v,\delta_v} := \operatorname{diag}((\varepsilon_v^{(1)})^{-1}, (\delta_v^{(1)})^{-1}, (\delta_v^{(2)})^{-1}, \ldots, (\delta_v^{(n)})^{-1}) \in \mathrm{GL}(n+1, K_v),$$

with $\varepsilon_v^{(i)}$ and $\delta_v^{(j)}$ satisfying the aforementioned conditions and

$$g_{\mathbf{f}^{(v)}(x_v)} := \begin{bmatrix} 1 & \mathbf{f}^{(v)}(x_v) \\ 0 & I_n \end{bmatrix} \in \mathrm{GL}(n+1, K_v).$$

To get our desired result, (4-4), we just need to verify the two conditions (C1) and (C2) of Theorem 5.1. Observe that to check Conditions (C1) and (C2), we first need explicit expressions for functions $\mathbf{x} \mapsto \operatorname{cov}(h(\mathbf{x})\Delta)$ in our setting.

We recall that given any $\Delta \in \mathfrak{P}(\mathcal{O}_S, n+1)$ of rank j, there exists an element $\mathbf{w} \in \bigwedge^j(\mathcal{O}_S^{n+1})$ such that

$$\operatorname{cov}(\Delta) = a_K c(\mathbf{w}),$$

where $a_K \in \mathbb{R}$ is a constant depending on K. Therefore, we have $\operatorname{cov}(h(\mathbf{x})\Delta) = a_K c(h(\mathbf{x})\mathbf{w})$.

Before proceeding further, let us fix some notation. We take $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n \in K_S^{n+1}$ as the standard basis of K_S^{n+1} over K_S, where

$$\mathbf{e}_i = (\mathbf{e}_i^{(v)})_v, \quad \mathbf{e}_i^{(v)} \in K_v^{n+1} \quad \text{for each } i = 0, \ldots, n;$$

and $\{\mathbf{e}_i^{(v)}\}_{i=0}^n$ forms the standard basis of K_v^{n+1} over K_v for each $v \in S$. Let $I = \{i_1, i_2, \ldots, i_j\} \subset \{0, 1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_j$ and $\mathbf{e}_I := \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \cdots \wedge \mathbf{e}_{i_j} \in \bigwedge^j(K_S^{n+1})$. Then, the standard basis of $\bigwedge^j(K_S^{n+1})$ is given by

$$\{\mathbf{e}_I : I = \{i_1, i_2, \ldots, i_j\} \subset \{0, 1, \ldots, n\} \text{ with } i_1 < i_2 < \cdots < i_j\},$$

and for any element $\mathbf{b} = \sum_{\substack{I \subseteq \{0,1,\ldots,n\}, \\ \#I = j}} b_I \mathbf{e}_I$, we define $\|\mathbf{b}\| = \max_I \|b_I\|$.

Hence, we can write the earlier mentioned $\mathbf{w} \in \bigwedge^j(\mathcal{O}_S^{n+1})$ as $\mathbf{w} = \sum_I w_I \mathbf{e}_I$, where $w_I \in \mathcal{O}_S$. We now find the expression of $(g_{\varepsilon_v,\delta_v})_v (g_{\mathbf{f}^{(v)}(x_v)})_v \mathbf{w}$ componentwise. Note that $g_{\mathbf{f}^{(v)}(x_v)}$ leaves $\mathbf{e}_0^{(v)}$ invariant and sends $\mathbf{e}_i^{(v)}$ to $f_i^{(v)}(x_v)\mathbf{e}_0^{(v)} + \mathbf{e}_i^{(v)}$ for all $i = 1, \ldots, n$. Therefore,

$$(g_{\mathbf{f}^{(v)}(x_v)})(\mathbf{e}_I^{(v)}) = \begin{cases} \mathbf{e}_I^{(v)} & \text{if } 0 \in I, \\ \mathbf{e}_I^{(v)} + \displaystyle\sum_{i\in I} \pm f_i^{(v)}(x_v)\mathbf{e}_{I\cup\{0\}\setminus\{i\}}^{(v)} & \text{otherwise,} \end{cases}$$

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and

$$(g_{\varepsilon_v,\delta_v})(g_{\mathbf{f}^{(v)}(x_v)})(\mathbf{e}_I^{(v)}) = \begin{cases} \varepsilon_v^{(1)-1}\Big(\prod_{i\in I\setminus\{0\}} \delta_v^{(i)-1}\Big)\mathbf{e}_I^{(v)} & \text{if } 0\in I,\\ \Big(\prod_{i\in I}\delta_v^{(i)-1}\Big)\mathbf{e}_I^{(v)} \pm \varepsilon_v^{(1)-1}\sum_{i\in I}\Big(\prod_{r\in I\setminus\{i\}}\delta_v^{(r)-1}\Big)f_i^{(v)}(x_v)\mathbf{e}^{(v)}_{I\cup\{0\}\setminus\{i\}} & \text{otherwise.}\end{cases}$$

Thus, $(g_{\varepsilon_v,\delta_v})_v(g_{\mathbf{f}^{(v)}(x_v)})_v\mathbf{w}$ is given by

$$((g_{\varepsilon_v,\delta_v})_v(g_{\mathbf{f}^{(v)}(x_v)})_v\mathbf{w})^{(v)} = \sum_{0\notin I}\Big(\prod_{i\in I}\delta_v^{(i)-1}\Big)w_I\mathbf{e}_I^{(v)} + \varepsilon_v^{(1)-1}\sum_{0\in I}\Big(\Big(\prod_{i\in I\setminus\{0\}}\delta_v^{(i)-1}\Big)w_I + \sum_{i\notin I}\pm\Big(\prod_{r\in I\cup\{i\}\setminus\{0\}}\delta_v^{(r)-1}\Big)f_i^{(v)}(x_v)w_{I\cup\{i\}\setminus\{0\}}\Big)\mathbf{e}_I^{(v)}. \tag{5-2}$$

We now check the conditions (C1) and (C2).

• *Checking Condition (C1).* Due to (5-2), we can form an explicit expression for $\operatorname{cov}(h(\mathbf{x})\Delta)$ and that is given by

$$\operatorname{cov}(h(\mathbf{x})\Delta) = a_K c((g_{\varepsilon_v,\delta_v})_v(g_{\mathbf{f}^{(v)}(x_v)})_v\mathbf{w}) = a_K\prod_v\Big\|\Big((g_{\varepsilon_v,\delta_v})_v(g_{\mathbf{f}^{(v)}(x_v)})_v\mathbf{w}\Big)^{(v)}\Big\|_{v,2}.$$

In view of (5-2), it is easy to observe that the components of all the coordinates of

$$((g_{\varepsilon_v,\delta_v})_v(g_{\mathbf{f}^{(v)}(x_v)})_v\mathbf{w})^{(v)}$$

are linear combinations of $1, f_1^{(v)},\dots,f_n^{(v)}$. Then, Condition (C1) immediately follows from Hypothesis (i) of Proposition 4.3, Lemma 3.9(b,c) and Lemma 3.10. So, there exists $C,\alpha>0$ such that $\mathbf{x}\mapsto\operatorname{cov}(h(\mathbf{x})\Delta)$ is (C,α)-good on $\tilde{B}$ with respect to μ, where C and α are explicitly computed in terms of $C_{v_1},\dots,C_{v_\ell},\alpha_{v_1},\dots,\alpha_{v_\ell}$.

• *Checking Condition (C2).* By the compactness of the unit sphere in K_v^{n+1} and Hypothesis (ii) of Proposition 4.3, there exists $\rho_v>0$ such that for any $\mathbf{a}=(a_0,a_1,\dots,a_n)\in K_v^{n+1}$,

$$\sup_{x_v\in B_v\cap\operatorname{supp}(\mu_v)}|a_0+a_1f_1^{(v)}(x_v)+\cdots+a_nf_n^{(v)}(x_v)|_v\ge\rho_v\|\mathbf{a}\|_{v,2}, \tag{5-3}$$

where $B=\prod_{v\in S}B_v$.

Using (5-2) and (5-3),

$$\sup_{x_v\in B_v\cap\operatorname{supp}(\mu_v)}\Big\|\Big((g_{\varepsilon_v,\delta_v})_v(g_{\mathbf{f}^{(v)}(x_v)})_v\mathbf{w}\Big)^{(v)}\Big\|_{v,2}\ge\rho_v|\varepsilon_v^{(1)}|_v^{-1}\Big(\prod_{i=1}^n|\delta_v^{(i)}|_v^{-1}\Big)\max_I|w_I|_v,$$

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since $|\delta_v^{(i)}|_v \geq 1$ for all $i = 1, \ldots, n$ and each $v \in S$. Therefore,

$$\sup_{x \in B \cap \operatorname{supp}(\mu)} c\Big((g_{\varepsilon_v,\delta_v})_v (g_{\mathbf{f}^{(v)}(x_v)})_v \mathbf{w}\Big) \geq \prod_{v\in S}\Big(\rho_v |\varepsilon_v^{(1)}|_v^{-1}\Big(\prod_{i=1}^{n} |\delta_v^{(i)}|_v^{-1}\Big) \max_I |w_I|_v\Big)$$
$$\geq \Big(\prod_{v\in S} |\varepsilon_v^{(1)}|_v |\delta_v^{(1)}|_v \cdots |\delta_v^{(n)}|_v\Big)^{-1} \max_I c(w_I)\Big(\prod_{v\in S}\rho_v\Big)$$

and, hence,

$$\|\operatorname{cov}(h(\mathbf{x}))\Delta\|_{\mu,B} \geq (\sqrt{D_K})^j \Big(\prod_{v\in S} |\varepsilon_v^{(1)}|_v |\delta_v^{(1)}|_v \cdots |\delta_v^{(n)}|_v\Big)^{-1}\Big(\prod_{v\in S}\rho_v\Big),$$

since $a_K \geq (\sqrt{D_K})^j$ and $w_I \in \mathcal{O}_S$ for each I.

Let

$$\tilde{\rho} \stackrel{\text{def}}{=} (\sqrt{D_K})^j \Big(\prod_{v\in S} |\varepsilon_v^{(1)}|_v |\delta_v^{(1)}|_v \cdots |\delta_v^{(n)}|_v\Big)^{-1}\Big(\prod_{v\in S}\rho_v\Big)$$
$$= (\sqrt{D_K})^j (\text{const}_K)^{-(n+1)}\Big(\prod_{v\in S}\rho_v\Big).$$

So, Condition (C2) holds with

$$\rho = \min\{1, \tilde{\rho}\}. \tag{5-4}$$

As a consequence of Theorem 5.1, Proposition 4.3 follows immediately with

$$\tilde{C} = C(N_X D^2)^{n+1}(n+1)^{1+((\alpha|S_r|)/2)+\alpha|S_c|}\Big(\frac{\sqrt{D_K}}{\rho}\Big)^{\alpha}. \tag{5-5}$$

□

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