Euler, the clothoid and $\int_0^\infty \frac{\sin x}{x}\,dx$

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Introduction

One of the many definite integrals that Euler was the first to evaluate was

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \tag{1}$$

He did this, almost as an afterthought, at the end of his short, seven-page paper catalogued as E675 in [1] and with the matter-of-fact title, *On the values of integrals from* $x = 0$ *to* $x = \infty$. It is a beautiful Euler miniature which neatly illustrates the unexpected twists and turns in the history of mathematics. For Euler's derivation of (1) emerges as the by-product of a solution to a problem in differential geometry concerning the clothoid curve which he had first encountered nearly forty years earlier in his paper E65, [1]. As highlighted in the recent *Gazette* article [2], E675 is notable for Euler's use of a complex number substitution to evaluate a real-variable integral. He used this technique in about a dozen of the papers written in the last decade of his life. The rationale for this manoeuvre caused much debate among later mathematicians such as Laplace and Poisson and the technique was only put on a secure footing by the work of Cauchy from 1814 onwards on the foundations of complex function theory, [3, Chapter 1]. Euler's justification was essentially pragmatic (in agreement with numerical evidence) and by what Dunham in [4, p. 68] characterises as his informal credo, 'Follow the formulas, and they will lead to the truth.' Smithies, [3, p. 187], contextualises Euler's approach by noting that, at that time, 'a function was usually thought of as being defined by an analytic expression; by the principle of the generality of analysis, which was widely and often tacitly accepted, such an expression was expected to be valid for all values, real or complex, of the independent variable'. In this article, we examine E675 closely. We have tweaked notation and condensed the working in places to reflect modern usage. At the end, we outline what is, with hindsight, needed to make Euler's arguments watertight: it is worth noting that all of his conclusions survive intact and that the intermediate functions of one and two variables that he introduces in E675 remain the key ingredients for much subsequent work on these integrals.

The clothoid

Euler begins E675 by reprising his earlier work from Additamentum 1 of E65, [1], on the *clothoid* (or *Euler spiral* or *Cornu spiral*, [5]). As lucidly explained in the very readable historical surveys [6, 7], this plane curve originated from a problem in elasticity studied by James Bernoulli and it is characterised intrinsically by the property that its radius of curvature, ρ, is

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inversely proportional to its arc length, s. Intuitively such a curve spirals in to a limiting point – the *pole* of the clothoid – and Euler's objective, his unfinished business from E65, is to locate the position of the pole.

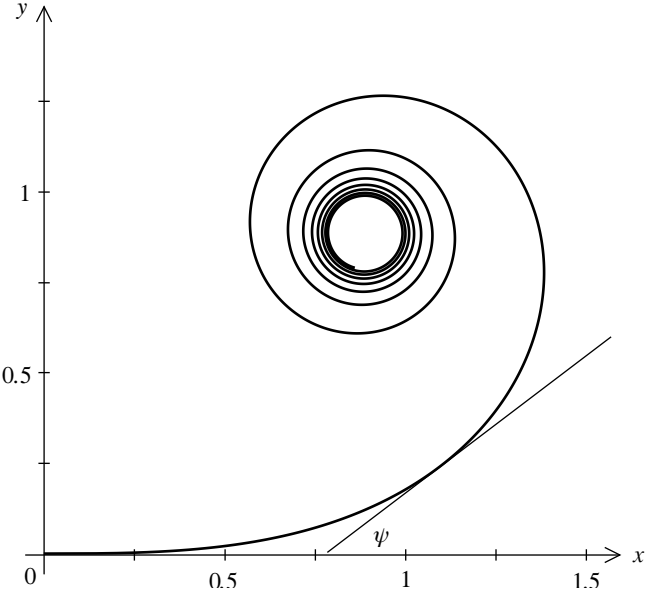


FIGURE 1: This clothoid (with $a = 1$) resembles a sketch which accompanies E675, [7, pp. 6-7]. Note that many modern authors switch the x- and y-axes and also continue the curve through negative values to produce a second pole, (e.g. [5]).

If $\rho s = a^2$ (with a constant) and if ψ is the angle that the tangent makes with the x-axis, as shown in Figure 1, then $\frac{ds}{d\psi}\, s = a^2$ so that $s = a\sqrt{2}\sqrt{\psi}$. Thus $\frac{dx}{d\psi} = \frac{dx}{ds}\cdot\frac{ds}{d\psi} = \frac{a}{\sqrt{2}}\cdot\frac{\cos\psi}{\sqrt{\psi}}$ and $\frac{dy}{d\psi} = \frac{a}{\sqrt{2}}\cdot\frac{\sin\psi}{\sqrt{\psi}}$ so that the clothoid is parametrised by the *Fresnel integrals* $x = \frac{a}{\sqrt{2}}\int_0^\psi \frac{\cos t}{\sqrt{t}}\,dt$ and $y = \frac{a}{\sqrt{2}}\int_0^\psi \frac{\sin t}{\sqrt{t}}\,dt$ with the coordinates of the pole given by

$$x = \frac{a}{\sqrt{2}}\int_0^\infty \frac{\cos t}{\sqrt{t}}\,dt, \qquad y = \frac{a}{\sqrt{2}}\int_0^\infty \frac{\sin t}{\sqrt{t}}\,dt. \tag{2}$$

It is these latter integrals that now absorb Euler's attention.

(In [8], Alex Bellos gives an entertaining account of the many uses of the clothoid as a smooth transition curve between straight and circular sections of railway tracks, road junctions and roller-coaster loop-the-loops. Indeed, he speculates on p. 229 that, 'If you were an alien flying low across a countryside of motorways and rail lines, you might well conclude that the clothoid is human civilization's favourite curve'.)

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Evaluating the Fresnel integrals

Here, typically, Euler generalises his formulae and considers the pair of integrals $C = \int_0^\infty x^{n-1}e^{-px}\cos qx\,dx$ and $S = \int_0^\infty x^{n-1}e^{-px}\sin qx\,dx$ with $n, p, q > 0$. Then

$$\begin{aligned} C + iS &= \int_0^\infty x^{n-1}e^{(-p+iq)x}dx \\ &= \frac{1}{(p-iq)^n}\int_0^\infty t^{n-1}e^{-t}dt, \text{ on substituting } t = (p-iq)x \quad (*) \\ &= \frac{(p+iq)^n}{(p^2+q^2)^n}\Gamma(n), \text{ by definition of } \Gamma(n) \\ &= \frac{\Gamma(n)}{(p^2+q^2)^{n/2}}\left(\cos n\theta + i\sin n\theta\right), \end{aligned}$$

on expressing $p + iq = \sqrt{p^2+q^2}\left(\cos\theta + i\sin\theta\right)$ with $\theta = \tan^{-1}\dfrac{q}{p}$.

Equating real and imaginary parts then gives

$$C = \frac{\Gamma(n)\cos n\theta}{(p^2+q^2)^{n/2}} \qquad \text{and} \qquad S = \frac{\Gamma(n)\sin n\theta}{(p^2+q^2)^{n/2}}. \tag{3}$$

Euler was well aware of the audacity of the key step – the complex substitution (with no change of limits) at (∗): 'only negative values of k ought to be excluded [in $t = kx$], so that even imaginary values could be used for k'. It is also worth noting that Euler has not abbreviated $\sqrt{-1}$ to i : this first occurs in one of the late bursts of papers on complex substitutions, E671, and, as Figure 2 shows, it was done quietly and without fanfare.

Quoniam mihi quidem alia adhuc via non patet istud praestandi, nisi per imaginaria procedendo, formulam $\sqrt{-1}$ littera i in posterum designabo, ita ut sit $ii = -1$, ideoque $\frac{1}{i} = -i$.

FIGURE 2: Euler's first use of i for $\sqrt{-1}$ in E671 [1].

Euler comments that the formulae in (3) are 'worthy of comment' even in the case where n is a whole number. For example, $n = 1$ gives the familiar integrals

$$\int_0^\infty e^{-px}\cos qx\,dx = \frac{p}{p^2+q^2} \qquad \text{and} \qquad \int_0^\infty e^{-px}\sin qx\,dx = \frac{q}{p^2+q^2}. \tag{4}$$

But his real goal is the Fresnel integral when $n = \frac{1}{2}$. The formulae in (3) then involve $\Gamma\left(\frac{1}{2}\right)$. Euler quotes the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ citing (without references) his previous work on this equivalent of the probability integral,

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$\int_0^\infty x^{-1/2}e^{-x}dx = 2\int_0^\infty e^{-t^2}dt$. Also,

$$\cos\tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1+\cos\theta)} = \sqrt{\frac{\sqrt{p^2+q^2}+p}{2\sqrt{p^2+q^2}}}$$

and

$$\sin\tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1-\cos\theta)} = \sqrt{\frac{\sqrt{p^2+q^2}-p}{2\sqrt{p^2+q^2}}},$$

so that (3) becomes

$$C = \int_0^\infty e^{-px}\frac{\cos qx}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\sqrt{p^2+q^2}+p}{\sqrt{p^2+q^2}}}$$

and

$$S = \int_0^\infty e^{-px}\frac{\sin qx}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\sqrt{p^2+q^2}-p}{\sqrt{p^2+q^2}}}. \tag{5}$$

As a limiting case, when $p = 0$ and $q = 1$,

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}} = \int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx \tag{6}$$

so, from (2), the pole of the clothoid above has coordinates $\left(\frac{q}{2}\sqrt{\pi}, \frac{q}{2}\sqrt{\pi}\right)$.

Evaluating $\int_0^\infty \frac{\sin x}{x}\,dx$

Finally, having used one limiting case ($p = 0$) to evaluate the sought-for Fresnel integrals, Euler now turns his attention to the trickier limiting case $n = 0$. Everything hinges on the behaviour of $\Gamma(n)\cos n\theta$ and $\Gamma(n)\sin n\theta$ as $n \to 0$. The first of these expressions diverges as $n \to 0$:

$$\Gamma(n)\cos n\theta = n\Gamma(n)\frac{\cos n\theta}{n} = \Gamma(n+1)\frac{\cos n\theta}{n} \to \infty.$$

But the second expression converges as $n \to 0$:

$$\Gamma(n)\sin n\theta = n\Gamma(n)\frac{\sin n\theta}{n} = \Gamma(n+1)\frac{\sin n\theta}{n} \to \Gamma(1)\,\theta = \theta.$$

This leads, via (3), to

$$\int_0^\infty e^{-px}\frac{\sin qx}{x}\,dx = \theta = \tan^{-1}\frac{q}{p}. \tag{7}$$

In particular, with $p = 0$ and $q = 1$, Euler (Figure 3) obtains $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, 'which … had seemed to surpass all techniques of calculation so far known'.

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T h e o r e m a III.

§. 139. Si litterae p et q denotent numeros positivos quoscunque, atque hinc quaeratur angulus ϕ, ut sit tang. $\phi = \frac{q}{p}$, habebitur sequens integratio maxime memorabilis

$$\int \frac{\partial x}{x} e^{-px} \text{ sin. } qx \left[\begin{matrix} ab\ x = 0 \\ ad\ x = \infty \end{matrix}\right] = \phi$$

cujus theorematis demonstratio dubito quin alio modo quam per approximationes investigari queat.

§. 140. Casus autem simplicissimus quo $p = 0$ et $q = 1$ jam omnia calculi artificia adhuc cognita superare videtur, quia autem hoc casu fit tang. $\phi = \frac{1}{0} = \infty$, erit $\phi = \frac{\pi}{2}$, unde oritur haec integratio $\int \frac{\partial x}{x}$ sin. $x = \frac{\pi}{2}$. Interim tamen de ejus veritate eo minus dubitare licet, quod approximationes adhibitae ad eundem valorem propemodum perducant. Quodsi hunc casum cum initio memorato $\int \frac{\partial x}{\sqrt{x}}$ sin. $x = \sqrt{\frac{\pi}{2}}$ comparemus, ingens similitudo summam attentionem meretur, cum hujus integrale sit praecise radix quadrata illius.

FIGURE 3: The final paragraphs of E675 showing Euler's evaluation of $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ using (7).

Smithies in [3, p. 14] comments on Euler's appeal to numerical approximation to corroborate (1), 'This hesitancy on Euler's part suggests that he may have had some qualms about whether his method was completely trustworthy. In fact, he describes his whole argument as being quite singular, and says that, since it appears to be potentially useful, he has expounded it in some detail.'

Later developments

A satisfactory treatment of the key step (∗) in deriving (3) had to wait for the work of Cauchy on complex function theory and his realisation of the importance of using a closed contour with due attention paid to the behaviour of the integrand at infinity and near singularities [3]. To establish (3) in the form $\int_0^\infty t^{n-1}e^{-\alpha t}dt = \frac{\Gamma(n)}{\alpha^n}$ for complex α with $\mathrm{Re}\,\alpha > 0$ and $n > 0$, it suffices to integrate $z^{n-1}e^{-z}$ around the boundary of the sector with vertices O, $R|\alpha|$, $R\alpha$ (indented at O if $0 < n < 1$) and show that the contributions from the circular arc(s) go to 0 as $R \to \infty$.

If $\mathrm{Re}\,\alpha = 0$, the same method works provided that $0 < n < 1$: full details are given by Jameson in [9, pp. 496-497]. This covers (6), but note that letting $p \to 0$ with $q = 1$ in (5) in order to infer (6) needs some care: see, for example, [13]. Similarly, it is tempting to start with the limiting case of (3) when $p = 0$, valid for $0 < n < 1$,

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$$\int_0^\infty x^{n-1} \sin x\,dx = \Gamma(n) \sin\frac{n\pi}{2} = \Gamma(n+1)\frac{\sin\frac{n\pi}{2}}{n} \tag{8}$$

and let $n \downarrow 0$ to conclude that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. However, making this watertight takes a bit more work as indicated in [9, pp. 497-498]. Other methods of evaluating the Fresnel integrals using both real and complex variables may be found in [10, p. 146, pp. 227-229, pp. 240-241] and [9, 11, 12, 13, 14]; Chen in [12] tweaks the double integral method below by considering $\int_0^\infty \int_0^\infty e^{-xy^2} \sin x\,dx\,dy$.

Contour integration gives a stand-alone proof of (1): integrate $\frac{e^{iz}-1}{z}$ around the semicircle centred on O passing through $-R$, R and Ri and let $R \to \infty$ to obtain $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. There is a plethora of proofs of (1) in the literature. In two famous *Gazette* articles, [15, 16], G. H. Hardy allocates marks to eleven such proofs according to their mathematical sophistication. Jameson includes three in his recent article, [17], and there are several more in Körner's book, [18]. A popular real variable method, [14, 19], uses a double integral (or an essentially equivalent differentiation under the integral sign) with the same integrand as that employed by Euler in (7), but with enough real analysis glue to make the details stick. Indeed, (7) may be established this way. For

$$\int_0^\infty \int_p^\infty e^{-xy} \sin qx\,dy\,dx = \int_0^\infty e^{-px}\frac{\sin qx}{x}\,dx$$

while

$$\int_p^\infty \int_0^\infty e^{-xy} \sin qx\,dy\,dx = \int_p^\infty \frac{q}{y^2+q^2}\,dy = \frac{\pi}{2} - \tan^{-1}\frac{p}{q} = \tan^{-1}\frac{q}{p}.$$

It is also worth noting that, for $p \geqslant q$, (7) can be established by using the Maclaurin expansion for $\sin qx$ and integrating term-by-term; this is a technique used by Euler for the case $p = q = 1$ [4, pp. 55-56]. Finally, just as with Fourier series where there are several specific examples in Euler's work, but no systematic theory beyond the evaluation of individual Fourier coefficients, so we have in E675 specific evaluations of Laplace transforms (with e^{-px} used as a mollifier in integrals) but no indication of any general theory.

Afterthought

Euler's papers are by no means free of typos and a glaring one occurs in the very first paragraph of E675 where he mentions two other definite integrals with limits of 0 and ∞, the well-known $\int_0^\infty \frac{1}{x^2+1}\,dx = \frac{\pi}{2}$ and his earlier result from E588 in [1],

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$$\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n\sin\frac{m\pi}{n}} \tag{9}$$

where, unfortunately, the LHS has been misprinted as $\int_0^\infty \frac{x^{m-1}}{(1+x)^n}\,dx$ (which evaluates as the beta function $B(m, n-m)$. In fact, we can connect (9) to E675 using a double integral calculation from [10, pp.130-132]. Consider $\int_0^\infty\int_0^\infty e^{-xy}y^{p-1}\sin x\,dx\,dy = \int_0^\infty\int_0^\infty e^{-xy}y^{p-1}\sin x\,dy\,dx$. The LHS simplifies to $\int_0^\infty \frac{y^{p-1}}{1+y^2}\,dy$ while, on setting $t = xy$, the RHS simplifies to $\Gamma(p)\int_0^\infty \frac{\sin x}{x^p}\,dx$. Thus

$$\int_0^\infty \frac{\sin x}{x^p}\,dx = \frac{1}{\Gamma(p)}\int_0^\infty \frac{y^{p-1}}{1+y^2}\,dy = \frac{\pi}{2\Gamma(p)\sin\frac{p\pi}{2}}, \tag{10}$$

on using (9). (It is worth noting that the latter integrals for the cases $p = 1$, corresponding to (1), and $p = \frac{1}{2}$, corresponding to (6), may be done directly.)

Formula (8) gives $\int_0^\infty \frac{\sin x}{x^p}\,dx = \Gamma(1-p)\sin(1-p)\frac{\pi}{2} = \Gamma(1-p)\cos\frac{p\pi}{2}$; this may be reconciled with (10) by recalling the reflection formula for the gamma function, $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$. (Jameson gives an alternative way of connecting (9) and (10) in [9, pp. 494-496].)

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Jordan Bell's recent verbatim translation of E675 into English, which is posted on [1] alongside the Latin original and also published on the arXiv website, [20], was immensely helpful in the preparation of this article.

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