NOTES ON PROBLEMS IN HEXAGONAL AEOLOTROPIC MATERIALS

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1. *Introduction.* Three-dimensional stress distributions in hexagonal aeolotropic materials have recently been considered by Elliott (1, 2), who obtained a general solution of the elastic equations of equilibrium in terms of two 'harmonic' functions, or, in the case of axially symmetric stress distributions, in terms of a single stress function. These stress functions are analogous to the stress functions employed to define stress systems in isotropic materials, and in the present note further problems in hexagonal aeolotropic media are solved, the method in each case being similar to that used for the corresponding problem in isotropic materials. Because of this similarity detailed explanations are unnecessary and only the essential steps in the working are given below.

Green and Willmore (3) have solved the elastic problem of an isolated line force uniformly distributed through an isotropic plate of finite thickness and acting parallel to the faces of the plate. The solution of the same problem in hexagonal aeolotropic material is found below, the faces of the plate being parallel to the hexagonal planes.

The solutions of the flat elliptical crack and punch problems in hexagonal aeolotropic media follow immediately from the solutions given by Green and Sneddon (4) for the corresponding problems in isotropic materials.

Finally, we consider the action of a point force on a semi-infinite body applied, in a direction perpendicular to the surface, to the interior of the body. The same problem for an isotropic material was first solved by Mindlin (5), but the method used by Dean, Parsons and Sneddon (6) is followed here.

2. *Fundamental formulae.* It is shown in (1) that the displacements and stresses for stress distributions in which there are no body forces can be expressed in terms of two 'harmonic' functions ϕ_1, ϕ_2 which are solutions of

$$\left(\nabla_1^2 + \frac{\partial^2}{\partial z_i^2}\right)\phi_i = 0 \quad (i = 1, 2), \tag{2·1}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad z_i = z\nu_i^{-\frac{1}{2}},$$

and ν_1 and ν_2 are the roots of the equation

$$c_{11}c_{44}\nu^2 + [c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}]\nu + c_{33}c_{44} = 0, \tag{2·2}$$

the axis of z being normal to the hexagonal planes. The roots ν_1, ν_2 may be real or complex depending upon the elastic constants $c_{11}, c_{12}, c_{33}, c_{13}, c_{44}$; they are real, for example, in the case of magnesium but are conjugate complex for zinc.

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The displacements and stresses in Cartesian coordinates (x, y, z) are

$$\left.\begin{aligned}
u &= \frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_2}{\partial x}, \quad v = \frac{\partial\phi_1}{\partial y}+\frac{\partial\phi_2}{\partial y}, \quad w = k_1\frac{\partial\phi_1}{\partial z}+k_2\frac{\partial\phi_2}{\partial z},\\
\widehat{xx} &= \left(c_{11}\frac{\partial^2}{\partial x^2}+c_{12}\frac{\partial^2}{\partial y^2}\right)(\phi_1+\phi_2)+c_{13}\left(k_1\frac{\partial^2\phi_1}{\partial z^2}+k_2\frac{\partial^2\phi_2}{\partial z^2}\right),\\
\widehat{yy} &= \left(c_{12}\frac{\partial^2}{\partial x^2}+c_{11}\frac{\partial^2}{\partial y^2}\right)(\phi_1+\phi_2)+c_{13}\left(k_1\frac{\partial^2\phi_1}{\partial z^2}+k_2\frac{\partial^2\phi_2}{\partial z^2}\right),\\
\widehat{zz} &= c_{13}\nabla_1^2(\phi_1+\phi_2)+c_{33}\left(k_1\frac{\partial^2\phi_1}{\partial z^2}+k_2\frac{\partial^2\phi_2}{\partial z^2}\right)\\
&= (k_1c_{33}-\nu_1c_{13})\frac{\partial^2\phi_1}{\partial z^2}+(k_2c_{33}-\nu_2c_{13})\frac{\partial^2\phi_2}{\partial z^2},\\
\widehat{xy} &= (c_{11}-c_{12})\frac{\partial^2}{\partial x\,\partial y}(\phi_1+\phi_2),\\
\widehat{yz} &= c_{44}\left\{(1+k_1)\frac{\partial^2\phi_1}{\partial y\,\partial z}+(1+k_2)\frac{\partial^2\phi_2}{\partial y\,\partial z}\right\},\\
\widehat{xz} &= c_{44}\left\{(1+k_1)\frac{\partial^2\phi_1}{\partial x\,\partial z}+(1+k_2)\frac{\partial^2\phi_2}{\partial x\,\partial z}\right\},
\end{aligned}\right\} \tag{2·3}$$

where k_1, k_2 are the roots of the equations

$$\frac{k(c_{13}+c_{44})+c_{44}}{c_{11}} = \frac{kc_{33}}{kc_{44}+(c_{13}+c_{44})} = \nu.$$

With cylindrical polar coordinates (r, θ, z) equations (2·3) become

$$\left.\begin{aligned}
u_r &= \frac{\partial}{\partial r}(\phi_1+\phi_2), \quad u_\theta = \frac{\partial}{r\partial\theta}(\phi_1+\phi_2), \quad w = k_1\frac{\partial\phi_1}{\partial z}+k_2\frac{\partial\phi_2}{\partial z},\\
\widehat{rr} &= \left[c_{11}\frac{\partial^2}{\partial r^2}+c_{12}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\right](\phi_1+\phi_2)+c_{13}\left(k_1\frac{\partial^2\phi_1}{\partial z^2}+k_2\frac{\partial^2\phi_2}{\partial z^2}\right),\\
\widehat{\theta\theta} &= \left[c_{12}\frac{\partial^2}{\partial r^2}+c_{11}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\right](\phi_1+\phi_2)+c_{13}\left(k_1\frac{\partial^2\phi_1}{\partial z^2}+k_2\frac{\partial^2\phi_2}{\partial z^2}\right),\\
\widehat{zz} &= (k_1c_{33}-\nu_1c_{13})\frac{\partial^2\phi_1}{\partial z^2}+(k_2c_{33}-\nu_2c_{13})\frac{\partial^2\phi_2}{\partial z^2},\\
\widehat{r\theta} &= (c_{11}-c_{12})\left[\frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta}\right](\phi_1+\phi_2),\\
\widehat{\theta z} &= c_{44}\left[(1+k_1)\frac{1}{r}\frac{\partial^2\phi_1}{\partial\theta\,\partial z}+(1+k_2)\frac{1}{r}\frac{\partial^2\phi_2}{\partial\theta\,\partial z}\right],\\
\widehat{rz} &= c_{44}\left[(1+k_1)\frac{\partial^2\phi_1}{\partial r\,\partial z}+(1+k_2)\frac{\partial^2\phi_2}{\partial r\,\partial z}\right].
\end{aligned}\right\} \tag{2·4}$$

3. *Line force in an infinite plate.* We suppose that a plate of hexagonal aeolotropic material, bounded by two plane faces which are free from traction and which are parallel to the hexagonal planes, is acted upon by a line force uniformly distributed through the plate and acting parallel to the faces of the plate. The plane-strain solution for an isolated force of intensity P per unit length directed along the x-axis and situated

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along the z-axis from $z = -\infty$ to $z = \infty$ is defined by the following displacements and stresses:

$$\left.\begin{aligned} u_r &= -\frac{P}{4\pi c_{11}(c_{11}-c_{12})}\{(3c_{11}-c_{12})\log r-(c_{11}+c_{12})\}\cos\theta,\\ u_\theta &= \frac{P}{4\pi c_{11}(c_{11}-c_{12})}\{(3c_{11}-c_{12})\log r\}\sin\theta, \quad w = 0,\\ \widehat{rr} &= -\frac{P(3c_{11}+c_{12})}{4\pi c_{11} r}\cos\theta, \quad \widehat{\theta\theta} = \frac{P(c_{11}-c_{12})}{4\pi c_{11} r}\cos\theta,\\ \widehat{zz} &= -\frac{Pc_{13}}{2\pi c_{11} r}\cos\theta, \quad \widehat{r\theta} = \frac{P(c_{11}-c_{12})}{4\pi c_{11} r}\sin\theta,\\ \widehat{\theta z} &= \widehat{rz} = 0. \end{aligned}\right\} \tag{3·1}$$

The generalized plane-stress solution is obtained from the plane-strain solution by replacing c_{11}, c_{12} by $(c_{11}c_{33}-c_{13}^2)/c_{33}$, $(c_{12}c_{33}-c_{13}^2)/c_{33}$ respectively and omitting the stress $\widehat{zz}$.

In considering an infinite plate of thickness $2h$, bounded by the planes $z = \pm h$, the boundary conditions are $\widehat{zz} = \widehat{\theta z} = \widehat{rz} = 0$ on the faces of the plate. We must therefore add to the plane-strain solution a stress system which cancels the stress $\widehat{zz}$ on the faces of the plate without introducing shearing stresses there and which has no singularities in the plate (except possibly at the points $r = 0, z = \pm h$). The additional stress system will be known if we determine ϕ_1, ϕ_2 such that they are even in z and

$$\left.\begin{aligned} (1+k_1)\frac{\partial\phi_1}{\partial z} + (1+k_2)\frac{\partial\phi_2}{\partial z} &= 0,\\ (k_1c_{33}-\nu_1c_{13})\frac{\partial^2\phi_1}{\partial z^2} + (k_2c_{33}-\nu_2c_{13})\frac{\partial^2\phi_2}{\partial z^2} &= \frac{Pc_{13}}{2\pi c_{11}}\frac{\cos\theta}{r}, \end{aligned}\right\} \tag{3·2}$$

when $z = \pm h$.

If we put $$\phi_i = \cos\theta\int_0^\infty f_i(u)\,J_1(ur)\cosh(uz_i)\,du \quad (i = 1, 2),$$

and use the result $$\int_0^\infty J_1(ur)\,du = \frac{1}{r} \quad (r \neq 0),$$

we find that the conditions (3·2) will be satisfied if

$$f_1(u) = \frac{P}{2\pi}\frac{c_{13}(1+k_2)}{c_{11}\sqrt{\nu_2}}\frac{\sinh(uh_2)}{u^2\Sigma}, \quad f_2(u) = -\frac{P}{2\pi}\frac{c_{13}(1+k_1)}{c_{11}\sqrt{\nu_1}}\frac{\sinh(uh_1)}{u^2\Sigma},$$

where

$$\Sigma = \frac{(k_1c_{33}-\nu_1c_{13})}{\nu_1}\frac{(1+k_2)}{\sqrt{\nu_2}}\sinh(uh_2)\cosh(uh_1) - \frac{(k_2c_{33}-\nu_2c_{13})}{\nu_2}\frac{(1+k_1)}{\sqrt{\nu_1}}\sinh(uh_1)\cosh(uh_2)$$

and $$h_i = h\nu_i^{-\frac{1}{2}}.$$

Thus we have

$$\phi_1 = \frac{P}{2\pi}\frac{c_{13}(1+k_2)}{c_{11}\sqrt{\nu_2}}\cos\theta\int_0^\infty\left\{\frac{\sinh(uh_2)}{u^2\Sigma}J_1(ur)\cosh(uz_1) - \frac{rh_2e^{-u}}{2u\gamma}\right\}du,$$

$$\phi_2 = -\frac{P}{2\pi}\frac{c_{13}(1+k_1)}{c_{11}\sqrt{\nu_1}}\cos\theta\int_0^\infty\left\{\frac{\sinh(uh_1)}{u^2\Sigma}J_1(ur)\cosh(uz_2) - \frac{rh_1e^{-u}}{2u\gamma}\right\}du,$$

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where $$\gamma = \frac{(k_1 c_{33} - \nu_1 c_{13})}{\nu_1} \frac{(1+k_2) h_2}{\sqrt{\nu_2}} - \frac{(k_2 c_{33} - \nu_2 c_{13})}{\nu_2} \frac{(1+k_1) h_1}{\sqrt{\nu_1}},$$

and where suitable terms which do not affect the stresses have been added to secure the convergence of the integrals at the lower limit.

On substituting these values of ϕ_1, ϕ_2 into the formulae (2·4) and adding the plane strain solution (3·1) the complete stress system is found to be

$$\left.\begin{aligned}
\frac{2\pi r \widehat{rr}}{P\cos\theta} &= -\frac{(3c_{11}+c_{12})}{2c_{11}} - \frac{c_{13} r}{4c_{11}} \left\{ \frac{(1+k_2)}{\sqrt{\nu_2}} [(3c_{11}+c_{12}) F_1 - (c_{11}-c_{12}) F_3] \right. \\
&\qquad \left. - \frac{(1+k_1)}{\sqrt{\nu_1}} [(3c_{11}+c_{12}) G_1 - (c_{11}-c_{12}) G_3] \right\} \\
&\qquad + \frac{c_{13}^2 r}{c_{11}} \left\{ \frac{k_1}{\nu_1} \frac{(1+k_2)}{\sqrt{\nu_2}} F_1 - \frac{k_2}{\nu_2} \frac{(1+k_1)}{\sqrt{\nu_1}} G_1 \right\}, \\
\frac{2\pi r \widehat{\theta\theta}}{P\cos\theta} &= \frac{(c_{11}-c_{12})}{2c_{11}} - \frac{c_{13} r}{4c_{11}} \left\{ \frac{(1+k_2)}{\sqrt{\nu_2}} [(3c_{12}+c_{11}) F_1 - (c_{12}-c_{11}) F_3] \right. \\
&\qquad \left. - \frac{(1+k_1)}{\sqrt{\nu_1}} [(3c_{12}+c_{11}) G_1 - (c_{12}-c_{11}) G_3] \right\} \\
&\qquad + \frac{c_{13}^2 r}{c_{11}} \left\{ \frac{k_1}{\nu_1} \frac{(1+k_2)}{\sqrt{\nu_2}} F_1 - \frac{k_2}{\nu_2} \frac{(1+k_1)}{\sqrt{\nu_1}} G_1 \right\}, \\
\frac{2\pi r \widehat{zz}}{P\cos\theta} &= -\frac{c_{13}}{c_{11}} + \frac{c_{13} r}{c_{11}} \left\{ \frac{(k_1 c_{33} - \nu_1 c_{13})}{\nu_1} \frac{(1+k_2)}{\sqrt{\nu_2}} F_1 - \frac{(k_2 c_{33} - \nu_2 c_{13})}{\nu_2} \frac{(1+k_1)}{\sqrt{\nu_1}} G_1 \right\}, \\
\frac{2\pi r \widehat{r\theta}}{P\sin\theta} &= \frac{c_{11}-c_{12}}{2c_{11}} + \frac{c_{13}(c_{11}-c_{12}) r}{4c_{11}} \left\{ \frac{(1+k_2)}{\sqrt{\nu_2}} (F_1+F_3) - \frac{(1+k_1)}{\sqrt{\nu_1}} (G_1+G_3) \right\}, \\
\frac{2\pi r \widehat{\theta z}}{P\sin\theta} &= -\frac{c_{44} c_{13}}{c_{11}} \frac{(1+k_1)(1+k_2)}{\sqrt{(\nu_1 \nu_2)}} \{F_2 - G_2\}, \\
\frac{2\pi r \widehat{rz}}{P\cos\theta} &= \frac{c_{44} c_{13} r}{2c_{11}} \frac{(1+k_1)(1+k_2)}{\sqrt{(\nu_1 \nu_2)}} \{(F_4 - F_5) - (G_4 - G_5)\},
\end{aligned}\right\} \tag{3·3}$$

where $$F_1(r,z) = \int_0^\infty \frac{\sinh(uh_2)}{\Sigma} \cosh(uz_1) J_1(ur)\, du,$$

$$F_2(r,z) = \int_0^\infty \frac{\sinh(uh_2)}{u\Sigma} \sinh(uz_1) J_1(ur)\, du,$$

$$F_3(r,z) = \int_0^\infty \frac{\sinh(uh_2)}{\Sigma} \cosh(uz_1) J_3(ur)\, du,$$

$$F_4(r,z) = \int_0^\infty \frac{\sinh(uh_2)}{\Sigma} \sinh(uz_1) J_0(ur)\, du,$$

$$F_5(r,z) = \int_0^\infty \frac{\sinh(uh_2)}{\Sigma} \sinh(uz_1) J_2(ur)\, du$$

and $G_i(r,z)$ is obtained from $F_i(r,z)$ by interchanging the suffixes 1 and 2 in the hyperbolic terms in the integrand.

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The average values of the stresses are found by integrating the stresses (3·3) throughout the plate, and after some reduction we find that

$$\left.\begin{aligned}
\frac{2\pi r\,\overline{\widetilde{rr}}}{P\cos\theta} &= -\frac{(3c_{11}+c_{12})}{2c_{11}}+\frac{\rho}{4}\sqrt{\left(\frac{c_{11}}{c_{33}}\right)}\frac{c_{13}^2(c_{11}-c_{12})}{c_{11}(c_{13}+c_{44})}(\nu_1-\nu_2)\,[F(\rho)+G(\rho)],\\
\frac{2\pi r\,\overline{\widetilde{\theta\theta}}}{P\cos\theta} &= \frac{(c_{11}-c_{12})}{2c_{11}}+\frac{\rho}{4}\sqrt{\left(\frac{c_{11}}{c_{33}}\right)}\frac{c_{13}^2(c_{11}-c_{12})}{c_{11}(c_{13}+c_{44})}(\nu_1-\nu_2)\,[3F(\rho)-G(\rho)],\\
\frac{2\pi r\,\overline{\widetilde{zz}}}{P\cos\theta} &= \frac{c_{13}}{c_{11}}\left\{\sqrt{\left(\frac{c_{11}}{c_{33}}\right)}\frac{(c_{11}c_{33}-c_{13}^2)}{(c_{13}+c_{44})}(\nu_1-\nu_2)\,\rho F(\rho)-1\right\},\\
\frac{2\pi r\,\overline{r\theta}}{P\sin\theta} &= \frac{c_{11}-c_{12}}{2c_{11}}+\frac{\rho}{4}\sqrt{\left(\frac{c_{11}}{c_{33}}\right)}\frac{c_{13}^2(c_{11}-c_{12})}{c_{11}(c_{13}+c_{44})}(\nu_1-\nu_2)\,[F(\rho)+G(\rho)],
\end{aligned}\right\} \tag{3·4}$$

where
$$\rho=\frac{r}{h},\quad F(\rho)=\int_0^\infty\frac{J_1(u\rho)}{u\Pi}\,du,\quad G(\rho)=\int_0^\infty\frac{J_3(u\rho)}{u\Pi}\,du,$$

and
$$\Pi(u)=\frac{(k_1c_{33}-\nu_1c_{13})}{\nu_1}\frac{(1+k_2)}{\sqrt{\nu_2}}\coth(u\nu_1^{-\frac{1}{2}})-\frac{(k_2c_{33}-\nu_2c_{13})}{\nu_2}\frac{(1+k_1)}{\sqrt{\nu_1}}\coth(u\nu_2^{-\frac{1}{2}}).$$

If we let $h\to\infty$ (i.e. $\rho\to0$) in (3·4) we obtain the plane-strain solution, while $h\to0$ (i.e. $\rho\to\infty$) gives the generalized plane-stress solution.

The problem of a line force in a plate of semi-infinite thickness can be solved in a similar manner but it will not be considered here.

4. *Elliptical crack and punch.* We suppose that the crack occupies the region

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant1,\quad z=0,$$

in a medium of infinite extent and that there is a uniform tension p at infinity normal to the crack. We may consider the resulting stress distribution to consist of the system $\widehat{zz}=p$, $\widehat{xx}=\widehat{yy}=\widehat{yz}=\widehat{zx}=\widehat{xy}=0$, together with a distribution of stress which vanishes at infinity and which satisfies the conditions

$$\left.\begin{aligned}
&\text{(i)}\ \ \widehat{xz}=\widehat{yz}=0\quad(z=0),\\
&\text{(ii)}\ \ \widehat{zz}=-p\quad\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}<1,z=0\right),\\
&\qquad w=0\quad\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}>1,z=0\right).
\end{aligned}\right\} \tag{4·1}$$

The first condition will be satisfied if we put

$$\phi_1=\frac{\sqrt{\nu_1}}{1+k_1}\Phi(x,y,z_1),\quad \phi_2=-\frac{\sqrt{\nu_2}}{1+k_2}\Phi(x,y,z_2), \tag{4·2}$$

where
$$\nabla^2\Phi(x,y,z)=0.$$

It should be noted that the expressions (4·2) for ϕ_1, ϕ_2 always give zero shearing stress on the plane $z=0$.

The remaining boundary condition requires

$$\left.\begin{aligned}
&\beta\frac{\partial^2\Phi}{\partial z^2}=-p\quad\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}<1,z=0\right),\\
&\text{and}\qquad\frac{\partial\Phi}{\partial z}=0\quad\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}>1,z=0\right),
\end{aligned}\right\} \tag{4·3}$$

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where $$\beta = \frac{(k_1 c_{33} - \nu_1 c_{13})}{\sqrt{(\nu_1)}\,(1+k_1)} - \frac{(k_2 c_{33} - \nu_2 c_{13})}{\sqrt{(\nu_2)}\,(1+k_2)}.$$

The boundary conditions on the harmonic function Φ are the same as those imposed upon the harmonic function used in (4), and the solution is there shown to be

$$\Phi = \frac{A}{2}\int_\lambda^\infty \left\{\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1\right\} \frac{ds}{\sqrt{\{Q(s)\}}}, \tag{4·4}$$

where A is a constant to be determined, $Q(s) = s(a^2+s)(b^2+s)$ and (λ, μ, ν) are the ellipsoidal coordinates of the point (x, y, z) and are the roots of

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{\theta} - 1 = 0 \quad (\infty > \lambda \geqslant 0 \geqslant \mu \geqslant -b^2 \geqslant \nu \geqslant -a^2).$$

In these coordinates the inside of the ellipse is given by $\lambda = 0$ and the outside by $\mu = 0$.

When $\lambda = 0$, that is, inside the ellipse, we have

$$\frac{\partial^2 \Phi}{\partial z^2} = -A\int_0^\infty \frac{(2s + a^2 + b^2)}{(a^2+s)(b^2+s)} \frac{ds}{\sqrt{\{Q(s)\}}} = -\frac{2A}{ab^2} E(K),$$

where $$E(K) = \int_0^K \mathrm{dn}^2 t\, dt,$$

and $\mathrm{sn}\, t$ is the Jacobian elliptic function which has real and imaginary periods $4K$, $2iK'$ respectively, corresponding to the modulus k and the complementary modulus k', where

$$ka = (a^2 - b^2)^{\frac{1}{2}}, \quad k'a = b.$$

To satisfy (4·3) we must have $A = ab^2 p / 2\beta E(K)$ and the solution is now complete.

We find that inside the ellipse

$$w = \alpha \left(\frac{\partial \Phi}{\partial z}\right)_{\lambda=0} = \frac{2\alpha A}{ab}\left\{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right\}^{\frac{1}{2}},$$

where $$\alpha = \frac{k_1}{1+k_1} - \frac{k_2}{1+k_2},$$

giving the normal displacement at all points of the crack. This displacement, apart from a multiplying constant, is the same as that for an isotropic medium. If we let $\nu_1, \nu_2 \to 1$ we find that the expression for w agrees with the result for isotropic materials.

The stresses and displacements can now be obtained from the formulae (4·4), (4·2) and (2·3). When ν_1, ν_2 are real and positive (which is the case for magnesium) the equation

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z_i^2}{\theta} - 1 = 0$$

has three real roots and the stresses and displacements can be obtained as in the isotropic case. For complex and negative values of ν_1, ν_2, however, the equation may have complex roots, but the displacements and stresses in the plane $z = 0$ can be determined without difficulty.

We now consider the indentation of a semi-infinite hexagonal aeolotropic medium $z \geqslant 0$ by a flat-ended elliptical punch. Since we assume that there is no shearing stress

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on the plane $z = 0$ we again take ϕ_1, ϕ_2 to be given by (4·2). The function Φ must be such that

$$\left.\begin{aligned} w = \alpha\frac{\partial\Phi}{\partial z} = \epsilon \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2} < 1, z = 0\right), \\ \frac{\partial^2\Phi}{\partial z^2} = 0 \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2} > 1, z = 0\right), \end{aligned}\right\}$$

where ϵ is a small constant.

These conditions show that $\partial\Phi/\partial z$ represents the velocity potential of the motion of a perfect fluid flowing through an elliptical hole in a thin rigid boundary. Hence, as in (4), we have

$$\frac{\partial\Phi}{\partial z} = \frac{\epsilon a}{2\alpha K}\int_\lambda^\infty \frac{ds}{\sqrt{\{Q(s)\}}}, \tag{4·5}$$

where K is defined above.

Although the harmonic function Φ such that $\partial\Phi/\partial z$ is given by (4·5) has not yet been found it is possible to obtain w, $\widehat{zz}$, $\widehat{xz}$, $\widehat{yz}$ and $\widehat{xx}+\widehat{yy}$ from the formulae (2·3). It is found that

$$\widehat{zz} = -\frac{\beta\epsilon}{\alpha b K}\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{-\frac{1}{2}}$$

over the area of indentation, while the total pressure exerted by the punch is

$$P = \frac{2\pi\beta a\epsilon}{\alpha K}.$$

If we let $\nu_1, \nu_2 \to 1$, the value of P then agrees with the value for isotropic materials.

Elliott (2) has solved circular punch and crack problems in hexagonal aeolotropic materials with the aid of the Hankel transform, but they could be solved in a similar manner to punch and crack problems considered in this section; the method would then be analogous to that given by Green (7) for isotropic media.

5. *Isolated force with plane boundary.* In this section we consider a semi-infinite hexagonal aeolotropic body $z \leqslant 0$ acted upon by a point force in the z direction in the interior of the body.

The functions

$$\left.\begin{aligned} \phi_1 = \frac{P}{8\pi(\nu_1-\nu_2)}\log\frac{R_1+z_1}{R_1-z_1}, \\ \phi_2 = -\frac{P}{8\pi(\nu_1-\nu_2)}\log\frac{R_2+z_2}{R_2-z_2}, \end{aligned}\right\} \tag{5·1}$$

where

$$P = F\frac{(c_{13}+c_{44})}{c_{11}c_{44}} \quad \text{and} \quad R_i^2 = r^2+z_i^2,$$

give the displacements and stresses due to a force F in the direction of the z-axis at the origin (1).

It follows from the principle of superposition that the functions

$$\left.\begin{aligned} \phi_1 = \frac{P}{8\pi(\nu_1-\nu_2)}\left\{\log\frac{\rho_1+z_1+h_1}{\rho_1-z_1-h_1}-\log\frac{\sigma_1+z_1-h_1}{\sigma_1-z_1+h_1}\right\}, \\ \phi_2 = -\frac{P}{8\pi(\nu_1-\nu_2)}\left\{\log\frac{\rho_2+z_2+h_2}{\rho_2-z_2-h_2}-\log\frac{\sigma_2+z_2-h_2}{\sigma_2-z_2+h_2}\right\}, \end{aligned}\right\} \tag{5·2}$$

where

$$\rho_i^2 = r^2+(z_i+h_i)^2, \quad \sigma_i^2 = r^2+(z_i-h_i)^2, \quad h_i = h\nu_i^{-\frac{1}{2}} \quad (i = 1, 2),$$

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give the stress distribution due to equal and opposite forces F at the points $(0, 0, \pm h)$. These functions give, on the plane $z = 0$,

$$\widehat{zz} = -\frac{P}{2\pi(\nu_1-\nu_2)}\left\{\frac{(k_1c_{33}-\nu_1c_{13})}{\nu_1}\frac{h_1}{(r^2+h_1^2)^{\frac{3}{2}}}-\frac{(k_2c_{33}-\nu_2c_{13})}{\nu_2}\frac{h_2}{(r^2+h_2^2)^{\frac{3}{2}}}\right\} = f(r) \quad \text{(say)},$$

$$\widehat{rz} = 0.$$

Since we require $\widehat{zz}$ to be zero on $x = 0$ we must add a stress system which cancels $\widehat{zz}$ on $z = 0$ without introducing shearing stresses there and which has no singularities for $z \leqslant 0$. Using the expressions (4·2) for the functions ϕ_1, ϕ_2 of this additional system, the harmonic function Φ is such that

$$\beta\left(\frac{\partial^2\Phi}{\partial z^2}\right)_{z=0} = -f(r),$$

and therefore

$$\Phi = -\frac{P}{2\pi(\nu_1-\nu_2)\beta}\left\{\frac{(k_1c_{33}-\nu_1c_{13})}{\nu_1}\log(\delta_1+h_1-z)-\frac{(k_2c_{33}-\nu_2c_{13})}{\nu_2}\log(\delta_2+h_2-z)\right\}, \quad (5\cdot3)$$

where $$\delta_i^2 = r^2+(z-h_i)^2 \quad (i = 1, 2).$$

Hence, by addition, the functions

$$\left.\begin{aligned}\phi_1 &= \frac{P}{8\pi(\nu_1-\nu_2)}\left\{\log\frac{\rho_1+z_1+h_1}{\rho_1-z_1-h_1}-\log\frac{\sigma_1+z_1-h_1}{\sigma_1-z_1+h_1}\right\}\\ &\quad -\frac{P}{2\pi\beta(\nu_1-\nu_2)}\left\{\frac{(k_1c_{33}-\nu_1c_{13})}{\sqrt{(\nu_1)}\,(1+k_1)}\log(\sigma_1+h_1-z_1)-\frac{(k_2c_{33}-\nu_2c_{13})}{\nu_2(1+k_1)}\sqrt{\nu_1}\log(\delta_{12}+h_2-z_1)\right\},\\ \phi_2 &= -\frac{P}{8\pi(\nu_1-\nu_2)}\left\{\log\frac{\rho_2+z_2+h_2}{\rho_2-z_2-h_2}-\log\frac{\sigma_2+z_2-h_2}{\sigma_2-z_2+h_2}\right\}\\ &\quad +\frac{P}{2\pi\beta(\nu_1-\nu_2)}\left\{\frac{(k_1c_{33}-\nu_1c_{13})}{\nu_1(1+k_2)}\sqrt{\nu_2}\log(\delta_{21}+h_1-z_2)-\frac{(k_2c_{33}-\nu_2c_{13})}{\sqrt{(\nu_2)}\,(1+k_2)}\log(\sigma_2+h_2-z_2)\right\},\end{aligned}\right\} \quad (5\cdot4)$$

where $$\delta_{12}^2 = r^2+(z_1-h_2)^2, \quad \delta_{21}^2 = r^2+(z_2-h_1)^2,$$

define the displacements and stresses in the semi-infinite body due to a force F acting at the point $(0, 0, -h)$ in the positive z direction.

We are mainly concerned with the conditions at the surface of the body. On the plane $z = 0$ we have

$$u_r = -\frac{P}{2\pi(\nu_1-\nu_2)}\left\{\frac{h_1}{r(r^2+h_1^2)^{\frac{1}{2}}}-\frac{h_2}{r(r^2+h_2^2)^{\frac{1}{2}}}\right\}-\frac{P}{2\pi\beta(\nu_1-\nu_2)}\left(\frac{\sqrt{\nu_1}}{1+k_1}-\frac{\sqrt{\nu_2}}{1+k_2}\right)$$
$$\times\left\{\frac{(k_1c_{33}-\nu_1c_{13})}{\nu_1}\frac{r}{[(r^2+h_1^2)^{\frac{1}{2}}+h_1](r^2+h_1^2)^{\frac{1}{2}}}-\frac{(k_2c_{33}-\nu_2c_{13})}{\nu_2}\frac{r}{[(r^2+h_2^2)^{\frac{1}{2}}+h_2](r^2+h_2^2)^{\frac{1}{2}}}\right\}, \quad (5\cdot5)$$

$$w = \frac{P\alpha}{2\pi\beta(\nu_1-\nu_2)}\left\{\frac{(k_1c_{33}-\nu_1c_{13})}{\nu_1}\frac{1}{(r^2+h_1^2)^{\frac{1}{2}}}-\frac{(k_2c_{33}-\nu_2c_{13})}{\nu_2}\frac{1}{(r^2+h_2^2)^{\frac{1}{2}}}\right\}. \quad (5\cdot6)$$

Letting $\nu_1, \nu_2 \to 1$ we find that these values agree with the corresponding results obtained in (6) for an isotropic medium.

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Since $\widehat{zz}$ is zero on $z = 0$,

$$\frac{\partial w}{\partial z} = -\frac{c_{13}}{c_{33}}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right)$$

on this plane, so that the surface stress can be found from (5·5); we have

$$\left.\begin{aligned} \widehat{rr} + \widehat{\theta\theta} &= (c_{11}+c_{12})\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right) + 2c_{13}\frac{\partial w}{\partial z} = \left(c_{11}+c_{12}-\frac{2c_{13}^2}{c_{33}}\right)\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right), \\ \widehat{rr} - \widehat{\theta\theta} &= (c_{11}-c_{12})\left(\frac{\partial u_r}{\partial r}-\frac{u_r}{r}\right) \end{aligned}\right\}$$

on $z = 0$. At the origin it is found that, after simplification,

$$\left.\begin{aligned} \widehat{rr} + \widehat{\theta\theta} &= \frac{F}{2\pi h^2}\left(c_{11}+c_{12}-\frac{2c_{13}^2}{c_{33}}\right)\frac{c_{11}c_{33}c_{44} + \sqrt{(c_{11}c_{33})}\,[c_{11}c_{33} - c_{13}(c_{13}+c_{44})]}{c_{11}c_{44}(c_{11}c_{33}-c_{13}^2)}, \\ \widehat{rr} - \widehat{\theta\theta} &= 0. \end{aligned}\right\} \tag{5·7}$$

For an isotropic medium the corresponding values are

$$\widehat{rr} + \widehat{\theta\theta} = \frac{F}{2\pi h^2}\frac{(1+\eta)(3-2\eta)}{(1-\eta)} \quad \text{and} \quad \widehat{rr} - \widehat{\theta\theta} = 0, \tag{5·8}$$

where η is Poisson's ratio.

Using the values of the elastic moduli of zinc and magnesium given by Seitz and Read (8) to determine the elastic constants, we find from (5·7) that the common values of $4\pi h^2\widehat{rr}/F$ and $4\pi h^2\widehat{\theta\theta}/F$ at the origin are 2·15 and 5·08 for zinc and magnesium respectively. The corresponding result for an isotropic medium of Poisson's ratio 0·25 is 4·17, and we see that magnesium is more nearly isotropic than zinc.

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