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Infinite matroids in tropical differential algebra

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Abstract. We consider a finite-dimensional vector space $W \subset K^E$ over a field K and a set E. We show that the set $\mathcal{C}(W) \subset 2^E$ of minimal supports of W are the circuits of a matroid on E. When the cardinality of K is large (compared to that of E), then the family of supports of W is a matroid. Afterwards we apply these results to tropical differential algebraic geometry (TDAG), studying the set of supports of spaces of formal power series solutions $\mathrm{Sol}(\Sigma)$ of systems of linear differential equations (LDES) Σ in variables $x_1, \dots, x_n$ having coefficients in $K[\![t_1, \dots, t_m]\!]$. If Σ is of differential type zero, then the set $\mathcal{C}(Sol(\Sigma)) \subset (2^{\mathbb{N}^m})^n$ of minimal supports defines a matroid on $E = [n] \times \mathbb{N}^m$, and if the cardinality of K is large enough, then the set of supports is also a matroid on E. By applying the fundamental theorem of TDAG (FTTDAG), we give a necessary condition under which the set of solutions $Sol(U)$ of a system U of tropical LDES is a matroid. We give a counterexample to the FTTDAG for systems Σ of LDES over countable fields for which is not a matroid.

1 Introduction

A fundamental concept in tropical algebraic geometry is the *tropicalization* $trop(X, v) \subseteq (\mathbb{R} \cup \{-\infty\})^n$ of an algebraic variety $X \subset K^n$ defined over a valued field $K = (K, v)$. Assume X is defined by an ideal I with constant coefficients (i.e., $I \subset k[x_1, \dots, x_n]$ where $k \subset K$ is a field with trivial valuation $v_0 : k \to \{-\infty, 0\}$). In this case, we may consider the K-points of X and tropicalize with respect to v which yields a polyhedral fan $trop(X, v)$. A classical result from tropical algebraic

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geometry states that $trop(X, v)$ coincides with the Bergman fan $\mathcal{B}(X)$ [26, Theorem 9.6]. If X is a linear space, the Bergman fan can be obtained from its matroid $M(X) = (\{1, \ldots, n\}, \mathcal{C}(X))$ (see, e.g., [21, p. 165] or [2]). On the other hand, we may consider the trivial valuation $v = v_0$. Note that the set

$$v_0(X) := \{(v_0(p_1), \ldots, v_0(p_n)) \in \{-\infty, 0\}^n \ : \ (p_1, \ldots, p_n) \in X\}, \tag{1.1}$$

consists of all the *supports* of the points of X: it suffices to consider the identification $\{-\infty, 0\}^n \cong 2^{[n]}$ as the set of all the indicator functions of the subsets of $[n]$. In general, we have $v_0(X) \subseteq trop(X, v_0)$, and this inclusion may be proper; for example, if $X \subset \mathbb{F}_2^3$ is the linear space spanned by $\{(0,1,1), (1,0,1)\}$, then

$$v_0(X) \cup \{(0,0,0)\} = trop(X, v_0),$$

which says that (1.1) carries potentially more information than the matroid $M(X)$. We show that, if $X \subset K^E$ is a finite-dimensional vector space, then the above situation cannot happen if the cardinality of K is large enough with respect to the cardinality of the set E; that is, the set of supports $v_0(X) \subset 2^E \cong \{0, -\infty\}^E$ and the matroid $M(X) = (E, \mathcal{C}(X))$ associated with X can be identified with each other via

$$(E, \mathcal{C}(X)) = M(X) \underset{\text{circuits}}{\overset{\text{scrawls}}{\rightleftarrows}} v_0(X) = (E, \mathcal{S}(X)). \tag{1.2}$$

To do this, we use the notion of infinite matroids under the cryptomorphisms of circuits [7] and scrawls [6]. Up to our knowledge, this result is new in both cases when E is finite and when E is infinite. While it is possible to obtain that $M(X) = (E, \mathcal{C}(X))$ is a matroid using the results of thin sums families and their associated thin sums systems from [1], we offer an independent proof which has the advantage of being more elementary. We believe that both proofs have their own advantages, so we have decided to include both.

Our interest in this type of questions comes from the theory of tropical differential algebraic (DA) geometry, where the tropicalization of the set of formal power series solutions of systems of homogeneous linear differential equations appears as a set of the form $v_0(X) \subset 2^E$, where E is infinite, as in (4.2). This is why in this article we deal mainly with duals of *representable matroids* over an infinite set of labels E (see Theorem 2.12), which are not representable in the usual sense (see Remark 2.13), but we show that under the above hypothesis they do arise as the semigroup of the set of supports of a vector space.

Also, we aim to further study the Boolean formal power series solutions of systems of tropical homogeneous linear differential equations. This study was initiated in [15] and it is a natural continuation for the tropical aspects of the DA theory of such systems, following the development of classical tropical geometry.

1.1 Analogies between classical and tropical DA geometry

The theory of tropical DA geometry was initiated by Grigoriev [15], the fundamental theorem was proved by Aroca, Garay, and Toghani in [3], and is becoming an established active field of research with several contributions, such as [12, 13, 16, 21].

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Since the beginning, a natural question arose: which concepts of classical tropical algebraic geometry over a valued field (K, v) can be generalized to the differential setting? More specifically, can tropical DA geometry be regarded as an infinite version of classical tropical algebraic geometry?

A cornerstone of classical tropical algebraic geometry is the fundamental theorem of tropical algebraic geometry (FTTAG, [22, Theorem 3.2.3], a generalization of Kapranov's Theorem [9] for hypersurfaces), which describes in three ways the tropicalization $trop(X, v) \subset \mathbb{R}^n$ coming from algebraic subvarieties $X \subset (K^*)^n$, where K is an algebraically closed field and $v : K^* \to \mathbb{R}$ is a non-trivial valuation. A tropical (partial) differential analog of this result (FTTDAG) was successfully constructed in [1, 10, 11] giving three different descriptions of the tropical space of Boolean formal power series $v_0(X) \subset \mathbb{B}[\![t_1, \ldots, t_m]\!]^n$ that come from DA varieties $X \subset K[\![t_1, \ldots, t_m]\!]^n$, where K is an uncountable algebraically closed field of characteristic zero and v_0 is the trivial valuation. Note that in [6] it is shown that the uncountability condition can be replaced by countably infinite transcendence degree over the field of definition of X.

A natural source of finite-dimensional vector spaces $X \subset K^E$ with E infinite are the sets of solutions of systems of homogeneous linear differential equations of differential type zero with coefficients in $K[\![t_1, \ldots, t_m]\!]$. We explore its consequences for distinct fields K satisfying or not the conditions of the FTTDAG. Our main results on matroids (Theorems 2.11 and 3.5) lead to the following result (Theorem 4.2).

Theorem 1.1 *Fix two positive integers m, n. Let $E = \mathbb{N}^m$ and let $\mathcal{T} : K^E \longrightarrow 2^E$ be the support map (Definition 2.4). Let $\Sigma \subset K_{m,n}$ be a system of homogeneous linear differential equations of differential type zero and let $\mathcal{T}(\mathrm{Sol}(\Sigma)) \subset (2^E)^n$ be the set of supports of $\mathrm{Sol}(\Sigma)$. Then*

(1) *the minimal elements $\mathcal{C}(\mathrm{Sol}(\Sigma))$ of $\mathcal{T}(\mathrm{Sol}(\Sigma))$ define the circuits of a matroid on $[n] \times \mathbb{N}^m$;*
(2) *if the cardinality of K is large enough, then $\mathcal{T}(\mathrm{Sol}(\Sigma))$ itself is the set of scrawls of $\mathcal{C}(\mathrm{Sol}(\Sigma))$.*

Obtaining that $\mathcal{T}(\mathrm{Sol}(\Sigma))$ is the set of scrawls of $\mathcal{C}(\mathrm{Sol}(\Sigma))$ (see item (1)), implies in particular that, if we have two supports of solutions of Σ, then their union appears also as the support of a solution of Σ; this is, $\mathcal{T}(\mathrm{Sol}(\Sigma))$ is a semigroup (with respect to the operation of union of sets). In item (2), the assumption on the cardinality of K is strict as we show in Section 6.

The previous result is about the structure of the tropicalization of the set of formal solutions of a classical system. On the other hand, following Grigoriev [14], one can study from an algebraic and combinatorial perspective the set of solutions $X = \mathrm{Sol}(U) \subset \mathbb{B}[\![t_1, \ldots, t_m]\!]^n$ associated with a system $U \subset \mathbb{B}_{m,n}$ of tropical linear differential equations, disregarding wether or not U is realizable in some field K, this is, independent of the existence of linear systems $\Sigma \subset K_{m,n}$ such that $U = trop(\Sigma)$ as in Definition 5.4. These sets of solutions are always semigroups even for the case of partial differential equations $m > 0$, as we show in Theorem 5.1.

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1.2 Statement of results

In Theorem 2.11, we show that if $\{0\} \neq W \subset K^E$ is a finite-dimensional K-vector subspace, then the pair $M(W) = (E, \mathcal{C}(W))$ is a matroid, where $\mathcal{C}(W) \subset 2^E$ is the set of vectors in W having minimal (nonempty) support. This seems to be known for experts, but we provide a rigorous proof of this. We also show that any element of $\mathcal{T}(W)$ is a union of circuits, and in Theorem 3.5, we show that if $\#E < \#S(K)$, where $S(K) = K \cup \{K\}$ denotes the set-successor of K, then also the converse holds true. In Theorem 5.1, we show that the set of solutions $\mathrm{Sol}(U) \subset \mathbb{B}[\![T]\!]^n$ associated with a system $U \subset \mathbb{B}_{m,n}$ of homogeneous linear tropical differential equations is a semigroup. In particular, using the FTTDAG, we give in Corollary 5.3 a necessary condition for the set of solutions $X = \mathrm{Sol}(U) \subset \mathbb{B}[\![t_1, \ldots, t_m]\!]^n$ to be a matroid (or a matroid of scrawls, as in Definition 2.3). As a consequence of the previous results, in Section 6, we give a counterexample for the FTTDAG in the case of linear differential equations over countable fields.

1.3 Roadmap

The article is organized as follows. In Section 2, we introduce standard preliminary material on matroid theory and we prove Theorem 2.11. In Section 3, we study the matroid of scrawls and prove Theorem 3.5. In Section 4, we discuss the theory of algebraic differential equations with coefficients in the ring $K[\![t_1, \ldots, t_m]\!]$ over an arbitrary field K, and we recast the result of the previous two sections for the case of homogeneous systems of linear differential equations of differential type zero. In Section 5, we discuss tropical differential equations, go further and analyze the special case in which K satisfies the hypotheses of the FTTDAG.

2 The infinite matroid induced by a finite-dimensional subspace of a vector space

In this section, we denote by $E \neq \varnothing$ an arbitrary set and by 2^E the power set of E, which is ordered by inclusion. We consider 2^E as a semigroup endowed with the set union as operation.

2.1 Basic theory of infinite matroids

A matroid on E may be given in terms of different collections of subsets of E: the circuits, the independent sets, or the bases; these are only some possibilities among the many ways of defining matroids when E is finite. The matroid axioms were shown to be equivalent by Whitney [27] in the finite case. For the infinite case, the Whitney axioms need to be completed (see [4]). In [3], another possibility to define matroids is exhibited by using scrawls.

Definition 2.1 Let $\mathcal{C} \subset 2^E$. We call $\mathcal{C}$ the *set of circuits* if it satisfies the following axioms:

(1) $\varnothing \notin \mathcal{C}$.
(2) No element of $\mathcal{C}$ is a subset of another.

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(3) Whenever $X \subset C \in \mathcal{C}$ and $\{C_x : x \in X\}$ is a family of elements of $\mathcal{C}$ such that $x \in C_y$ iff $x = y$ for all $x, y \in X$, then for every $z \in C \backslash (\bigcup_x C_x)$, there is an element $C' \in \mathcal{C}$ such that $z \in C' \subset (C \cup \bigcup_x C_x) \backslash X$.
(4) The set of $\mathcal{C}$-independents $\mathcal{I}(\mathcal{C}) := \{I \subset E : C \nsubseteq I \, \forall C \in \mathcal{C}\}$ satisfies

$$\text{if } I \subset X \subset E \text{ and } I \in \mathcal{I}, \text{ then } \{I' \in \mathcal{I} : I \subset I' \subset X\} \text{ has a maximal element.}$$

In this case, $M = (E, \mathcal{C})$ is a *matroid*, and $\mathcal{C}$ is called the *set of circuits of M*.

Definition 2.2 Let $M = (E, \mathcal{C})$ be a matroid given in terms of its circuits. An *independent set* is a subset of E that contains no circuits (compare item (4) in Definition 2.1). A *basis* is a maximal independent set.

With abuse of notation, a matroid M is also denoted by $M = (E, \mathcal{B})$, where the elements in $\mathcal{B}$ are the *bases* of M. We also have the following definition of matroid via sets of *scrawls*, which are the unions of circuits [3, Section 2.2].

Definition 2.3 Let $\mathcal{S} \subset 2^E$. We call $\mathcal{S}$ the *set of scrawls* if it satisfies the following axioms:

(1) $\mathcal{S}$ is a semigroup, i.e., any union of elements in $\mathcal{S}$ is in $\mathcal{S}$;
(2) $\mathcal{S}$ satisfies the conditions (3) and (4) in Definition 2.1, where $\mathcal{I}(\mathcal{S}) := \{I \subset E : C \nsubseteq I \, \forall C \in \mathcal{S} \backslash \{\varnothing\}\}$.

In this case, $M = (E, \mathcal{S})$ is a *matroid* and $\mathcal{S}$ is called the *set of scrawls of M*.

Definitions 2.1 and 2.3 are equivalent, indeed, given a set of circuits, we get a set of scrawls by taking all the finite unions of them, conversely, given a set of scrawls, we get a set of circuits by taking the minimal non-empty elements with respect to inclusion. Notice that a set of scrawls naturally carries the structure of a semigroup and simultaneously encodes the information of a matroid.

Let $M = (E, \mathcal{B})$ be a matroid with basis elements $\mathcal{B}$. By [4, Theorem 3.1], the complements $\mathcal{B}^* := \{E \backslash B : B \in \mathcal{B}\}$ form another matroid $M^* = (E, \mathcal{B}^*)$, called the *dual matroid* to M. The circuits of M^* are called *cocircuits* of M. A matroid M is called *tame* if every intersection of a circuit and a cocircuit is finite [4]; otherwise, M is called *wild*.

If all the circuits of a matroid M (respectively, M^*) are finite then M is called finitary (respectively, cofinitary). These matroids are tame [15]. The matroids considered in this article are cofinitary as we will show in Theorem 2.12.

2.2 The set of supports of a finite-dimensional vector space

In this section, we consider the vector space K^E, where E is as above and K is any field. We will consider finite-dimensional K-vector subspaces $\{0\} \neq W \subset K^E$.

Definition 2.4 The *support map* $\mathcal{T} : K^E \longrightarrow 2^E$ is the mapping $(a_i)_{i \in E} \mapsto \{i \in E : a_i \neq 0\}$. We call $\mathcal{T}(v)$ the *support* of v, and $\mathcal{T}(W)$ is defined as the set $\{\mathcal{T}(\phi) \mid \phi \in W\} \subseteq 2^E$.

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It seems commonly known among experts in matroid theory that $\mathcal{T}(W)$ has a natural matroid structure by considering the elements with minimal support as the circuits. For the sake of completeness, we provide a proof by describing the circuits.

Definition 2.5 Let $\{0\} \neq W \subset K^E$ be a finite-dimensional K-vector subspace. We define $\mathcal{C}(W) \subset 2^E$ to be the sets in $\mathcal{T}(W)\backslash\varnothing$ that are minimal with respect to set inclusion.

Example 2.1 The set $\mathcal{C}(W)$ may consist of infinitely many elements. Consider for instance W generated by $\varphi_1 = 1 + \sum_{i\geq 2} t^i$, $\varphi_2 = \sum_{i\geq 1} i\, t^i$. Then, for every $n > 1$, we obtain that $\phi_n := n \cdot \varphi_1 - \varphi_2 \in W$ has the support $\mathcal{T}(\phi_n) = \mathbb{N}\backslash\{n\}$. Since there is no non-zero element in W whose support is a subset of $\mathcal{T}(\phi_n)$, $\mathcal{T}(\varphi_1)$ or $\mathcal{T}(\varphi_2)$, we obtain that

$$\mathcal{C}(W) = \{\mathbb{N}\backslash\{n\} \,:\, n \in \mathbb{N}\}.$$

Note that, by definition, $\mathcal{C}(W)$ satisfies items (i) and (ii) from Definition 2.1 by explicitly excluding the empty set and only considering minimal sets which cannot be subsets of other minimal sets. The next result shows that $\mathcal{C}(W) \neq \varnothing$.

Lemma 2.2 *Let W be as above. For every element $0 \neq \varphi \in W$, there is $0 \neq \psi \in W$ of minimal support such that $\mathcal{T}(\psi) \subset \mathcal{T}(\varphi)$.*

Proof Since $\{0\} \neq W$, we have $s = \dim_K(W) > 0$. If $s = 1$, then $W = K \cdot \varphi_1$ with $0 \neq \varphi_1 \in K^E$, and it is clear that $\mathcal{C}(W) = \{S_1 = \mathcal{T}(\varphi_1) \neq \varnothing\}$.

Suppose that $s > 1$. Consider $0 \neq \varphi_1 \in W$ and let $S_1 = \mathcal{T}(\varphi_1)$. If S_1 is minimal, we are done. Otherwise, by definition of minimality, there should exist $\varnothing \subsetneq S_2 \subsetneq S_1$ corresponding to some $0 \neq \varphi_2 \in W$. Then, $\{\varphi_1, \varphi_2\} \subset W$ is linearly independent; otherwise, an expression $\lambda_1\varphi_1 + \lambda_2\varphi_2 = 0$ with $(\lambda_1, \lambda_2) \neq (0, 0)$ would imply that $\lambda_1, \lambda_2 \neq 0$, and this would yield $S_1 = S_2$. Now repeat the process: if S_2 is minimal, we are done; otherwise, there exists $\varnothing \subsetneq S_3 \subsetneq S_2$ corresponding to some $0 \neq \varphi_3 \in W$. Then, the chain $\varnothing \subsetneq S_3 \subsetneq S_2 \subsetneq S_1$ implies that $\{\varphi_1, \varphi_2, \varphi_3\} \subset W$ is linearly independent; otherwise, an expression $\lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 = 0$ with $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$ would yield at least one equality on the chain $S_3 \subsetneq S_2 \subsetneq S_1$. This process eventually finishes since the dimension of W is finite. ∎

Given W as above, below we show that $\mathcal{C}(W)$ also satisfies (3) and (4) from Definition 2.1, so $M(W) = (E, \mathcal{C}(W))$ is indeed a matroid.

Remark 2.3 Suppose that $W = K \cdot \varphi_1$ with $0 \neq \varphi_1 \in K^E$. Then, $\mathcal{C}(W) = \{\mathcal{T}(\varphi_1)\}$, and so $(E, \mathcal{C}(W))$ is a matroid.

Take a basis $\{\varphi_1, \ldots, \varphi_s\} \subset W \subset K^E$ of W as a K-vector space. Then, $W = \{\lambda \cdot \underline{\varphi} := \sum_{i=1}^{s} \lambda_i \varphi_i \,:\, \lambda \in K^s\}$. Each element of the basis has an expression in the standard basis of K^E of the form $\varphi_i = (a_{ij})_{j\in E}$ with $a_{ij} \in K$. For each $j \in E$, set $u^{(j)} := (a_{1j}, \ldots, a_{sj}) \in K^s$. With this notation, an index $j \in E$ lies in the support of $\lambda \cdot \underline{\varphi} \in W$ if and only if $\lambda \cdot u^{(j)} \neq 0$, that is:

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$$\mathcal{T}(\lambda \cdot \underline{\varphi}) = \{j \in E \,:\, \lambda \cdot u^{(j)} \neq 0\}. \tag{2.1}$$

We keep this notation for the following lemmas.

Lemma 2.4 *Given $X \subset E$, there exists $0 \neq \phi \in W$ with $\mathcal{T}(\phi) \subset X$ if and only if the K-linear subspace of K^s generated by $\{u^{(j)}\}_{j \notin X}$ is a proper subspace of K^s.*

Proof As a consequence of (2.1), we have that $\mathcal{T}(\lambda \cdot \underline{\varphi}) \subset X$ if and only if $0 \neq \lambda \in K^s$ is a solution of the system

$$\{u^{(i)} \cdot \lambda = 0\}_{i \notin X}.$$

This system has non-zero solutions if and only if the K-linear subspace of K^s generated by $\{u^{(j)}\}_{j \notin X}$ is a proper subspace of K^s. Since $W = \{\lambda \cdot \underline{\varphi} \,:\, \lambda \in K^s\}$, the statement follows, as the dimension of the space of solutions of a linear system $AX = 0$ in s unknowns is s minus the rank of the matrix A. ∎

Lemma 2.5 *Given $0 \neq \phi \in W$, let $L \subset K^s$ be the K-linear subspace generated by $\{u^{(i)}\}_{i \notin \mathcal{T}(\phi)}$. Then, L is a proper subspace of K^s and $u^{(i)} \notin L$ for all $i \in \mathcal{T}(\phi)$.*

Proof As a direct consequence of Lemma 2.4, L is a proper subspace of K^s. Now, ϕ is an element of W if and only if it is of the form $\lambda \cdot \underline{\varphi}$ and an element $i \in E$ is in the support of $\lambda \cdot \underline{\varphi}$ if and only if $\lambda \cdot u^{(i)} \neq 0$. Since L is generated by the set $\{u^{(i)}\}_{i \notin \mathcal{T}(\phi)}$, we have that $\lambda \cdot u^{(i)} = 0$ for all elements in L. ∎

Lemma 2.6 *Given $C \subset E$, let $L \subset K^s$ be the space spanned by $\{u^{(i)}\}_{i \notin C}$. Then, $C \in \mathcal{C}(W)$ if and only if L is $(s-1)$-dimensional and $u^{(i)} \notin L$ for all $i \in C$.*

Proof Suppose that $C \in \mathcal{C}(W)$, then $C = \mathcal{T}(\phi)$ for some $0 \neq \phi \in W$, and it follows from Lemma 2.5 that L is a proper subspace of K^s and $u^{(i)} \notin L$ for all $i \in C$. If the dimension of L is $d < s-1$, we take $i_1, i_2, \ldots i_{s-d}$ in C such that the subspace spanned by L and $\{u^{(i_k)}\}_{k=1,\ldots,s-d}$ is K^s. The system $\{u^{(i)} \cdot \lambda = 0\}_{i \notin C} \cup \{u^{(i_k)} \cdot \lambda = 0\}_{k=2,\ldots,s-d} \cup \{u^{(i_1)} \cdot \lambda = 1\}$ has a unique solution $0 \neq \lambda \in K^s$. For this solution, $\mathcal{T}(\lambda \cdot \underline{\varphi}) \subset C$. Since $\{i_k\}_{k=2,\ldots,s-d} \subset C$ and $i_k \notin \mathcal{T}(\lambda \cdot \underline{\varphi})$ for $k = 2, \ldots, s-d$, the set inclusion is strict (so C is not minimal). So, we deduce that L is $(s-1)$-dimensional.

Conversely, suppose that L is $(s-1)$-dimensional. Then, a solution of the system $\{u^{(i)} \cdot \lambda = 0\}_{i \notin C}$ is unique up to scalar multiplication. If $0 \neq \lambda \in K^s$ is such a solution, then the only elements of W with support contained in C are scalar multiples of $\phi := \lambda \cdot \underline{\varphi}$, so they all have the same support. Now, $\mathcal{T}(\phi) \subset C$ implies that the K-linear subspace generated by $\{u^{(i)}\}_{i \notin \mathcal{T}(\phi)}$ is a proper subspace of K^s containing L, and since L is $(s-1)$-dimensional, it is then the whole L.

Now, if $i \in C$ implies that $u^{(i)} \notin L$, then $u^{(i)} \in L$ if and only if $i \notin C$; indeed, the converse $i \notin C$ implies $u^{(i)} \in L$ is implied by the construction of L. Likewise, $u^{(i)} \in L$ if and only if $i \notin \mathcal{T}(\phi)$ by Lemma 2.5 and the previous paragraph. This yields $C = \mathcal{T}(\phi)$ and thus $C \in \mathcal{C}(W)$. ∎

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Lemma 2.7 *Given $0 \neq \phi \in W$, for every $z \in \mathcal{T}(\phi) \subset E$, there exists a minimal element $C \in \mathcal{T}(W)\backslash\varnothing$ such that $z \in C \subset \mathcal{T}(\phi)$.*

Proof Let L be the subspace generated by $\{u^{(i)}\}_{i\notin\mathcal{T}(\phi)}$. By Lemma 2.4, the subspace L is of dimension $0 < d < s$. Moreover, since ϕ is a combination of the φ_i's, for all $i \in \mathcal{T}(\phi)$, it holds that $u^{(i)} \notin L$.

(1) If $d = s - 1$, then, by Lemma 2.6, $\mathcal{T}(\phi)$ is minimal.
(2) Suppose that the dimension of L is $d \leq s - 2$. Take $i_2, \ldots i_{s-d}$ in C such that the subspace generated by L, $u^{(z)}$ and $\{u^{(i_k)}\}_{k=2,\ldots,s-d}$ is K^s. The system $\{u^{(i)} \cdot \lambda = 0\}_{i\notin C} \cup \{u^{(i_k)} \cdot \lambda = 0\}_{k=2,\ldots,s-d} \cup \{u^{(z)} \cdot \lambda = 1\}$ has a unique solution $0 \neq \lambda \in K^s$. For this solution, $z \in \mathcal{T}(\lambda \cdot \underline{\varphi}) \subset C\backslash\{i_k)\}_{k=2,\ldots,s-d}$. Since the subspace generated by L and $\{u^{(i_k)}\}_{k=2,\ldots,s-d}$ is of dimension $s - 1$, again by Lemma 2.6, $\mathcal{T}(\lambda \cdot \underline{\varphi})$ is minimal. ∎

Lemma 2.8 *Let $X \subset E$ and let $\{\phi_x : x \in X\} \subset W$ be a set such that $\mathcal{T}(\phi_x) \cap X = \{x\}$ for all $x \in X$. Then, the vectors $\{u^{(x)} : x \in X\}$ are linearly independent.*

Proof For each $x \in X$, let $\lambda_x \in K^s$ be such that $\phi_x = \lambda_x \cdot \underline{\varphi}$ and let L_x be the linear subspace generated by $\{u^{(i)}\}_{i\notin\mathcal{T}(\phi_x)}$. By Lemma 2.5, since $x \in \mathcal{T}(\phi_x)$, then $u^{(x)} \notin L_x$. Now, $\mathcal{T}(\phi_x) \cap X = \{x\}$ implies that $u^{(y)} \in \{u^{(i)}\}_{i\notin\mathcal{T}(\phi_x)}$ for $y \in X\backslash\{x\}$, then $u^{(x)}$ is not in the space generated by $\{u^{(i)}\}_{i\in X\backslash\{x\}} \subset L_x$. ∎

Lemma 2.9 *Property (3) of Definition 2.1 holds for $(E, \mathcal{C}(W))$.*

Proof Suppose that $X \subset C \in \mathcal{C}$ and $\{C_x : x \in X\}$ is a family of elements of $\mathcal{C}$ such that $x \in C_y$ iff $x = y$ for all $x, y \in X$. Then, since W is finitely generated, by Lemma 2.8, X is a finite set.

Choose $\varphi = (b_j)_{j\in E} \in W$ such that $\mathcal{T}(\varphi) = C$ and, for each $x \in X$, choose $\varphi_x = (a_{xj})_{j\in E} \in W$ such that $\mathcal{T}(\varphi_x) = C_x$.

Set $\phi := \varphi - \sum_{x\in X} \frac{b_x}{a_{xx}}\varphi_x$. We have that $C\backslash(\bigcup_x C_x) \subset \mathcal{T}(\phi) \subset (C \cup \bigcup_x C_x)\backslash X$ and the result follows from Lemma 2.7. ∎

Lemma 2.10 *Property (4) of Definition 2.1 holds for $(E, \mathcal{C}(W))$.*

Proof Let $I \subset X \subset E$ with $I \in \mathcal{I}$. If $X \in \mathcal{I}$, then X is the maximal element we are looking for. Otherwise, there exists $\varphi \in W$ such that $\mathcal{T}(\varphi) \subset X$. Then, by Lemma 2.4, the dimension of the subspace of K^s generated by $\{u^{(j)} : j \notin I\}$ is s and the dimension of the subspace of K^s generated by $\{u^{(j)} : j \notin X\}$ is r with $r < s$. Then, there exist $\{i_{r+1}, \ldots, i_s\} \subset X\backslash I$ such that $\{u^{(i_{r+1})}, \ldots, u^{(i_s)}\}$ together with $\{u^{(j)} : j \notin X\}$ generate K^s.

If $\bar{I} := X\backslash\{u^{(i_{r+1})}, \ldots, u^{(i_s)}\}$ then $\bar{I}$ is a maximal element of $\{I' \in \mathcal{I} : I \subset I' \subset X\}$. ∎

Theorem 2.11 *Let $\{0\} \neq W \subset K^E$ be a finite-dimensional K-vector subspace. Then, $M(W) = (E, \mathcal{C}(W))$ is a matroid. Moreover, any element of $\mathcal{T}(W)$ is a union of circuits.*

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Proof Items (i) and (ii) are fulfilled as stated after Definition 2.5, and Lemmas 2.9 and 2.10 show the remaining items. The fact that any element of $\mathcal{T}(W)$ is a union of circuits follows from Lemma 2.7. ∎

The above result is closely related to [15, Theorem 4.12] where the language of thin sum matroids is used, for more details see Section 2.3.

Recall from Example 2.1 that the set $\mathcal{C}(W)$ may consist of infinitely many elements. Nevertheless, the matroid $(E, \mathcal{C}(W))$ has some finite structure in the following sense.

Theorem 2.12 *The matroid $M = (E, \mathcal{C}(W))$ is cofinitary. In particular, M is tame.*

Proof By Lemma 2.4, we have that $X \subset E$ is independent if and only if the K-linear subspace generated by $\{u^{(j)}\}_{j\notin X}$ is K^s, which means that there exist $\{j_1, \dots, j_s\} \subset E\backslash X$ such that $\{u^{(j_1)}, \dots, u^{(j_s)}\}$ is a linearly independent set. Then, the bases of the matroid are exactly $\mathcal{B}_M = \{E\backslash\{j_1, \dots, j_s\} : Span\{u^{(j_1)}, \dots, u^{(j_s)}\} = K^s\}$, and $\mathcal{B}_M^* = \{\{j_1, \dots, j_s\} : Span\{u^{(j_1)}, \dots, u^{(j_s)}\} = K^s\}$. Thus, the circuits of the dual matroid will have at most $s+1$ elements. ∎

Remark 2.13 The fact that $\mathcal{B}_M^* = \{\{j_1, \dots, j_s\} : Span\{u^{(j_1)}, \dots, u^{(j_s)}\} = K^s\}$ says that the dual M^* of $M = (E, \mathcal{C}(W))$ is precisely the matroid defined by the linear independence of the family of vectors $\{u^{(j)} : j \in E\}$ (see [15, Definition 2.6].

A matroid defined by linear independence of a family of vectors is always finitary (even if the space spanned by the vectors is infinite-dimensional, see [15, p. 2], [4, Section 2.6]). Thus, our original matroid $(E, \mathcal{C}(W)) = M = (M^*)^*$ will in general not be representable.

2.3 Relationship with thin sums matroids

Again, take a basis $\{\varphi_1, \dots, \varphi_s\} \subset W \subset K^E$ of W as a K-vector space of the form $\varphi_i = (a_{ij})_{j\in E}$. In Remark 2.13, we showed that the dual of our matroid $M = M(W) = (E, \mathcal{C}(W))$ is the matroid $N = M^*$ represented by the family $\{u^{(j)} : j \in E\} \subset K^s$ where $u^{(j)} := (a_{1j}, \dots, a_{sj})$.

Having this in mind, it is possible to use the theory of thin families and their associated thin sums matroids to give a much shorter proof of the fact that M is a matroid. The necessary concepts and constructions are rather technical, and the interested reader can find the details in [15]. But since the proof is rather short, we include it here as an alternative.

The dual $N^* = M$ arises as a thin sums matroid over a thin family for the field K by [15, Theorem 1.2], and [15, Theorem 3.3] gives an explicit way to compute N^*: let C be the family of all linear dependencies of the family $\{u^{(j)} : j \in E\}$, which in this case consists of all the finitely supported elements c of K^E which are orthogonal to the set $\{\varphi_1, \dots, \varphi_s\}$. Then, $p : E \to K^C$ defined by $p(e)(c) := c(e)$ is a thin family, and the corresponding thin sums matroid $M_{ts}(p)$ is N^*.

If $v \in K^E$ and $F \subset E$, we denote by $v|_F$ the element of K^E having coordinates $(v|_F)_e = v_e$ if $e \in F$, and $(v|_F)_e = 0$ otherwise. Now, we have the following result.

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Lemma 2.14 *The set of vectors $v \in K^E$ which are orthogonal to all $c \in C$ is precisely W.*

Proof Since $\{\varphi_1, \ldots, \varphi_s\}$ generate W and are orthogonal to all the elements of C by definition, then all the elements of W are also orthogonal to all the elements of C.

Conversely, let $v \in K^E$ be an orthogonal vector to all the elements of C, and let $F \subset E$ be a finite set large enough that the set $\{\varphi_1|_F, \ldots, \varphi_s|_F\}$ is linearly independent. Then, $v|_F$ is orthogonal to every element orthogonal to the set $\{\varphi_1|_F, \ldots, \varphi_s|_F\}$, and so we have an expression $v|_F = \sum_i l_i \varphi_i|_F$ for uniquely determined $l_1, \ldots, l_s \in K$ since $\{\varphi_1|_F, \ldots, \varphi_s|_F\}$ is linearly independent.

Given $e \in E \backslash F$, we know that $v|_{F\cup\{e\}}$ is orthogonal to every element orthogonal to the set $\{\varphi_1|_{F\cup\{e\}}, \ldots, \varphi_s|_{F\cup\{e\}}\}$, and so we have an expression $v|_{F\cup\{e\}} = \sum_i l_i' \varphi_i|_{F\cup\{e\}}$. Now, we restrict this expression to $F \subset F \cup \{e\}$ to find that $l_i' = l_i$ for all i, and since this is true for every $e \in E\backslash F$, we deduce that $v = \sum_i l_i \varphi_i \in W$. ∎

The previous lemma shows that the set of thin dependencies D_p of the thin family p is precisely W, now [15, Theorem 3.4] says that $M_{ts}(p) = M^*(\overline{p})$, where $\overline{p} : E \to k^W$ is defined by $\overline{p}(e)(v) = v(e)$ for $e \in E$ and $v \in W$.

3 Scrawls and cardinality

In the previous section, we have shown that the minimal elements of $\mathcal{T}(W)$ satisfy the axioms of circuits. The scrawls of the matroid on E given in terms of these circuits are unions of circuits. In this section, we investigate the conditions on E and W upon which the collection of scrawls coincide with $\mathcal{T}(W)$. For this purpose, let us denote by $\#C$ the cardinality of a set C and by $S(C) = C \cup \{C\}$ the successor-set. Note that $\#S(C) = \#C + 1$ for finite sets C and $\#S(C) = \#C$ if C is infinite.

3.1 Matroids of scrawls

Denote by $\mathrm{Lin}(W)$ the set of K-linear subspaces $L \subset K^s$ with $L \neq K^s$ such that L is generated by a set of the form $\{u^{(i)}\}_{i\in X}$ for some $X \subset E$. We will denote by Ψ_W the map given by

$$\begin{array}{rccc} \Psi_W : & \mathrm{Lin}(W) & \longrightarrow & 2^E \\ & L & \mapsto & \{i \in E \,:\, u^{(i)} \notin L\}. \end{array} \tag{3.1}$$

Notice that $\mathrm{Lin}(W)$ is in fact independent of the choice of basis $\{\varphi_1, \ldots, \varphi_s\}$. To see this consider another basis $\{\psi_1, \ldots, \psi_s\}$ of W and let $\psi_i = (b_{ij})_{j\in E}$. Denote for $j \in E$, $w^{(j)} := (b_{1j}, \ldots, b_{sj})$. Then, there exists an invertible $s \times s$ matrix $\lambda = (\lambda_{ij})_{1\leq i,j\leq s}$ encoding the change of basis: $\psi_i = \sum \lambda_{ij}\varphi_j$. In particular, for all $j \in E$, we have

$$\lambda u^{(j)} = w^{(j)}.$$

Now consider $X \subset E$ and $L = \langle u^{(j)} \,:\, j \in X\rangle$. We have

$$\langle u^{(i)}\rangle_{i\in X} = \langle \lambda^{-1} w^{(i)}\rangle_{i\in X} = \langle u^{(j)}\rangle_{j\in \mathcal{T}(\{\lambda^{-1}w^{(i)}\}_{i\in X})}.$$

So the base change matrix λ induces a natural bijection between the linear spaces generated by subsets of $\{u^{(i)}\}_{i\in E}$ and those of $\{w^{(i)}\}_{i\in E}$.

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Lemma 3.1 *The morphism Ψ_W induces a one-to-one correspondence between the circuits of the matroid induced by W and the spaces of codimension one in* $\mathrm{Lin}\, W$.

Proof This is a direct consequence of Lemma 2.6. ∎

Lemma 3.2 *The collection $\mathcal{T}(W) \subset 2^E$ is closed under union if and only if Ψ_W is a natural one-to-one correspondence between* $\mathrm{Lin}(W)$ *and* $\mathcal{T}(W)$.

Proof Lemma 2.5 implies that $\mathcal{T}(W) \subset \Psi_W(\mathrm{Lin}(W))$.

Now, $\mathrm{Lin}(W)$ is closed under intersection and $\Psi_W(L \cap L') = \Psi_W(L) \cup \Psi_W(L')$. Then, $\Psi_W(\mathrm{Lin}(W))$ is closed under union. All elements of $\mathrm{Lin}(W)$ can be written as intersection of elements of codimension one. Then, all the elements in $\Psi_W(\mathrm{Lin}(W))$ may be written in terms of images of elements of codimension one. By Lemma 3.1, images of elements of codimension one are circuits. By Theorem 2.11, all elements of $\mathcal{T}(W)$ are union of circuits. ∎

Lemma 3.3 *Let L be a K-linear subspace of K^s with $\dim_K(L) = d \leq s-2$ and let $\{u^{(i)}\}_{i\in X} \subset K^s$ with $X \subset E$ be such that $\{u^{(i)}\}_{i\in X} \cap L = \emptyset$. If $\#X < \#S(K)$, then there exists a K-linear subspace $\overline{L} \supset L$ of K^s of dimension $d+1$ with $\{u^{(i)}\}_{i\in X} \cap \overline{L} = \emptyset$.*

Proof Let A be the collection of K-linear subspaces of dimension $d+1$ of K^s containing L. The collection A is isomorphic to $\mathbb{P}_K^{s-d-1}$, where $\mathbb{P}_K$ denotes the projective space over K. Then, the cardinality of A is greater than or equal to the cardinality of $S(K)$. For every non-zero vector $u^{(i)} \notin L$, $i \in X$, since the dimension of L is d, there exists exactly one $L_i \in A$ such that $u^{(i)} \in L_i$. Thus, if $\#X < \#S(K)$, there is $\overline{L} \in A \backslash \{L_i\}_{i\in X}$ and $\overline{L}$ does not contain any of the $u^{(i)} \notin L$. ∎

Let us note that for infinite K the proof could be simplified by using $\#\mathbb{P}_K^{s-d-1} = \#\mathbb{P}_K = \#K$.

Lemma 3.4 *Let W be a subspace of K^E and let Ψ_W be as in (3.1). If $\#E < \#S(K)$, then Ψ_W is a natural one-to-one correspondence between* $\mathrm{Lin}(W)$ *and* $\mathcal{T}(W)$.

Proof We start by showing that, for $L \in \mathrm{Lin}(W)$, $\Psi_W(L)$ is in $\mathcal{T}(W)$.

(1) If L is of codimension one it is a consequence of Lemma 2.6 (with no assumption about cardinality).

(2) Suppose that the dimension of L is $d < s-1$. Then, the cardinality of $X := \{i \in E : u^{(i)} \notin L\}$ satisfies $\#S^d(X) \leq \#E$, where S^d denotes the d-th successor-set. Applying $s-d-1$ times Lemma 3.3, there exists a K-linear hyperplane $\overline{L} \supset L$ that does not contain any element of $\{u^{(i)}\}_{i\in X}$. Take $\lambda \in K^s$ such that $\overline{L} = \{v \in K^s : \lambda \cdot v = 0\}$. Since $L \subset \overline{L}$ and $\{u^{(i)}\}_{i\in E, u^{(i)}\notin L} \cap \overline{L} = \emptyset$ we have that $\lambda \cdot u^{(i)} = 0$ for all $u^{(i)} \in L$ and $\lambda \cdot u^{(i)} \neq 0$ for all $u^{(i)} \notin L$. Then, by (2.1), we have that $\mathcal{T}(\lambda \cdot \underline{\varphi}) = \Psi_W(L)$.

That the mapping is injective is straightforward and, to show surjectivity, it is enough to see that $\mathcal{T}(\phi) = \Psi_W(L)$ for any $\phi \in W$, where L is the subspace generated by $\{u^{(i)}\}_{i\in E\backslash \mathcal{T}(\phi)}$. ∎

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Theorem 3.5 *Let W be a subspace of K^E. If $\#E < \#S(K)$, then $\mathcal{T}(W)$ is a set of scrawls for E.*

Proof By Theorem 2.11, any element of $\mathcal{T}(W)$ is a union of circuits. That any union of circuits is in $\mathcal{T}(W)$ is a consequence of Lemma 3.4 together with Lemma 3.2. ■

The following example shows that the condition in Theorem 3.5 about cardinality is optimal.

Example 3.6 Let E be a finite set and let K be a finite field with $\#K < \#E$. Let $i_0 \in E$ and set $a_{i_0} := 1$. Choose, for $i \in E$, the $a_i \in K$ such that $K = \{a_i\}_{i \in E \setminus \{i_0\}}$. Let $i_1 \in E$ be such that $a_{i_1} = 0$. Set $\varphi_1 := \{b_i\}_{i \in E} \in K^E$, where $b_i := 1$ for $i \neq i_0$ and $b_{i_0} = 0$ and set $\varphi_2 := \{a_i\}_{i \in E} \in K^E$ and let W be the subspace of K^E spanned by φ_1 and φ_2.

In this case, the image of the mapping (3.1) is not contained in $\mathcal{T}(W)$ since $\Psi(\{(0, \ldots, 0)\}) = E$ and E is not the support of any element of W. In order to see this, take an element $\phi := \lambda_1\varphi_1 + \lambda_2\varphi_2 \in W$. If $\lambda_1\lambda_2 = 0$ then either $i_0 \notin \mathcal{T}(\phi)$ or $i_1 \notin \mathcal{T}(\phi)$. If $\lambda_1\lambda_2 \neq 0$, let $i \in E$ be such that $a_i = -\frac{\lambda_1}{\lambda_2}$ (it exists because $K = \{a_i\}_{i \in E \setminus \{i_0\}}$). Since $\mathcal{T}(\phi) = \mathcal{T}(\frac{1}{\lambda_2}\phi) = \mathcal{T}(\frac{\lambda_1}{\lambda_2}\varphi_1 + \varphi_2) = \mathcal{T}(\{\frac{\lambda_1}{\lambda_2} + a_i\}_{i \in E}) \cup \{i_0\}$ we have that $i \notin \mathcal{T}(\phi)$.

Note that if K, E, and $W \subset K^E$ satisfy the conditions of Theorem 3.5, then $\mathcal{T}(W)$ is in particular a semigroup.

Corollary 3.7 *Let K and E be arbitrary. Let $s \in \mathbb{N}$, for each $j \in E$, set $u^{(j)} := (a_{1j}, \ldots, a_{sj}) \in K^s$, such that $\{\varphi_i = (a_{ij})_{j \in E} : i = 1, \ldots, s\}$ is linearly independent.*

Let M be the representable matroid induced by the family $\{u^{(j)} : j \in E\}$. Then, M^* is the matroid of scrawls of $LinSpan\{\varphi_1, \ldots, \varphi_s\}$ if $\#E < \#S(K)$.

Proof Let $W = LinSpan\{\varphi_1, \ldots, \varphi_s\}$. By Theorem 2.12, if M is the representable matroid induced by the family $\{u^{(j)} : j \in E\}$, then M^* is its matroid of scrawls, and $M^* = \mathcal{T}(W)$ if $\#E < \#S(K)$ by Theorem 3.5. ■

Remark 3.8 We have that Corollary 3.7 says that even if a cofinitary matroid M is representable by a family of column vectors, it does not follow automatically that its dual M^* is the matroid of scrawls of the linear span of the row vectors.

Remark 3.9 Our concept of set of scrawls is stronger than that of a semigroup of $(2^E, \cup)$, since it is clear that a semigroup, being closed under unions, is spanned by its minimal elements, but it does not necessarily follow that this set of minimal elements satisfy the axioms of the circuits of a matroid. The concept of set of scrawls is also stronger than that of the circuits of a matroid on 2^E, since it may happen that $a, b \in \mathcal{C}(W)$, but $a \cup b \notin \mathcal{T}(W)$ (see Theorem 3.5).

4 Tropical linear spaces in the differential algebra setting

We apply the previous theory to the case in which the set of formal solutions of a homogeneous system of linear differential equations is a finite-dimensional vector space. The dimension of such solution spaces could be stated with the usage of

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D-modules [25] or jet-spaces [17]. When considering differential equations with polynomial coefficients instead of formal power series coefficients, one would speak of D-finite solutions [20]. In this article, however, we give a presentation by using the so-called differential type applicable to every system of algebraic (non-linear) differential polynomials as those considered in the FTTDAG [11].

Throughout this section, we will consider K to be a field of characteristic zero and $m, n \in \mathbb{N}\backslash\{0\}$.

4.1 Preliminaries on differential algebra

We denote by $K[\![T]\!] = K[\![t_1, \ldots, t_m]\!]$ the ring of multivariate formal power series, by $D = \{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_m}\}$, the set of standard partial derivatives, and we denote for $J \in \mathbb{N}^m$ the differential operator $\Theta(J) = \frac{\partial^{|J|}}{\partial t_1^{j_1} \cdots \partial t_m^{j_m}}$. We denote by $K_{m,n}$ the polynomial ring $K[\![T]\!][x_{i,J} : i = 1, \ldots, n,\ J \in \mathbb{N}^m]$, where x_i are differential indeterminates and $x_{i,J} := \Theta(J)x_i$. For $\Sigma \subset K_{m,n}$, we set $\Theta\Sigma = \{\Theta(J)f : J \in \mathbb{N}^m, f \in \Sigma\}$.

Definition 4.1 Let $\Sigma \subset K_{m,n}$. The differential ideal $[\Sigma] \subset K_{m,n}$ spanned by Σ is the minimal ideal containing Σ and being closed under taking derivatives.

An element $P \in K_{m,n}$ is called a *differential polynomial*, and the *order* of P is defined as the maximum of the $|J| = J_1 + \cdots + J_m$ effectively appearing in P. The variables $x_{i,J}$, $i = 1, \ldots, n$, $J \in \mathbb{N}^m$, in $K_{m,n}$ denote differential variables. We can define a map $P : K[\![T]\!]^n \to K[\![T]\!]$ in which a monomial $E_M = \prod_{i,J} x_{i,J}^{m_{i,J}}$ sends the vector $\varphi = (\varphi_1, \ldots, \varphi_n) \in K[\![T]\!]^n$ to $E_M(\varphi) = \prod_{i,J}(\frac{\partial^{|J|}\varphi_i}{\partial t_1^{j_1} \cdots \partial t_m^{j_m}})^{m_{i,J}}$.

Definition 4.2 We say that $\varphi = (\varphi_1, \ldots, \varphi_n) \in K[\![T]\!]^n$ is a *solution* of $P \in K_{m,n}$ if $P(\varphi) = 0$. We denote by $\mathrm{Sol}(P)$ the set of solutions of $P \in K_{m,n}$. Let $\Sigma \subset K_{m,n}$. The differential (algebraic) variety defined by Σ is the set of common solutions $\mathrm{Sol}(\Sigma) = \bigcap_{P \in \Sigma} \mathrm{Sol}(P) \subset K[\![T]\!]^n$.

Let $\Sigma \subset K_{m,n}$ be a system of differential equations such that the radical differential ideal generated by Σ is prime. We need to require that Σ is in a reduced form such that system such as $\{y, y' - 1\}$ are further simplified before continuation. Formally, this can for instance be achieved by imposing that Σ is autoreduced [5, 18, 24]. Let L denote the set of leaders of Ω. Then, the transcendence degree of the general solution of Σ is equal to the cardinality d of the set $\Theta\{x_1, \ldots, x_n\}\backslash\Theta L$. We say that Σ is of *differential type zero* if and only if d is finite. For the general notion of differential type and more details on the dimension of the solution set, we refer to [18, 19].

4.2 The linear case

In this section, we apply the results of the previous sections to the space of solutions $\mathrm{Sol}(\Sigma)$ of a homogeneous linear system of differential equations $\Sigma \subset K_{m,n}$. We start with some definitions.

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Definition 4.3 An (algebraic) linear differential equation is a linear polynomial $P \in K_{m,n}$, i.e., $P = \sum_{i,J} \alpha_{i,J} x_{i,J} + \alpha$, with $\alpha_{i,J}, \alpha \in K[\![T]\!]$. We say that P is homogeneous if $\alpha = 0$.

If $P \in K_{m,n}$ is linear and homogeneous, then it is easy to see that $\mathrm{Sol}(P) \subset K[\![T]\!]^n$ is a K-vector space. Thus, if $\Sigma \subset K_{m,n}$ is a system of homogeneous linear differential equations, then $\mathrm{Sol}(\Sigma)$ is also a K-vector space.

Remark 4.1 Let us note that for homogeneous linear systems of differential equations Σ, the differential ideal generated by Σ is always prime and every autoreduced set Ω of Σ is homogeneous and linear as well. Moreover, the transcendence degree of the general solution d, if it is finite, is the dimension of $\mathrm{Sol}(\Sigma)$ [18, Chapter 3, Section 5].

With abuse of notation, we will denote by $\mathbb{N}^m$ the idempotent monoid $(\mathbb{N}^m, \cup, \varnothing)$, and we will denote by $\mathcal{T} : K[\![T]\!] \to 2^{\mathbb{N}^m}$ the support map.[1] If $n \geq 1$ and $X \subset K[\![T]\!]^n$, its set of supports (see [10]) is

$$\mathcal{T}(X) = \{(\mathcal{T}(w_1), \ldots, \mathcal{T}(w_n)) \in (2^{\mathbb{N}^m})^n \ : \ (w_1, \ldots, w_n) \in X\}. \tag{4.1}$$

The canonical order of the idempotent monoid $(2^{\mathbb{N}^m})^n$ defined by $(S_1, \ldots, S_n) \leq (T_1, \ldots, T_n)$ whenever $S_i \cup T_i = T_i$ for all $i = 1, \ldots, n$ coincides with the *product* or *cartesian order* $\leq_{prod}$ defined by $(S_1, \ldots, S_n) \leq_{prod} (T_1, \ldots, T_n)$ when $S_i \subseteq T_i$ for $i = 1, \ldots, n$. Note that if $n \geq 2$, then this order differs from the inclusion order, thus, with the purpose of being able to apply the theory of the previous sections, first, we need to perform the following transformation.

We only need the following extra procedures whenever $n \geq 2$. Let us denote by $K[\![T^I \ : \ I \in [n] \times \mathbb{N}^m]\!]$ the set of all maps $\varphi : [n] \times \mathbb{N}^m \to K$ endowed with the sum given by point-wise addition of maps. We can represent the elements of this K-module as $\sum_{I \in [n] \times \mathbb{N}^m} a_I t^I$.

We construct an isomorphism of K-modules $\Phi : (K[\![t_1, \ldots, t_m]\!]^n, +) \to (K[\![T^I \ : \ I \in [n] \times \mathbb{N}^m]\!], +)$ by sending the vector $\varphi = (\varphi_1, \ldots, \varphi_n)$ with $\varphi_i(I) = a_{i,I}$ to the map $\Phi(\varphi) : [n] \times \mathbb{N}^m \to K$ sending (i, I) to $a_{i,I}$. The inverse map of Φ sends $\varphi \in K[\![T^I \ : \ I \in [n] \times \mathbb{N}^m]\!]$ to $(\varphi_1, \ldots, \varphi_n) \in K[\![t_1, \ldots, t_m]\!]^n$, where $\varphi_i := \sum_{I \in \mathbb{N}^m} \varphi(i, I) T^I$.

This, in turn, induces an isomorphism of monoids $\phi : (2^{\mathbb{N}^m})^n \to 2^{[n] \times \mathbb{N}^m}$, where now $2^{[n] \times \mathbb{N}^m}$ is ordered by inclusion. The isomorphism sends the vector $(S_1, \ldots, S_n)$ to $\{1\} \times S_1 \cup \cdots \cup \{n\} \times S_n$.

Then, we have

$$\phi \circ \mathcal{T}(X) = \{\{1\} \times \mathcal{T}(w_1) \cup \cdots \cup \{n\} \times \mathcal{T}(w_n) \in 2^{[n] \times \mathbb{N}^m} \ : \ (w_1, \ldots, w_n) \in X\}. \tag{4.2}$$

Thus, if $W \subset K[\![t_1, \ldots, t_m]\!]^n$ is a linear space, then $\Phi(W) \subset K[\![T^I \ : \ I \in [n] \times \mathbb{N}^m]\!]$ is also a linear space which is isomorphic to W, and we have that $\phi \circ \mathcal{T}(W) = \mathcal{T} \circ \Phi(W)$, so we can apply the theory of the previous sections to the images under ϕ of set of supports (4.1) associated with finitely-dimensional vector spaces $W \subset K[\![t_1, \ldots, t_m]\!]^n$.

[1] The formal power series in the argument of $\mathcal{T}$ can be identified as the list of coefficients such that this is consistent with Definition 2.4.

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Theorem 4.2 *Let $\Sigma \subset K_{m,n}$ be a system of homogeneous linear differential equations of differential type zero and let $\mathcal{T}(Sol(\Sigma)) \subset (2^{\mathbb{N}^m})^n$ be the set of supports of $Sol(\Sigma)$. Then*

(1) *the minimal elements $\mathcal{C}(\mathrm{Sol}(\Sigma))$ of $\mathcal{T}(\mathrm{Sol}(\Sigma))$ define the circuits of a matroid $M(Sol(\Sigma))$ on $[n] \times \mathbb{N}^m$;*
(2) *If $\#[n] \times \mathbb{N}^m < \#K$, then $\phi \circ \mathcal{T}(Sol(\Sigma)) \subset 2^{[n]\times\mathbb{N}^m}$ is a set of scrawls of the matroid $M(Sol(\Sigma))$.*

Proof Since Σ is of differential type zero, we have that $\{0\} \neq \mathrm{Sol}(\Sigma) \subset K[\![T]\!]^n$ is a finite-dimensional K-vector space.

We also have that the minimal elements of $\phi \circ \mathcal{T}(\mathrm{Sol}(\Sigma)) \subset (2^{[n]\times\mathbb{N}^m}, \subseteq)$ coincide with the minimal elements of $\mathcal{T}(\mathrm{Sol}(\Sigma)) \subset ((2^{\mathbb{N}^m})^n, \leq_{prod})$, where $\leq_{prod}$ is the product order, since they are isomorphic as posets. Thus, $\phi(\mathcal{C}(\mathrm{Sol}(\Sigma))) = \mathcal{C}(\phi(\mathrm{Sol}(\Sigma)))$, so (i) follows from Theorem 2.11, and (ii) follows from Theorem 3.5 after applying the inverse homomorphism ϕ^{-1} to the semigroup $\phi \circ \mathcal{T}(\mathrm{Sol}(\Sigma)) \subset 2^{[n]\times\mathbb{N}^m}$. ∎

Remark 4.3 If $n \geq 2$, one must use the isomorphism $\phi : (2^{\mathbb{N}^m})^n \to 2^{[n]\times\mathbb{N}^m}$ in order to unveil the matroidal structures of the sets $\mathrm{Sol}(\Sigma)$. The isomorphism of posets yields that condition (iii) in Definition 2.1 can be stated directly in terms of the poset $\mathcal{T}(\mathrm{Sol}(\Sigma)) \subset ((2^{\mathbb{N}^m})^n, \leq_{prod})$. It should be interesting to see if the same can be done for condition (iv) in Definition 2.1.

5 Applications to tropical DA geometry

We have that Theorem 4.2 from Section 4 is valid for the set of supports $\mathcal{T}(\mathrm{Sol}(\Sigma)) \subset (2^{\mathbb{N}^m})^n$ of a system $\Sigma \subset K_{m,n}$ of homogeneous linear differential equations of differential type zero over an arbitrary field K. An important case occurs when K *satisfies the hypotheses of the* FTTDAG [6], namely, when K is an algebraically closed field of characteristic zero and has infinite transcendence degree over the field of definition of Σ.[2]

5.1 Tropical algebra preliminaries

If K satisfies the hypotheses of the FTTDAG, then we can express our results in a more algebraic form using the formalism of tropical algebra. Recall that $2^{\mathbb{N}^m} = (2^{\mathbb{N}^m}, \cup)$ is a semigroup. The tropical counterparts of the underlying algebraic structures are as follows.

(1) Consider the Minkowski set sum $+ : 2^{\mathbb{N}^m} \times 2^{\mathbb{N}^m} \to 2^{\mathbb{N}^m}$;
(2) Define the *(tropical) differential operators* $D = \{\frac{\partial}{\partial t_i} : \mathbb{N}^m \to \mathbb{N}^m \ : \ i = 1, \ldots, m\}$ by shifting the support accordingly as

$$\tfrac{\partial}{\partial t_i}(S) := \{(j_1, \ldots, j_{i-1}, j_i - 1, j_{i+1}, \ldots, j_m) \ : \ (j_1, \ldots, j_m) \in S, j_i > 0\}.$$

Then, the tuple $(2^{\mathbb{N}^m}, \cup, +, D)$ is an (idempotent) differential semiring (see [10]).

[2] Note that the latter is always fulfilled for uncountable fields.

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We give an alternative presentation of this algebraic structure. More details and the connection to the approach above are presented in [7].

Let $\mathbb{B} = \{0 < 1\}$ be the Boolean semifield with the usual tropical addition $a + b := min(a, b)$ and tropical multiplication $ab := a + b$. If $\mathbb{B}[\![t_1, \ldots, t_m]\!]$ denotes the semiring of Boolean formal power series endowed with the standard operations of sum and product of Boolean power series, and $D = \{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_m}\}$ denotes the set of standard partial derivations on $\mathbb{B}[\![t_1, \ldots, t_m]\!]$, then we have an isomorphism of (idempotent) differential semirings

$$(\mathbb{B}[\![t_1, \ldots, t_m]\!], +, \times, D) \cong (2^{\mathbb{N}^m}, \cup, +, D). \tag{5.1}$$

We denote by $\mathbb{B}_{m,n}$ the polynomial semiring $\mathbb{B}[\![T]\!][x_{i,J} : i = 1, \ldots, n,\ J \in \mathbb{N}^m]$. An element $P \in \mathbb{B}_{m,n}$ is called a differential polynomial, and the variables $x_{i,J}$, $i = 1, \ldots, n$, $J \in \mathbb{N}^m$ in $\mathbb{B}_{m,n}$ denote differential variables.

We define a map $P : \mathbb{B}[\![T]\!]^n \to \mathbb{B}[\![T]\!]$ in which a monomial $E_M = \prod_{i,J} x_{i,J}^{m_{i,J}}$ sends the vector $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{B}[\![T]\!]^n$ to $E_M(\varphi) = \prod_{i,J}(\frac{\partial^{|J|}\varphi_i}{\partial t_1^{j_1} \cdots \partial t_m^{j_m}})^{m_{i,J}}$. The solutions of P are defined in a tropical way as follows.

Definition 5.1 Given $A \in 2^{\mathbb{N}^m}$, its Newton polyhedron $\mathrm{New}(A)$ is the convex hull of the set $\{I + J : I \in A, J \in \mathbb{N}^m\} \subseteq \mathbb{R}^m_{\geq 0}$. We define the Newton polyhedron $\mathrm{New}(\varphi)$ of $\varphi \in \mathbb{B}[\![t_1, \ldots, t_m]\!]$ by using the isomorphism from (5.1).

The *semiring of vertex polynomials* is defined as the quotient $V\mathbb{B}[T] := \mathbb{B}[\![T]\!]/\mathrm{New}$, where $\mathrm{New} \subset \mathbb{B}[\![T]\!] \times \mathbb{B}[\![T]\!]$ denotes the semiring congruence composed of pairs of Boolean power series with equal Newton polyhedra. We denote by $V : \mathbb{B}[\![T]\!] \to V\mathbb{B}[T]$ the resulting quotient homomorphism of semirings.

In the following, we will denote the sum of equivalence classes in $V\mathbb{B}[T]$ by "$\oplus$".

Definition 5.2 Given a sum

$$s = a_1 \oplus \cdots \oplus a_k \tag{5.2}$$

in $V\mathbb{B}[T]$ involving $k \geq 2$ summands, let $s_{\widehat{i}} := a_1 \oplus \cdots \oplus \widehat{a_i} \oplus \cdots \oplus a_k$ denote the sum obtained by omitting the i-th summand, $i = 1, \ldots, k$.

The sum (5.2) *tropically vanishes* in $V\mathbb{B}[T]$ if $s = s_{\widehat{i}}$ for every $i = 1, \ldots, k$.

Given $P \in K_{m,n}$, the definition of a solution of P can be given in a tropical way as follows.

Definition 5.3 We say that $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{B}[\![T]\!]^n$ is a *solution* of $\sum_M a_M E_M = P \in \mathbb{B}_{m,n}$ if

$$V(P(\varphi)) = \bigoplus_M V(a_M E_M(\varphi))$$

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vanishes tropically in $V\mathbb{B}[T]$. We denote by $\mathrm{Sol}(P)$ the set of solutions of P. Recall that $E_M = \prod_{i,J} x_{i,J}^{m_{i,J}}$ represents a monomial and $a_M \in \mathbb{B}[\![T]\!]$ its corresponding coefficient.

Given $U \subset \mathbb{B}_{m,n}$ a system of tropical differential equations, we will denote by $\bigcap_{p\in U} \mathrm{Sol}(p) = \mathrm{Sol}(U) \subset \mathbb{B}[\![T]\!]^n$ its set of common solutions.

We will be mostly interested in the case where $P \in \mathbb{B}_{m,n}$ is linear as in Definition 4.3, where $\alpha_{i,J}, \alpha \in \mathbb{B}[\![T]\!]$. The tropical analog of the fact that the set of solutions of a system of homogeneous linear differential equations is a vector space also holds for the case of homogeneous linear tropical differential equations in $\mathbb{B}_{m,n}$. Note that under the isomorphism (5.1), the closure under taking unions translates to the closure under taking sums.

Theorem 5.1 *Let $U \subset \mathbb{B}_{m,n}$ be a system of homogeneous linear tropical differential equations. Then, $\mathrm{Sol}(U) \subset \mathbb{B}[\![T]\!]^n$ is a semigroup.*

Proof Since $\mathrm{Sol}(U) = \bigcap_{p\in U} \mathrm{Sol}(p)$, it suffices to show that $\mathrm{Sol}(p)$ is a semigroup for linear $p \in \mathbb{B}_{m,n}$. Let $\varphi, \psi \in \mathrm{Sol}(p)$ and set $\alpha = \varphi + \psi$, then $p(\alpha) = p(\varphi) + p(\psi)$, and it follows that $V(p(\alpha)) = V(p(\varphi) + p(\psi)) = V(p(\varphi)) \oplus V(p(\psi)) \subset V(p(\varphi)) \cup V(p(\psi))$, which finishes the proof. ∎

So, if $U \subset \mathbb{B}_{m,n}$ is as in Theorem 5.1, then $\mathrm{Sol}(U) \subset \mathbb{B}[\![T]\!]^n$ is a semigroup and thus, the union of tropical solutions is again tropical solutions. The structure of the semigroup $\mathrm{Sol}(U)$ was studied in [14] for the ordinary case ($m = 1$) and U finite. Following Remark 4.3, if $n \geq 2$, we can consider the image of $\mathrm{Sol}(U)$ under the map $\phi : \mathbb{B}[\![T]\!]^n \to \mathbb{B}[\![T^I : I \in [n] \times \mathbb{N}^m]\!]$, which is also a semigroup, but it is not necessarily the set of scrawls of a matroid (see Theorem 4.2).

In Corollary 5.3, we give a necessary condition for a set $\phi(\mathrm{Sol}(U))$ to be the set of scrawls of a matroid. It would be interesting to find sufficient conditions under which a semigroup $\phi(\mathrm{Sol}(U))$ is the set of scrawls of a matroid.

5.2 Connections with the fundamental theorem

In this section, we analyze the special case when the coefficient field K fulfills the hypotheses of the FTTDAG, that is, K is an uncountable algebraically closed field of characteristic zero.

To start with, by (5.1), we have an isomorphism of semirings $2^{\mathbb{N}^m} \cong \mathbb{B}[\![t_1, \ldots, t_m]\!]$, and we will denote by $\mathcal{T} : K[\![t_1, \ldots, t_m]\!] \to \mathbb{B}[\![t_1, \ldots, t_m]\!]$ the support map. If $n \geq 1$ and $X \subset K[\![T]\!]^n$, its set of supports $\mathcal{T}(X) \subset \mathbb{B}[\![t_1, \ldots, t_m]\!]^n$ is defined as in (4.1).

Consider now a system $\Sigma \subset K_{m,n}$ of homogeneous linear differential equations of differential type zero over K. Then, by Theorem 4.2, the set $\phi \circ \mathcal{T}(\mathrm{Sol}(\Sigma))$ is the set of scrawls of a matroid.

Now, the FTTDAG can be used to give a sufficient condition for the solution set of tropical differential equations to be the set of scrawls of a matroid (see 5.2). This theorem gives an equality between the set $\mathcal{T}(\mathrm{Sol}(\Sigma))$ and the set of formal Boolean power series solutions $\mathrm{Sol}(trop([\Sigma]))$ as in Definition 5.3 of some system $trop([\Sigma]) \subset \mathbb{B}_{m,n}$.

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Definition 5.4 Given $P = \sum_M a_M E_M$ in $K_{m,n}$, we denote by $trop(P)$ the polynomial $trop(P) = \sum_M \mathcal{T}(a_M)E_M$ in $\mathbb{B}_{m,n}$.

Recall that if $\Sigma \subset K_{m,n}$, then $[\Sigma]$ denotes the differential ideal spanned by it. The following result defines DA tropical varieties in three different ways, and also justifies the name.

Theorem 5.2 [11, Fundamental Theorem] *Let $\Sigma \subset K_{m,n}$. Then, the following three subsets of $\mathbb{B}[\![t_1, \ldots, t_m]\!]^n$ coincide.*

(1) $X = \mathcal{T}(\mathrm{Sol}(\Sigma))$;
(2) $X = \mathrm{Sol}(trop([\Sigma])) = \bigcap_{P\in[\Sigma]} \mathrm{Sol}(trop(P))$;
(3) $X = \{w \in \mathbb{B}[\![T]\!]^n : in_w([\Sigma])\}$ *contains no monomial.*

Definition 5.5 Any subset $X \subset \mathbb{B}[\![t_1, \ldots, t_m]\!]^n$ satisfying one of the characterizations of the above theorem is called a DA tropical variety.

For a variety of examples and further discussion on the FTTDAG, see [6].
We now present the following result.

Corollary 5.3 *Let $U \subset \mathbb{B}_{m,n}$ and $X = \bigcap_{p\in U} \mathrm{Sol}(p)$. If $U = trop([\Sigma])$, where $\Sigma \subset K_{m,n}$ is a system of homogeneous linear differential equations of differential type zero, then $\phi(X)$ is the set of scrawls of $\phi(\mathcal{C}(\mathrm{Sol}(\Sigma)))$.*

Proof Follows from the above result after applying the Fundamental Theorem 5.2, since $\mathcal{T}(\mathrm{Sol}(\Sigma)) = \mathrm{Sol}(trop([\Sigma])) = \bigcap_{P\in[\Sigma]} \mathrm{Sol}(trop(P)) = X$. Then, we apply ϕ to both ends of the equality. ∎

We have shown that if $W = \mathrm{Sol}([\Sigma])$, where Σ is as in Corollary 5.3, then $M(W) = (2^{[n]\times\mathbb{N}^m}, \phi \circ \mathcal{T}(W))$ is a matroid. Since a satisfactory theory of duality exists for infinite matroids, and the dual $M(W)^*$ of $M(W)$ is representable and finitary, it should be interesting to know if its set of circuits $\mathcal{C}^*(W)$ relate to some notion of tropical basis for the ideal $[\Sigma]$.

Example 5.4 Let $\Sigma \subset K_{1,1}$ have the solutions W generated by $\varphi_1 = \sum_{i\geq 0} t^{2i}$, $\varphi_2 = \sum_{i\geq 0} t^{2i+1} \in K[\![t]\!]$. Since $n = 1$, there is no need to consider the map ϕ, and it follows that

$$\mathcal{T}(W) = \{\mathcal{T}(\varphi_1) = 2\cdot\mathbb{N}, \mathcal{T}(\varphi_2) = 2\cdot\mathbb{N}+1, \varnothing, \mathbb{N}\}.$$

Thus, $\mathcal{C} = \{C_1 = \mathcal{T}(\varphi_1), C_2 = \mathcal{T}(\varphi_2)\}$. The bases of $\mathcal{T}(W)$ are the maximal subsets of $\mathbb{N}$ that do not contain the set of even numbers nor the set of odd numbers. So they are complements of sets of the form $\{e, o\}$, where e is an even number and o is an odd number. In particular, the bases of the dual are these pairs $\{e, o\}$. The circuits of the dual (cocircuits) are now given as the pairs $\{e_1, e_2\}$, $\{o_1, o_2\}$, where e_1, e_2 are even and o_1, o_2 are odd numbers.

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6 Counterexample: the fundamental theorem of tropical DA geometry over a countable field

The article [1] by Aroca, Garay, and Toghani proves a fundamental theorem for tropical DA geometry over uncountable fields. Using a non-constructive result in [8], it was shown that there exists a system of algebraic partial differential equations over a countable field for which the fundamental theorem does not hold [10, Remark 7.3]. The question of whether the fundamental theorem holds for ordinary or even linear differential equations over countable fields remained open. As consequence of the results given in the previous sections, we give a negative answer to this question by constructing a counterexample as follows.

Fix a countable field $\Bbbk = \{a_0, a_1, a_2, \dots\}$, and consider the linear differential polynomial

$$\Sigma := \{y'' + \gamma(t)y' + \beta(t)y\} \subset \Bbbk_{1,1} \tag{6.1}$$

with $\gamma(t) = \sum_{i\geq 0} c_i t^i$ and $\beta(t) = \sum_{i\geq 0} b_i t^i$. In the following, we want to build b_i and c_i such that $\mathrm{Sol}(\Sigma)$ is generated by the two power series solutions (see Example 3.6)

$$\varphi_1 = 1 + \sum_{i\geq 2} t^i \quad \text{and} \quad \varphi_2 = t + \sum_{i\geq 2} a_i t^i.$$

Their supports are $\mathcal{T}(\varphi_1) = \mathbb{N}\backslash\{1\}$ and $\mathcal{T}(\varphi_2) = \mathbb{N}\backslash\{0\}$. For every $a_1, a_2 \in \Bbbk$, also $a_1\phi_1 - a_2\phi_2$ is a solution of Σ and

$$\mathcal{T}(a_1\varphi_1 - a_2\varphi_2) = \mathbb{N}\backslash\{j\}$$

for some $j \in \mathbb{N}$ (or $\varnothing$ in the case of $a_1 = a_2 = 0$). Note that there are no further solutions of Σ. The union $\mathcal{T}(\varphi_1) \cup \mathcal{T}(\varphi_2) = \mathbb{N}$ is a tropical solution (see Theorem 5.1), but cannot be realized by any solution of (6.1).

Let us now construct the coefficients b_i, c_i such that Σ has a solution set generated by φ_1, φ_2. By plugging φ_1 into (6.1), we obtain

$$\begin{aligned} 0 &= \sum_{i\geq 0}(i+2)(i+1)t^i + \Big(\sum_{i\geq 0} c_i t^i\Big)\cdot\Big(\sum_{i\geq 0}(i+1)t^i - 1\Big) + \Big(\sum_{i\geq 0} b_i t^i\Big)\cdot\Big(\sum_{i\geq 0} t^i - t\Big) \\ &= \sum_{i\geq 0}\Big((i+2)(i+1) + \sum_{j=0}^{i}(i+1-j)c_j - c_i + \sum_{j=0}^{i} b_j - b_{i+1}\Big)t^i. \end{aligned}$$

Similarly, by plugging-in φ_2 into (6.1) and setting $a_0 = 0, a_1 = 1$,

$$\begin{aligned} 0 &= \sum_{i\geq 0}(i+2)(i+1)a_{i+2}t^i + \Big(\sum_{i\geq 0} c_i t^i\Big)\cdot\Big(\sum_{i\geq 0}(i+1)a_{i+1}t^i\Big) + \Big(\sum_{i\geq 0} b_i t^i\Big)\cdot\Big(\sum_{i\geq 0} a_i t^i\Big) \\ &= \sum_{i\geq 0}\Big((i+2)(i+1)a_{i+2} + c_i + \sum_{j=0}^{i-1}(i+1-j)c_j a_{i+1-j} + \sum_{j=0}^{i-1} b_j a_{i-j}\Big)t^i. \end{aligned}$$

By coefficient comparison in both equations, seen as polynomials in t, we obtain the system of recurrence equations

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$$b_{i+1} = (i+2)(i+1) + \sum_{j=0}^{i}(i+1-j)c_j - c_i + \sum_{j=0}^{i} b_j,$$

$$c_{i+1} = -(i+3)(i+2)a_{i+3} - \sum_{j=0}^{i}(i+2-j)c_j a_{i+2-j} - \sum_{j=0}^{i} b_j a_{i+1-j}$$

for every $i \in \mathbb{N}$. By choosing b_0, c_0 as any element in $\Bbbk$, $\gamma(t), \beta(t)$ are uniquely determined from the given solutions φ_1, φ_2. Let us fix $b_0, c_0 \in \Bbbk$.

Then, since φ_1, φ_2 are linearly independent and Σ consists of a single linear ordinary differential equation of order two, the solution set is indeed given exactly as the linear combinations of φ_1 and φ_2.

Remark 6.1 Notice that $\Bbbk$ can be chosen as the algebraic closure of the rational numbers. Then, $\gamma(t)$ and $\beta(t)$ are (non-convergent) formal power series and Σ is a system of linear differential equations involving formal power series coefficients. We thus have shown that for this frequently used case, the fundamental theorem does not hold. For holononomic systems, i.e., when all coefficients of the elements in Σ are polynomial, this question remains open.

Remark 6.2 Within this article, we have worked with homogeneous linear differential systems together with linear spaces. We expect that the results can be generalized to affine representable matroids [23, Section 1.5] and (non-homogeneous) linear differential systems of differential type zero.

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