A note on integral functions. By Mr R. E. A. C. PALEY, Trinity College.

[*Received* 7 March, *read* 16 May 1932.]

1. Let $f(z)$ denote an integral function of finite order ρ. We write

$$T(r, f(z)) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta,$$

$$M(r, f(z)) = \max_{0\leqslant\theta\leqslant 2\pi} |f(re^{i\theta})|.$$

It has been shown* that

$$\limsup \frac{T(r, f(z))}{\log M(r, f(z))} \geqslant h_\rho, \tag{1·1}$$

where h_ρ is a constant which depends only on ρ. We are naturally led to enquire whether some equation of the form (1·1) may be true with lim sup replaced by lim inf. In this note we show that the reverse is true. We construct an integral function of zero order for which

$$\liminf \frac{T(r, f(z))}{\log M(r, f(z))} = 0. \tag{1·2}$$

The proof may easily be modified to construct a function of any finite order or of infinite order for which (1·2) is satisfied.

2. We employ the function†

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)} \qquad (0<\alpha<2). \tag{2·1}$$

The function (2·1) is an integral function of order $1/\alpha$, and satisfies the inequalities

$$\left| E_\alpha(z) - \frac{1}{\alpha} e^{z^{1/\alpha}} \right| < \frac{M_\alpha}{|z|} \qquad (|\arg z| \leqslant \tfrac{1}{2}\alpha\pi),$$

$$|E_\alpha(z)| < M_\alpha \qquad (|\arg z| \geqslant \tfrac{1}{2}\alpha\pi),$$

where the constants M_α depend only on α. We have

$$T(r, E_\alpha(z)) \sim \frac{\alpha}{\pi} r^{1/\alpha}$$

and

$$\log M(r, E_\alpha(z)) \sim r^{1/\alpha},$$

* For $\rho \leqslant \frac{1}{2}$, the best possible value of h_ρ is known to be $(\sin \pi\rho)/\pi\rho$. For $\rho \geqslant \frac{1}{2}$ it seems likely that the correct constant is $1/(\pi\rho)$, which is the value for the function (2·1), with $\alpha = 1/\rho$. Miss M. L. Cartwright has shown me a proof of (1·1) with a constant h_ρ which for large values of ρ is asymptotically $1/(2e\rho)$.

† See e.g. L. Bieberbach, *Lehrbuch der Functionentheorie*, 2 (1927), 265–269.

https://doi.org/10.1017/S0305004100010100 Published online by Cambridge University Press

so that, for the function $E_\alpha(z)$, we have

$$\lim \frac{T(r)}{\log M(r)} = \frac{\alpha}{\pi}, \tag{2·2}$$

which is arbitrarily small by choice of α.

Now let $F(r)$ denote the function $r^{\log^2 r} = e^{\log^3 r}$ $(r \geqslant e)$. In the investigation which follows, $F(r)$ may be replaced by a function of the form $e^{\beta r}$ $(\beta > 0)$, and other functions of a quite general type, but, to avoid complexity of detail, we assume that $F(r)$ is of the form stated.

Let R be a fixed number. We write

$$a = R\left(\frac{2e}{\alpha k}\right)^\alpha \log^{-2\alpha} R,$$

where $k = k(\alpha)$ remains to be chosen. Consider the polynomial

$$E_\alpha'(z) = E'_{\alpha, R}(z) = \sum_{n=0}^{[k\log^2 R]} \frac{(z/a)^n}{\Gamma(1+\alpha n)}.$$

For $|z| \leqslant R$, we have

$$|E_\alpha'(z)| \leqslant \frac{1}{\alpha} e^{\left(\frac{|z|}{a}\right)^{1/\alpha}} \leqslant \frac{1}{\alpha} e^{\frac{\alpha k}{2e}\log^2 R \left(\frac{|z|}{R}\right)^{1/\alpha}} \leqslant \tfrac{1}{2} F(|z|),$$

if R is sufficiently large. If $|z| \geqslant R$, then

$$\begin{aligned} |E_\alpha'(z)| &\leqslant \sum_{n=0}^{[k\log^2 R]} \frac{(R/a)^n}{\Gamma(1+\alpha n)} \left(\frac{|z|}{R}\right)^{k\log^2 R} \\ &\leqslant \frac{1}{\alpha} \exp\left(\frac{\alpha k}{2e}\log^2 R + k\log|z|\log^2 R - k\log^3 R\right) \\ &\leqslant \tfrac{1}{2} F(|z|), \end{aligned}$$

if $k(\alpha)$ is chosen sufficiently small, and R sufficiently large. It follows that, for all z,

$$|E'_{\alpha, R}(z)| \leqslant \tfrac{1}{2} F(|z|) \qquad (R \geqslant R_0(\alpha)). \tag{2·3}$$

On the circle $|z| = R$, we have

$$|z|/a = \left(\frac{\alpha k}{2e}\right)^\alpha \log^{2\alpha} R,$$

and

$$\begin{aligned} \left|E_\alpha\left(\frac{z}{a}\right) - E_\alpha'(z)\right| &\leqslant \sum_{n=[k\log^2 R]+1}^{\infty} \frac{\left(\frac{\alpha k}{2e}\right)^{\alpha n} \log^{\alpha n} R}{\Gamma(1+\alpha n)} \\ &\leqslant B_1 \sum_{n=[k\log^2 R]+1}^{\infty} \frac{\left(\frac{\alpha k}{2e}\right)^{\alpha n} \log^{2\alpha n} R}{(\alpha n)^{\alpha n}} e^{\alpha n} \\ &\leqslant B_2 2^{-\alpha n}, \end{aligned} \tag{2·4}$$

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where B_1, B_2 denote positive constants which depend only on α. On the circle $|z| = R$,

$$\left| E_\alpha\left(\frac{z}{a}\right) - \frac{1}{\alpha} e^{\frac{ak}{2e}\log^2 R e^{i\theta/\alpha}} \right| \leqslant M_\alpha \qquad (|\arg z| \leqslant \tfrac{1}{2}\alpha\pi), \tag{2·5}$$

$$\left| E_\alpha\left(\frac{z}{a}\right) \right| \leqslant M_\alpha \qquad (|\arg z| \geqslant \tfrac{1}{2}\alpha\pi), \tag{2·6}$$

where the constants M_α depend only on α. It follows from (2·2), (2·4), (2·5), (2·6) that, if R is sufficiently large, then

$$\frac{T(R, \alpha E_\alpha'(z))}{\log M(R, \alpha E_\alpha'(z))} \leqslant \tfrac{1}{2}\alpha. \tag{2·7}$$

3. We now write

$$E^*(z) = \sum_{n=1}^{\infty} \alpha_n E'_{a_n, R_n}(z),$$

where $\alpha_n = n^{-2}$, $R_1 = 10$, and $R_2, R_3, \ldots, R_n, \ldots$ remain to be chosen. We denote by k_n the number $k(\alpha_n)$ defined in the last paragraph. Suppose that $R_1, R_2, \ldots, R_{n-1}$ have already been chosen. We give an inductive method for constructing R_n. We observe that we can choose b so large that

$$\sum_{m=0}^{\lambda} \frac{(|z|/b)^m}{\Gamma(1+\alpha_n m)} \leqslant 2 \qquad (|z| \leqslant R_{n-1},\ 0 \leqslant \lambda < \infty). \tag{3·1}$$

We choose $R_n \geqslant 2R_{n-1}$ sufficiently large to ensure that the inequalities (2·3), (2·7) are satisfied with $R = R_n$, $\alpha = \alpha_n$, that

$$R_n \left(\frac{2e}{\alpha_n k_n}\right)^{\alpha_n} \log^{-2} R_n \geqslant b, \tag{3·2}$$

in virtue of (2·5) that

$$\log M(R_n, \alpha_n E'_{a_n, R_n}(z)) \geqslant \frac{\alpha_n k_n}{4e} \log^2 R_n, \tag{3·3}$$

that

$$\frac{1}{\alpha_n k_n} < 8e\sqrt{\log R_n}, \tag{3·4}$$

and finally, since

$$\sum_{m=1}^{n-1} \alpha_m E'_{a_m, R_m}(z)$$

is a polynomial, that

$$\log M\left(R_n, \sum_{m=1}^{n-1} \alpha_m E'_{a_m, R_m}(z)\right) \leqslant \frac{\alpha_n^2 k_n}{8e} \log^2 R_n. \tag{3·5}$$

We observe that, in virtue of (3·1), (3·2), we have

$$|E'_{a_n, R_n}(z)| \leqslant 2 \qquad (|z| \leqslant R_{n-1}). \tag{3·6}$$

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In virtue of (2·3) we have

$$|E^*(z)| \leqslant \sum_{n=1}^{\infty} \tfrac{1}{2} n^{-2} F(|z|) \leqslant F(|z|),$$

so that $E^*(z)$ is an integral function of zero order. Now consider the behaviour of $E^*(z)$ on the circles $|z| = R_n$. We have first, in virtue of (3·3), (3·5), (3·6),

$$\log M(R_n, E^*(z)) \geqslant \frac{\alpha_n k_n}{8e} \log^2 R_n \qquad (n \geqslant n_0), \tag{3·7}$$

$$\log M(R_n, E^*(z)) \geqslant \tfrac{1}{2} \log M(R_n, \alpha_n E'_{a_n, R_n}(z)) \qquad (n \geqslant n_0). \tag{3·8}$$

The inequalities (3·4), (3·7) show that

$$\log M(R_n, E^*(z)) \geqslant \log^{3/2} R_n,$$

so that $E^*(z)$ is certainly not a polynomial. Next we have, by (3·5), (3·6),

$$T\left(R_n, \sum_{m=1}^{n-1} \alpha_m E'_{a_m, R_m}(z)\right) \leqslant \frac{\alpha_n^2 k_n}{8e} \log^2 R_n, \tag{3·9}$$

$$T\left(R_n, \sum_{m=n+1}^{\infty} \alpha_m E'_{a_m, R_m}(z)\right) \leqslant 4, \tag{3·91}$$

so that by (2·7), (3·7), (3·8), (3·9), (3·91) we have, for $n \geqslant n_0$,

$$\begin{aligned} &\frac{T(R_n, E^*(z))}{\log M(R_n, E^*(z))} \\ &\leqslant \left[T\left(R_n, \sum_{m=1}^{n-1} \alpha_m E'_{a_m, R_m}(z)\right) + T(R_n, \alpha_n E'_{a_n, R_n}(z)) \right. \\ &\qquad \left. + T\left(R_n, \sum_{m=n+1}^{\infty} \alpha_m E'_{a_m, R_m}(z)\right) + \log 3 \right] \Big/ \log M(R_n, E^*(z)) \\ &\leqslant \alpha_n + \alpha_n + \frac{6}{\log M(R_n, E^*(z))}, \end{aligned}$$

which is arbitrarily small as n tends to infinity.

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