Area formulas for certain convex quadrilaterals

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1 *Introduction*

The purpose of this paper is to collect for the first time area formulas for all convex quadrilaterals that can be expressed in terms of their four sides alone. There are presently 23 such classes that we know of. Many mathematics teachers would probably find it challenging just to name that many different classes of quadrilaterals, and even more difficult to tell how to calculate their different areas. Several of these classes of quadrilaterals are included neither in MathWorld, Wikipedia nor in any other encyclopedia that we know of. The formulas can be sorted into four categories as either well known, a special case of another formula, less known, or previously unknown (at least to us). The last category only includes two formulas and we will prove these. The other formulas are not proved here, but we either give references to where such proofs can be found or we give a hint on how the derivations can be made when a formula is a special case of another formula. We encourage the reader both to try the derivations themselves, and also to draw figures for each type of quadrilateral based on their definitions. Note that we only consider *convex* quadrilaterals without stating this in every subsection. In this entire paper we use the notations that a convex quadrilateral $ABCD$ has the sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ and area K.

2 *A collection of area formulas*

2.1 *Right quadrilateral*

There exist at least two different definitions for what is meant by a right quadrilateral: a quadrilateral with just one right angle or a quadrilateral with two opposite right angles. In this paper we use the former definition (in contrast to what we did in a paper on tilted kites a year ago in another journal). In this case a right quadrilateral has area

$$K = \tfrac{1}{2}ab + \tfrac{1}{4}\sqrt{4c^2d^2 - (c^2 + d^2 - a^2 - b^2)^2}$$

when the right angle is between the sides a and b. This formula is a special case of a formula for the area of a convex quadrilateral stated at [1] (without a reference, but it is correct). You just set the angle α between the sides a and b in that formula equal to $\frac{\pi}{2}$. Another easy way of obtaining this right quadrilateral area formula is by adding the area of a right triangle with legs a and b, and the area of a triangle whose sides are c, d and $\sqrt{a^2 + b^2}$, which is directly obtained using a semi-factorised version of Heron's formula.

2.2 *Tilted kite*

A tilted kite is a quadrilateral with at least one pair of opposite equal angles [1, p. 192]. If $\angle A = \angle C$ and it is neither a kite nor a parallelogram,

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then a tilted kite has area

$$K = \frac{ad + bc}{4|ad - bc|} Q_1$$

where

$$Q_1 = \sqrt{(-a+b+c-d)(a-b+c-d)(a+b-c-d)(a+b+c+d)}.$$

This formula was included in a collection of geometry problems by Meier Hirsch in 1805. A proof can be found in [2, pp. 97-98]. In the next three area formulas there are similar square root expressions.

2.3 *Cyclic quadrilateral*

A cyclic quadrilateral is a quadrilateral that is inscribed in a circle. Its area is given by the famous Brahmagupta's formula, which states that

$$K = \tfrac{1}{4} Q_2$$

where

$$Q_2 = \sqrt{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}.$$

The derivation of this formula can be found in many places, for instance in [3, p. 58] and at numerous websites.

2.4 *Trapezium*

A general trapezium (trapezoid in American English) is a quadrilateral with at least one pair of opposite parallel sides (the bases). If these are a and c, where $a \neq c$, then the trapezium has area

$$K = \frac{a + c}{4|a - c|} Q_3$$

where

$$Q_3 = \sqrt{(-a+b+c+d)(a+b-c+d)(a-b-c+d)(a+b-c-d)}.$$

This formula is a direct consequence of a formula for the trapezium height (see [4, p. 35], which needs to be factorised) and the well-known fact that the area of a trapezium is equal to the arithmetic mean of the bases times the height.

2.5 *Bisect-diagonal quadrilateral*

A bisect-diagonal quadrilateral is a quadrilateral where at least one of its diagonals bisects the other diagonal [5, p. 214]. If diagonal AC is bisected by diagonal BD, and the quadrilateral is neither a kite nor a parallelogram, then the bisect-diagonal quadrilateral has area

$$K = \frac{Q_2 Q_4}{4|a^2 - b^2 - c^2 + d^2|}$$

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where

$$Q_2 = \sqrt{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)},$$

$$Q_4 = \sqrt{(a+b+c+d)(a-b-c+d)(a-b+c-d)(a+b-c-d)}.$$

This formula was derived for the first time in [5, p. 220], but unfortunately that publication had a misprint: the factor 4 in the denominator was missing.

2.6 *Cyclic orthodiagonal quadrilateral*

A cyclic orthodiagonal quadrilateral is a quadrilateral with perpendicular diagonals that is inscribed in a circle. It has area

$$K = \tfrac{1}{2}(ac + bd)$$

which was derived in two different ways in [6, p. 222].

2.7 *Bicentric quadrilateral and exbicentric quadrilateral*

A bicentric quadrilateral is a cyclic quadrilateral with an incircle (the sides satisfy $a + c = b + d$) whereas an exbicentric quadrilateral is a cyclic quadrilateral with an excircle: a circle tangent to the extensions of all four sides (the sides satisfy $a + b = c + d$ or $a + d = b + c$). Their areas are given by the same formula, which we consider to be the most beautiful of all quadrilateral area formulas:

$$K = \sqrt{abcd}.$$

One proof of this formula can be found in [7, p. 155] for bicentric quadrilaterals and in [8, p. 21] for exbicentric quadrilaterals.

2.8 *Tangential trapezium*

A tangential trapezium is a trapezium with an incircle. Using Pitot's theorem $a + c = b + d$ in the formula for a general trapezium yields that a tangential trapezium has area

$$K = \frac{a+c}{|a-c|}\sqrt{ac\,(a-b)(b-c)}$$

when the unequal bases are a and c.

2.9 *Right trapezium*

A right trapezium is a trapezium with at least two adjacent right angles. If its bases are a and c, and the side d is perpendicular to them (which makes it the height), then the right trapezium has area

$$K = \tfrac{1}{2}d\,(a + c).$$

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2.10 *Hjelmslev quadrilateral*

A Hjelmslev quadrilateral is a quadrilateral with at least one pair of opposite right angles [2. pp. 429-430]. Thus one of its diagonals divide it into two right triangles. If this diagonal is AC, then it has area

$$K = \tfrac{1}{2}(ab + cd).$$

This formula is a special case of the formula for the are of a right quadrilateral, where $a^2 + b^2 = c^2 + d^2$.

2.11 *Orthodiagonal trapezium*

An orthodiagonal trapezium is a trapezium with perpendicular diagonals. Since the area of an orthodiagonal quadrilateral is half the product of the diagonals, and the diagonals in a trapezium have known formulas (see [4, p. 31]), an orthodiagonal trapezium with unequal parallel sides a and c has area

$$K = \frac{\sqrt{(ac(a-c) + ad^2 - cb^2)(ac(a-c) + ab^2 - cd^2)}}{2|a-c|}.$$

2.12 *Isosceles trapezium*

The isosceles trapezium can be defined in several different ways. The two best definitions, according to us, are that it is a quadrilateral with two distinct pairs of adjacent equal angles, or a trapezium with two equal base angles. Its area formula is a special case of the formula for a general trapezium, using the property that $b = d$. Thus it has area

$$K = \tfrac{1}{4}(a + c)\sqrt{4b^2 - (a-c)^2}.$$

If also $a = c$, then we have a rectangle and the formula simplifies to $K = ab$.

2.13 *Bisect-equidiagonal quadrilateral*

A bisect-equidiagonal quadrilateral is a quadrilateral with diagonals that are both equal in length and where at least one of them is bisecting the other [5, p. 224]. (It was called an 'equidiagonal sliding-kite' in [10, p. 74], but we think bisect-equidiagonal quadrilateral is a less ambiguous name.) Suppose the pair of sides a, b and c, d are on different sides of the bisecting diagonal. Then the area is

$$K = \tfrac{1}{8}\sqrt{(3a^2 - b^2 + 3c^2 - d^2 + Q)(-a^2 + 3b^2 - c^2 + 3d^2 + Q)}$$

where

$$Q = \sqrt{8(a^2 + d^2)(b^2 + c^2) - (a^2 + b^2 + c^2 + d^2)^2}.$$

This formula and the next one will be derived at the end of this paper.

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2.14 *Equitriangular kite*

In [9, p. 207] Michael de Villiers called a kite with at least three equal angles a 'triangular kite'. We however prefer the name *equitriangular kite* which emphasises the defining condition. If the fourth angle is between the two sides of length b and the other two sides have length a, then the area is given by

$$K = \tfrac{1}{2}a\sqrt{(a+b)(3b-a)}.$$

2.15 *Midsquare quadrilateral*

A midsquare quadrilateral is a quadrilateral with diagonals that are both equal in length and perpendicular [10, p. 74] (they have also been called 'skewsquares'). The midpoints of its four sides are the vertices of a square (hence the name). This quadrilateral has area

$$K = \tfrac{1}{4}\left(a^2 + c^2 + \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2}\right),$$

which was derived in [11, pp. 139-140]. The formula seems unsymmetrical, but note that $a^2 + c^2 = b^2 + d^2$ holds in all quadrilaterals with perpendicular diagonals.

2.16 *Trilateral trapezium*

A trilateral trapezium (has also been called 'trisosceles trapezium') is a special case of an isosceles trapezium where there are at least three equal sides $b = c = d$. It has area

$$K = \tfrac{1}{4}\sqrt{(a+b)^3(3b-a)}.$$

If also the fourth side is equal to the other three, the formula yields $K = a^2$ which is the area of a square.

2.17 *Midsquare trapezium*

A midsquare trapezium is a special case of an orthodiagonal trapezium with diagonals of equal length [10, p. 80]. This means that it also has equal lateral sides $b = d$, so its area is

$$K = \tfrac{1}{2}\left(ac + b^2\right).$$

2.18 *Bicentric trapezium*

A bicentric trapezium is a trapezium with both an incircle and a circumcircle. Since it is a special case of an isosceles trapezium with $2b = a + c$, it has area

$$K = \tfrac{1}{2}(a+c)\sqrt{ac}$$

where a and c are the lengths of the bases. It can also be stated as $K = b\sqrt{ac}$ where b is the common length of the lateral sides, which is a special case of the formula for the area of a bicentric quadrilateral when $b = d$.

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2.19 *Midsquare kite*

A midsquare kite is a quadrilateral with diagonals that have the triple properties of being equal in length, perpendicular and where at least one of them bisects the other [10, p. 74]. It is also a kite in which the midpoints of its sides form a square. Its area is given by

$$K = \tfrac{1}{4}\left(a^2 + c^2 + \sqrt{4a^2c^2 - (a^2 - c^2)^2}\right)$$

since it is a special case of the formula for the area of a midsquare quadrilateral with $b = c$ and $d = a$. When also $a = c$, we get the square formula $K = a^2$.

2.20 *Right kite, rectangle and square*

A right kite is a kite with at least one pair of opposite right angles. Since it is composed of two congruent right triangles, it has the area $K = ab$, the same as the rectangle. A square is a special case of both a right kite and a rectangle with $a = b$, so it has area $K = a^2$.

3 *Proofs*

We conclude by deriving two formulas, which are neither special cases of other included formulas we have listed nor formulas for which we have references.

3.1 *Bisect-equidiagonal quadrilateral*

Since the diagonals are of equal length and one of them is bisecting the other, we can use Apollonius' theorem to first get an expression for the diagonal length p. Thus

$$\tfrac{1}{2}\sqrt{2(a^2 + d^2) - p^2} + \tfrac{1}{2}\sqrt{2(b^2 + c^2) - p^2} = p.$$

Multiplying both sides by 2, squaring, rewriting and then dividing both sides by 2 yields

$$\sqrt{2(a^2 + d^2) - p^2}\sqrt{2(b^2 + c^2) - p^2} = 3p^2 - \left(a^2 + b^2 + c^2 + d^2\right).$$

Squaring again, expanding and collecting similar terms, we get

$$8p^4 - 4\left(a^2 + b^2 + c^2 + d^2\right)p^2 + \left(a^2 + b^2 + c^2 + d^2\right)^2 - 4\left(a^2 + d^2\right)\left(b^2 + c^2\right) = 0.$$

This quadratic equation in p^2 has solutions

$$p^2 = \tfrac{1}{4}\left(a^2 + b^2 + c^2 + d^2 \pm \sqrt{8(a^2 + d^2)(b^2 + c^2) - (a^2 + b^2 + c^2 + d^2)^2}\right).$$

To decide which sign is correct we study the special case of a square. By inserting $a = b = c = d$ we get

$$p^2 = \tfrac{1}{4}\left(4a^2 \pm 4a^2\right)$$

and we see that plus is obviously the correct sign since $p^2 \neq 0$. It is quite well known that the area K of any convex quadrilateral with diagonals p and q is given by the following formula (derived independently by the German

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mathematicians von Staudt and Bretschneider in 1842):

$$16K^2 = 4p^2q^2 - \left(a^2 - b^2 + c^2 - d^2\right)^2.$$

Factorising the right-hand side and using $p = q$ gives

$$16K^2 = \left(2p^2 + a^2 - b^2 + c^2 - d^2\right)\left(2p^2 - a^2 + b^2 - c^2 + d^2\right).$$

Multiplying both sides by 2×2, inserting the expression for $4p^2$ and solving for K yields

$$K = \tfrac{1}{8}\sqrt{(3a^2 - b^2 + 3c^2 - d^2 + Q)(-a^2 + 3b^2 - c^2 + 3d^2 + Q)}$$

where we define

$$Q = \sqrt{8\,(a^2 + d^2)(b^2 + c^2) - (a^2 + b^2 + c^2 + d^2)^2}.$$

3.2 *Equitriangular kite*

An equitriangular kite with angles α, α, α, β and sides a, a, b, b, where β is between the two sides of length b, has area

$$K = \tfrac{1}{2}a^2 \sin\alpha + \tfrac{1}{2}b^2 \sin\beta = \tfrac{1}{2}a^2 \sin\alpha - \tfrac{1}{2}b^2 \sin 3\alpha$$

since $\beta = 2\pi - 3\alpha$. But the area can also be expressed by dividing the kite along the other diagonal, so

$$K = 2 \times \tfrac{1}{2}ab \sin\alpha = ab \sin\alpha.$$

Using the triple angle formula for sine and equating these area expressions yields

$$\tfrac{1}{2}a^2 \sin\alpha - \tfrac{1}{2}b^2 \sin\alpha\left(3 - 4\sin^2\alpha\right) = ab \sin\alpha.$$

Rewriting this using $\sin\alpha \neq 0$, we get

$$\sin^2\alpha = \frac{3b^2 + 2ab - a^2}{4b^2}.$$

Thus the area of an equitriangular kite is given by

$$K = ab \sin\alpha = \tfrac{1}{2}a\sqrt{3b^2 + 2ab - a^2} = \tfrac{1}{2}a\sqrt{(a + b)(3b - a)}.$$

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The answers to the *Nemo* page from November on parabolas were:

1.	Edgar Rice Burroughs	Tarzan the Untamed	Chapter 1
2.	Amy Lowell:	A Roxbury Garden	
3.	Charles Reade	The Cloister and the Hearth	Chapter 63
4.	George Orwell	Charles Dickens	Chapter 6
5.	DH Lawrence	Bat	
6.	Thomas Hardy	Far from the Madding Crowd	Chapter 46

Congratulations to Ian Anderson and Martin Lukarewski on tracking all of these down. Working through the conics, the theme this time is ellipses. The quotations are to be identified by reference to author and work. One quotation is a translation from the French. Solutions are invited to the Editor by 24th May 2018.

1. His world was now an ellipse, with a dual centrality, of which his own was not the major.

2. We sleep and wake and sleep, but all things move;
The Sun flies forward to his brother Sun;
The dark Earth follows wheel'd in her ellipse;
And human things returning on themselves
Move onward, leading up the golden year.

Continued on page 86.

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