Ergod. Th. & Dynam. Sys. (2017), **37**, 1238–1252 © *Cambridge University Press, 2016*
doi:10.1017/etds.2015.73

Fixed points of local actions of nilpotent Lie groups on surfaces

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(*Received* 11 *May* 2014 *and accepted in revised form* 26 *July* 2015)

Abstract. Let G be a connected nilpotent Lie group with a continuous local action on a real surface M, which might be non-compact or have non-empty boundary ∂M. The action need not be smooth. Let φ be the local flow on M induced by the action of some one-parameter subgroup. Assume K is a compact set of fixed points of φ and U is a neighborhood of K containing no other fixed points.

THEOREM. *If the Dold fixed-point index of* $\varphi_t|U$ *is non-zero for sufficiently small* $t > 0$, *then* $\mathsf{Fix}(G) \cap K \neq \varnothing$.

1. *Introduction*

Poincaré [**29**] published a seminal result on surface dynamics in 1885, that was extended to higher dimensions by Hopf [**17**] in 1925, and generalized to maps in compact polyhedra and topological manifolds by Lefschetz [**20**] in 1926. We will use the following version.

THEOREM. (Poincaré–Hopf–Lefschetz) *Every flow on a compact manifold* N *of non-zero Euler characteristic* $\chi(N)$ *has a fixed point.*

In his pioneering 1964 paper, Lima [**21**] generalized this result to actions of connected abelian Lie groups on compact surfaces. In 1986, this was extended to nilpotent groups by Plante [**28**].

THEOREM. [**28**] *Every action of a connected nilpotent Lie group on a compact surface of non-zero Euler characteristic has a fixed point.*

Our main result is Theorem 1.1, which is an extension of Plante's theorem to local actions of nilpotent Lie groups on arbitrary surfaces.

Throughout this article, G denotes a connected Lie group of positive dimension having unit element e_G. The Lie algebra $\mathfrak{g}$ of G is identified with the set of homomorphisms $\mathbb{R} \to G$.

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A *local action* (G, M, α) of G on M assigns to each $g \in G$ a local homeomorphism

$$\alpha(g) : \mathcal{D}\alpha(g) \approx \mathcal{R}\alpha(g)$$

between open subsets of M, satisfying certain conditions (see §2). We may abuse notation by omitting 'α' and writing g for $\alpha(g)$.

Define the *fixed-point sets*

$$\mathsf{Fix}(g) := \{x \in \mathcal{D}g : g(x) = x\},$$
$$\mathsf{Fix}(G) := \bigcap_{g \in G} \mathsf{Fix}(g).$$

The local action is *effective* if

$$\mathsf{Fix}(g) = \mathcal{D}g \implies g = e_G.$$

For each $X \in \mathfrak{g}$, composing the homomorphism $X : \mathbb{R} \to G$ with the maps $\alpha(g)$ produces a local action $(\mathbb{R}, M, X^\alpha)$, called a *local flow*, defined by

$$X^\alpha(t) : p \mapsto \alpha(X(t))(p) \quad (p \in \mathcal{D}\,\alpha(X(t)).$$

Define

$$\mathsf{Fix}(X) := \bigcap_{t \in \mathbb{R}} \mathsf{Fix}(\alpha(X(t))).$$

The following terms are made precise in §3. Assume (G, M, α) is given. A *block* for X (or an X*-block*) is a compact, relatively open set $K \subset \mathsf{Fix}(X)$. An open neighborhood $U \subset M$ of K is *isolating* for X if its closure $\overline{U}$ is compact and $\mathsf{Fix}(X) \cap \overline{U} = K$.

The *index* $\mathsf{i}_K(X)$ is an integer, defined in §3 as Dold's fixed-point index of $\alpha_t^X | U$ for any isolating neighborhood U of K and any sufficiently small $t > 0$. This index is completely determined by M, X and K. The X-block K is *essential* if $\mathsf{i}_K(X) \neq 0$.

THEOREM 1.1. *Let G be a connected nilpotent Lie group with an effective local action on a surface M, and let $X \in \mathfrak{g}$ be arbitrary. Then* $\mathsf{Fix}(G)$ *meets every essential X-block.*

This reduces to Plante's theorem when M is compact, but even in that case it gives further information, as follows.

COROLLARY 1.2. *Let G, M and X be as in Theorem 1.1.*

(i) *If $\Gamma \subset M$ is a compact attractor for X and $\chi(\Gamma) \neq 0$, then* $\mathsf{Fix}(G) \cap \Gamma \neq \varnothing$.

(ii) *If X has ν essential blocks, then* $\mathsf{Fix}(G)$ *has ν components.*

1.1. *Discussion.* Theorem 1.1 is inspired by the following remarkable result of Bonatti [**4**].

THEOREM. *Assume* $\dim(M) \leq 4$, $\partial M = \varnothing$, *and X, Y are commuting analytic vector fields on M. If K is an essential block for the local flow generated by X, then* $\mathsf{Z}(Y) \cap K \neq \varnothing$†.

† 'The demonstration of this result involves a beautiful and quite difficult local study of the set of zeros of X, as an analytic Y-invariant set. Of course, analyticity is an essential tool in this study, and the validity of this type of result in the smooth case remains an open and apparently hard question.' —Molino [**24**].

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When $\partial M = \varnothing$ and X is a C^1 vector field on M, our definition of the index of X in K extends Bonatti's, which runs as follows. If $U \subset M$ is an isolating neighborhood of K then $\mathsf{i}_K(X)$ equals the intersection number of the images in the tangent bundle of M of $X|U$ and the trivial vector field on U.

Plante's theorem does not extend to Lie groups that are soluble, or even supersoluble†, according to Lima [**21**] and Plante [**28**].

Related work on the dynamics of Lie group actions can be found in the papers [**1**, **2**, **5**, **8–11**, **13–15**, **18**, **19**, **22**, **23**, **25**, **27**, **30**, **31**, **33**, **34**], which are listed in the references.

Open questions. Are there smooth commuting vector fields on the compact 3-ball that point inward at the boundary and have no common fixed points?

Does every compact surface support a smooth fixed-point free action by a solvable Lie group?

Is there an example of a connected nilpotent Lie group acting without fixed-point on a compact 4-manifold of non-zero characteristic?

Does Bonatti's theorem generalize to three or more vector fields, or to manifolds of dimension five or greater?

1.2. *Terminology.* The empty set is denoted by $\varnothing$. The sets of integers, natural numbers and positive natural numbers are $\mathbb{Z}$, $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+$, respectively. The set of real numbers is $\mathbb{R}$.

The closure of a set S in a topological space $\mathcal{X}$ is $\overline{S}$, and its frontier is $\mathsf{Fr}(S) := \overline{S} \cap \overline{X \setminus S}$.

Maps are continuous unless the contrary is indicated. The domain of f is $\mathcal{D}f$ and its range is $\mathcal{R}f := f(\mathcal{D}f)$. We may write $f \cdot x$ for $f(x)$, with the tacit assumption that $x \in \mathcal{D}f$.

Every ordered pair (C, C') of oriented circles (Jordan curves) in an oriented surface M has an intersection number $C\#C' \in \mathbb{Z}$. It depends only on the homology classes $[C], [C'] \in H_1(M)$ represented by C and C', and is denoted also by $[C] \# [C']$.

A connected oriented surface M has finite genus $\mathsf{g}(M) := k \in \mathbb{N}_+$ if k is the largest integer such that M contains k mutually disjoint circles. An equivalent condition is that M contains k, but not $k+1$, mutually disjoint sets, each of which is the union of two oriented circles having positive intersection number. If no such k exists, we set $\mathsf{g}(M) = \infty$. The genus of non-orientable connected surface is defined as the genus of its orientable double covering space. The genus of a disconnected surface is the sum of the genera of its components.

M has genus 0 if and only if the intersection pairing $H_1(M) \times H_1(M) \to \mathbb{R}$ is trivial. A compact surface has genus 0 if and only if it embeds in the unit sphere $\mathbf{S}^2$ or the projective plane $\mathbf{P}^2$.

2. *Local actions*

If $f : A \to B$ denotes a map, its domain is $\mathcal{D}f := A$ and its range is $\mathcal{R}f := f(A)$.

Let g, f denote maps. Regardless of their domains and ranges, the composition $g \circ f$ is defined as the map $g \circ f : x \mapsto g(f(x))$ whose domain, perhaps empty, is $f^{-1}(\mathcal{D}g)$.

† Supersoluble: isomorphic to a multiplicative group of upper triangular real matrices.

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The associative law holds for these compositions. The maps $(h \circ g) \circ f$ and $h \circ (g \circ f)$ have the same domain

$$D := \{x \in \mathcal{D}f : f(x) \in \mathcal{D}g,\ g(f(x)) \in \mathcal{D}h\},$$

and

$$x \in D \implies (h \circ g)(f(x)) = h((g \circ f)(x)).$$

A *local homeomorphism* f on a topological space Q is a homeomorphism between open subsets of Q. The set of these homeomorphisms is denoted by $\mathsf{LH}(Q)$.

Definition 2.1. A *local action* of G on a manifold M is a triple (G, M, α), where $\alpha : G \to \mathsf{LH}(M)$ is a function having the following properties.

- The set $\Omega(\alpha) := \{(g, p) \in G \times M : p \in \mathcal{D}\alpha(g)\}$ is an open neighborhood of $\{e_G\} \times M$.
- The *evaluation map*
$$\Omega(\alpha) \to M, \quad (g, p) \mapsto \alpha(g) \cdot p$$
is continuous.
- $\alpha(e_G)$ is the identity map of M.
- The maps $\alpha(fg) \circ \alpha(h)$ and $\alpha(f) \circ \alpha(gh)$ agree on the intersection of their domains.
- $\alpha(g^{-1}) = \alpha(g)^{-1}$.

Notation of α may be omitted.

The *orbit* of p is the set

$$G \cdot p := \{g \cdot p : g \in G,\ p \in \mathcal{D}g\}.$$

A set $S \subset M$ is *invariant* if it contains the orbits of all its elements.

If $\Omega(\alpha) = G \times M$ the local action is a *global action*. When G is simply connected and M is compact, every local action extends to a unique global action.

A local action is often specified by a Lie algebra homomorphism $\theta : \mathfrak{g} \to \mathfrak{v}^\infty(M)$. The trajectories of the local flow X^α are the integral curves of $\theta(X)$. See [**26**, Theorem II.11] and [**35**, Theorem 2.16.6].)

The proof of the next result is left to the reader.

PROPOSITION 2.2. *If $H \subset G$ is a normal subgroup, then $\mathsf{Fix}(H)$ is invariant under G.*

2.1. *Local flows.*

Definition 2.3. A *local flow* on a space M is a local action $(\mathbb{R}, M, \varphi)$, sometimes referred to simply as φ. If $(t, p) \in \mathbb{R} \times M$ we may denote $\varphi(t)(p)$ by $\varphi_t(p)$ or $t \cdot p$.

If (G, M, α) denotes a local action, each one-parameter subgroup $X \in \mathfrak{g}$ determines a local flow X^α on M defined by

$$t \cdot p := \alpha(X(t)) \cdot p \quad (t \in \mathbb{R},\ p \in \mathcal{D}\alpha(X(t))).$$

In the rest of this section, α denotes a local flow on a surface M.

For each $p \in M$ there is a unique maximal open interval $J_p \subset \mathbb{R}$, containing 0, on which the *trajectory* of p,

$$\alpha^p : J_p \to M, \quad t \mapsto \alpha_t(t),$$

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is defined. The orbit of p is $\alpha^p(J_p)$, usually denoted by $O(p)$. The restrictions of the trajectory of p to the subintervals on which $t \geq 0$ (respectively $t \leq 0$) are the forward and backward *semitrajectories* of p. Their images are the forward and backward *semiorbits* $O_+(p)$ and $O_-(p)$.

A set is *invariant* if it contains the orbits of its members.

If $p = t \cdot p$ for some $t > 0$, then p and its orbit are *periodic*. If p is not a fixed point it is *cyclic* and its orbit is a *cycle*. The *period* of a cyclic point is the smallest $t > 0$ such that $\alpha^t(p) = p$.

An *orbit arc* $A \subset M$ is an arc of the form $\alpha^p(J)$ for some (non-degenerate) interval $J \subset \mathbb{R}$.

If $x \neq y \in O_+(x)$ we write $x \prec y$. In this case there is a smallest $s > 0$ such that $s \cdot x = y$, and we define orbit arcs

$$[xy] := \{t \cdot x : t \in [0, s]\},$$
$$(xy] := \{t \cdot x : t \in (0, s]\},$$

and so forth.

Definition 2.4. A *flowbox* with domain $S \subset M$ and chart h is a homeomorphism

$$h = (h_1, h_2) : S \approx I \times J \subset \mathbb{R} \times \mathbb{R}$$

such that:

- I and J are intervals;
- h_2 is constant on each orbit arc in S; and
- the map $h_1 : S \to J$ converts the action of α on each orbit into the following translation.

$$\textit{If } s > 0,\ t \cdot p \in S \textit{ for all } t \in [0, s] \textit{ and } q = \alpha_r(p),\ \textit{then}\ \ h_1(q) = h_1(p) + r.$$

Whitney [**36**] proved that every non-fixed point lies in the interior of a flowbox.

An open arc $T \subset M$ is a *transversal* if there is a flowbox $h : S \hookrightarrow \mathbb{R}^2$ such that $T \subset S$ and $h_2|T$ is injective.

2.2. *Minimal sets of local flows.* A point p is *recurrent* if there is a sequence $\{t_i\}$ in $\mathbb{R}$ such that

$$\lim_{i \to \infty} t_i \cdot p = p, \quad |t_i| \to \infty.$$

A recurrent point that is not periodic is *exceptional*.

A *minimal set* is a non-empty, compact invariant set $\mathbf{m}$ satisfying the following equivalent conditions.

- No proper subset of $\mathbf{m}$ is compact and invariant.
- Every orbit in $\mathbf{m}$ is dense in $\mathbf{m}$.
- Every point of $\mathbf{m}$ is recurrent.

Periodic orbits (including fixed points) are minimal sets; all others are *exceptional minimal sets*.

The proof of the following result is left to the reader.

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LEMMA 2.5. *Let* $\mathbf{m} \subset M$ *be a minimal set for the local flow* α *on* M.

(a) $\mathbf{m}$ *is connected.*

(b) *Let* $\pi : \tilde{M} \to M$ *be a finite-sheeted covering space of* M. *Let* $\tilde{\alpha}$ *be the local flow on* $\tilde{M}$ *such that* $\pi \circ \tilde{\alpha}_t = \alpha_t \circ \pi$. *There is a minimal set* $\tilde{\mathbf{m}}$ *for* $\tilde{\alpha}$ *such that* $\pi(\tilde{\mathbf{m}}) = \mathbf{m}$.

The next two results are adapted from results of Lima [**21**] and Plante [**28**].

THEOREM 2.6. *If* $H_1(M)$ *has finite rank, every exceptional minimal set has a neighborhood not meeting any cycle.*

Question. Can the assumption of finite rank be dropped?

Proof. We assume M is orientable, otherwise passing to an oriented finite-sheeted covering space. We also assume $\partial M = \varnothing$, because exceptional minimal sets cannot meet ∂M.

If $B \subset M$ is an oriented topological circle, its homology class is denoted by $[B] \in H_1(M)$.

Let $\{C_j\}$ be an infinite sequence of cycles in M and $\{p_j\}$ a sequence such that

$$p_j \in C_j, \quad \lim_{j\to\infty} p_j = p \in M. \tag{1}$$

As $\mathbf{m}$ is exceptional,

$$C_j \cap \mathbf{m} = \varnothing \quad (j \in \mathbb{N}_+). \tag{2}$$

We will prove $p \notin \mathbf{m}$.

Each C_j is a 1-manifold having a natural orientation, specified by a generator $\omega_j \in H_1(C_j) \cong \mathbb{Z}$. Let $[C_j] \in H_1(M)$ denote the image of ω_j under the homology homomorphism of the inclusion $C_j \hookrightarrow M$. Let $H \subset H_1(M)$ be the free abelian subgroup generated by the homology classes $[C_j]$. Since $H_1(M)$ has finite rank, there exists $n \geq 1$ such that

$$H \text{ is generated by } \{[C_1], \ldots, [C_n]\}. \tag{3}$$

By (2), there are arbitrarily small precompact flowbox neighborhoods S of p satisfying

$$S \subset M \backslash (C_1 \cup \cdots \cup C_n). \tag{4}$$

Let $T \subset S$ be a transversal through p. Then T is a *cross-section* for the local flow [**36**]: There exists $\delta > 0$ and a homeomorphism

$$F : (-\delta, \delta) \times T \approx S, \quad F(s, y) = s \cdot y. \tag{5}$$

As $\overline{O}_+(p)$ is compact, p is recurrent but not periodic, and T is a transversal, it follows that $\overline{O}_+(p) \cap T$ is a Cantor set. Therefore there are points $x, y, q \in V$ such that:

(a) $x, y, q \in O_+(p) \cap T$;

(b) $x \prec y \prec q$; and

(c) $(xq) \cap T = \{y\}$.

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Therefore equation (4) implies that

$$[xy] \cap (C_1 \cup \cdots \cup C_n) = \varnothing. \tag{6}$$

Let $L \subset T$ be the compact arc such that $\partial L = \{x, y\}$. Then $q \in L \backslash \partial L$, and the forward semitrajectory of x meets L first at x, next at y, and next at q. The compact arcs $[xy]$ and L meet only at their common boundary $\{x, y\}$, and therefore the set

$$\Sigma := [xy] \cup L \tag{7}$$

is a circle containing q.

Note that $C_j \cap [xy] = \varnothing$ because C_j is a periodic orbit, $[xy]$ lies in the non-periodic orbit $O(p)$, and distinct orbits are disjoint. Therefore

$$C_j \cap \Sigma = C_j \cap L \quad (j \in \mathbb{N}_+), \tag{8}$$

and hence from equation (6)

$$C_i \cap \Sigma = \varnothing \quad (i = 1, \ldots, n). \tag{9}$$

Assume *per contra* that $p \in \mathbf{m}$. Choose $r \geq 0$ such that $q = r \cdot p$. Set $q_j := r \cdot p_j$. By equation (1),

$$q_j \in C_j \quad \text{and} \quad \lim_{j \to \infty} q_j = q. \tag{10}$$

Therefore, there exists a smallest $k > n$ such that

$$C_k \cap L \neq \varnothing. \tag{11}$$

Equation (5) shows that there are orientations for the surface M such that the oriented curves C_k and L have local intersection number $+1$ at each intersection point. Consequently, equation (11) implies that

$$[C_k] \# [\Sigma] > 0. \tag{12}$$

Equation (6) shows there are integers a_i such that

$$[C_k] = \sum_{i=1}^{m} a_i [C_i].$$

Therefore

$$\begin{aligned} [C_k] \# [\Sigma] &= \sum_{i=1}^{n} a_i [C_i] \# [\Sigma], \\ &= 0 \quad \text{by equation (9).} \end{aligned}$$

This contradicts equation (12). □

THEOREM 2.7.

(a) *If M contains $r \geq 1$ exceptional minimal sets, then $\mathsf{g}(M) \geq r$.*

(b) *Every exceptional minimal set has a neighborhood containing no other minimal set.*

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Proof. (a) By Lima [**21**, Lemma 4], $\mathsf{g}(M) \geq 2r - 1 \geq 1$†. Assertion (a) follows because distinct minimal sets are contained in disjoint surfaces of genus ≥ 1, and the genus function is additive on unions of disjoint surfaces‡.

(b) An exceptional minimal set **m** has a neighborhood containing no fixed point. By (a), **m** has a neighborhood meeting no other minimal set and, by Theorem 2.6, **m** has a neighborhood meeting no cycle. The intersection of these neighborhoods proves the conclusion. □

3. *The fixed-point index*

The late Dold [**6**, **7**] defined an integer-valued fixed-point index $I(f)$ for a large class of maps f having compact fixed-point sets. We use this to define an index for blocks of fixed points in local flows.

Dold's index is defined for data f, V, $\mathcal{S}$ such that:

- V is an open set in a topological space $\mathcal{S}$;
- $f : V \to \mathcal{S}$ is continuous and $\mathsf{Fix}(f)$ is compact; and
- V is a *Euclidean neighborhood retract* (ENR): some open set in a Euclidean space retracts onto a homeomorph of V§.

We will use the following properties of $I(f)$.

(D1) $I(f) = I(f|V_0)$ if $V_0 \subset V$ is an open neighborhood of $\mathsf{Fix}(f)$.

(D2) $I(f) = \begin{cases} 0 \text{ if } \mathsf{Fix}(f) = \varnothing, \\ 1 \text{ if } f \text{ is constant.} \end{cases}$

(D3) $I(f) = \sum_i I(f|V_i)$ if V is the union of a finite number of disjoint open sets V_i.

(D4) $I(f_0) = I(f_1)$ if there is a homotopy $f_t : V \to \mathcal{S}$, $(0 \leq t \leq 1)$ such that $\bigcup_t \mathsf{Fix}(f_t)$ is compact.

These correspond to (5.5.11)–(5.5.13) and (5.5.15) in Chapter VII of Dold's book [**7**].

(D5) Assume $\mathcal{S}$ is a manifold, f is C^1, $\mathsf{Fix}(f)$ is a singleton $p \in M \backslash \partial M$, and $\mathsf{Det}\ f'(p) \neq 0$. Then $I(f) = (-1)^\nu$, where ν is the number of distinct eigenvalues λ of $f'(p)$ such that $\lambda > 1$ (see [**7**, VII.5.17, Example 3]).

(D6) Suppose $\mathcal{S}$ is compact, $V = \mathcal{S}$, and f is homotopic to the identity. Then $I(f) = \chi(\mathcal{S})$ (see [**7**, VII.6.22]).

Now let $\varphi := \{\varphi_t\}_{t \in \mathbb{R}}$ be a local flow on $\mathcal{S}$.

A compact set $K \subset \mathsf{Fix}(\varphi)$ is a *block* for φ, or a *φ-block*, if it has an open, precompact ENR neighborhood $U \subset \mathcal{S}$ such that $\overline{U} \cap \mathsf{Fix}(\varphi) = K$. Such a U is *isolating for* φ, and for (φ, K).

PROPOSITION 3.1. *If U is isolating for φ, there exists $\tau := \tau(U) > 0$ such that for all $t \in (0, \tau]$:*

(a) $\mathsf{Fix}(\varphi_t) \cap U$ *is compact; and*

(b) $I(\varphi_t|U) = I(\varphi_\tau|U)$.

† This also follows from Seibert and Tulley [**32**]: *Every recurrent point for a local flow in an arbitrary subset of $\mathbb{R}^2$ is periodic.* These authors credit this elegant result to an obscure 1936 work by Bohr and Fenchel [**3**].

‡ Plante [**28**, Lemma 2.3] proved, for compact M, that the number of exceptional minimal sets is bounded by the rank of $H_1(M)$.

§ The class of ENRs includes metrizable topological manifolds and their triangulable subsets.

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Proof. If (a) fails, there exist sequences $\{t_k\}$ in $(0, \infty)$ and $\{p_k\}$ in U such that

$$t_k \to 0, \quad p_k \in \mathsf{Fix}(\varphi_{t_k}) \quad \text{and} \quad p_k \to q \in \mathsf{Fr}(U).$$

Joint continuity of $(t, x) \mapsto \varphi_t(x)$ implies $q \in \mathsf{Fix}(\varphi) \cap \mathsf{Fr}(U)$, which is a contradiction because U is isolating for φ. Assertion (b) is a consequence of (a) and (D4). □

PROPOSITION 3.2. *If U_1 and U_2 are isolating for (φ, K), there exists $\sigma > 0$ such that, for all $t \in (0, \sigma]$,*

$$I(\varphi_t|U_1) = I(\varphi_t|U_2) = I(\varphi_\sigma|U_1 \cap U_2).$$

Proof. Evidently $U_1 \cap U_2$ is isolating for (φ, K). Set

$$\sigma := \min\{\tau(U_1), \tau(U_2), \tau(U_1 \cap U_2)\} > 0,$$

where $\tau(\cdot)$ is defined as in Proposition 3.1. The conclusion follows from Proposition 3.1 and (D1). □

Definition 3.3. Let $K \subset \mathsf{Fix}(\varphi)$ be a φ-block. By Propositions 3.1, 3.2 there is a unique integer $\mu(\phi, K, U)$ having the following property.

$$\textit{If } U \textit{ is isolating for } (\varphi, K) \textit{ and } 0 < t \le \tau(U), \textit{ then } I(\varphi_t|U) = \mu(\phi, K, U). \tag{13}$$

The *index* of φ in U, and at K, is defined as

$$\mathsf{i}(\varphi, U) = \mathsf{i}_K(\varphi) := \mu(\phi, K, U).$$

We call K *essential* for φ provided $\mathsf{i}_K(\varphi) \neq 0$. This implies $K \neq \varnothing$, by (D2).

Now let (G, M, α) be a local action of the Lie group G on a surface M. Every $X \in \mathfrak{g}$ generates a local flow X^α on M, often denoted simply by X. A block K of fixed points for X is called an X-block.

Using equation (13), define the index of X at K, and in U, as

$$\mathsf{i}_K(X) = \mathsf{i}(X, U) := \mathsf{i}(X^\alpha, U).$$

If $\mathsf{i}_K(X) \neq 0$, K is *essential* for X.

PROPOSITION 3.4. *Let $U \subset M$ be isolating for X.*

(a) *The set*

$$\mathfrak{N}(X, U, \mathfrak{g}) := \{Y \in \mathfrak{g} : U \textit{ is isolating for } Y \textit{ and } \mathsf{i}(Y, U) = \mathsf{i}(X, U)\}$$

is an open neighborhood of X in $\mathfrak{g}$.

(b) *If $\overline{U}$ is a compact invariant manifold then*

$$Y \in \mathfrak{N}(X, U, \mathfrak{g}) \implies \mathsf{i}(Y, U) = \chi(\overline{U}).$$

Proof. (a) Compactness of $\mathsf{Fr}(U)$ implies that the set

$$\mathfrak{U} := \{Y \in \mathfrak{g} : \mathsf{Fix}(Y) \cap \mathsf{Fr}(U) = \varnothing\}$$

is an open neighborhood of X, and U is isolating for every $Y \in \mathfrak{U}$. If $Y \in \mathfrak{U}$ is sufficiently close to X and $0 \le s \le 1$, then $Y_s := (1 - s)X + sY$ also lies in $\mathfrak{U}$. Therefore $\mathsf{i}(Y, U) = \mathsf{i}(X, U)$ by (D4).

(b) This follows from (D6). □

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4. *Dynamics of nilpotent local action*

In this section and the next, G denotes a connected nilpotent non-trivial Lie group, M a surface, and (G, M, α) an effective local action. Connectedness of G implies $\mathsf{Fix}(G) = \mathsf{Fix}(\mathfrak{g})$.

The following result is trivial but useful.

PROPOSITION 4.1. *If $p \in M$ and $\mathfrak{u}, \mathfrak{w} \subset \mathfrak{g}_p$ are linear subspaces such that $\mathfrak{u} + \mathfrak{w} = \mathfrak{g}$, then $p \in \mathsf{Fix}(\mathfrak{g})$.*

The *isotropy group* of $p \in M$ is the closed Lie subgroup $G_p \subset G$ generated by

$$\{g \in G : g \cdot p = p\}.$$

If $S \subset M$, set $G_S := \bigcap_{p \in S} G_p$. The *stabilizer of p* is the Lie algebra of G_p, equal to the algebra

$$\mathfrak{g}_p := \{X \in \mathfrak{g} : X_p = 0\},$$

and the *stabilizer of S* is $\mathfrak{g}_S := \bigcap_{p \in S} \mathfrak{g}_p$, which is the Lie algebra of G_S.

LEMMA 4.2. *If $\mathsf{dim}\,(\mathfrak{g}) \geq 2$, every element of $\mathfrak{g}$ lies in an ideal of codimension one.*

Proof. If $\mathsf{dim}\,(\mathfrak{g}) = 2$ the conclusion is trivial because $\mathfrak{g}$ is abelian. Assume inductively that $\mathsf{dim}\,(\mathfrak{g}) = d \geq 3$ and the lemma holds for Lie algebras of lower dimension. Let $Y \in \mathfrak{g}$ be arbitrary. Fix a one-dimensional central ideal $\mathfrak{j}$ and a surjective Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}.$$

By the inductive assumption, $\pi(Y)$ belongs to a codimension-one ideal $\mathfrak{f} \subset \mathfrak{g}/\mathfrak{j}$ and hence Y belongs the codimension-one ideal $\pi^{-1}(\mathfrak{f}) \subset \mathfrak{g}$. □

The set $\mathcal{C}(\mathfrak{g})$ of codimension-one ideals has a natural structure as a projective variety in the real projective space $\mathbf{P}^{d-1}$, $d = \mathsf{dim}\,(\mathfrak{g})$, and is given the corresponding metrizable topology.

PROPOSITION 4.3. [**28**]

(a) *Every component of $\mathcal{C}(\mathfrak{g})$ has positive dimension.*

(b) *Every codimension-one subalgebra is an ideal.*

(c) *Assume $\mathsf{dim}\,(G \cdot p) = 1$. Then $\mathfrak{g}_p$ is an ideal of codimension one, and*

$$\mathfrak{g}_x = \mathfrak{g}_p \quad (x \in G \cdot p).$$

Moreover, we state the following.

(d) *There is a one-dimensional subgroup $R \subset G$ such that $G \cdot p = R \cdot p$.*

Proof. (a)–(c) are proved in Plante [**28**, Lemmas 2.5, 2.6].

The isotropy subgroup G_p of p is normal (Proposition 4.3), and hence the action of G on the orbit of p factors through the canonical surjection $\pi : G \to G/G_p$. As the latter group is connected and one-dimensional, it equals $\pi(R)$ with $R \subset G$ a connected one-dimensional subgroup meeting G_p only at e_G. This implies (d)†. □

† While the results in [**28**] cited in this proof refer to a global action on a compact surface, their proofs apply unchanged to local actions on arbitrary surfaces.

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4.1. *Minimal sets for nilpotent local actions.* A *minimal set* $\mathbf{m} \subset M$ for the local surface action (G, M, α) is a non-empty, compact, invariant set containing no smaller such set. Equivalently, $\mathbf{m}$ is compact and the orbit of each of its points is dense in $\mathbf{m}$. Compact orbits are minimal sets; all other minimal sets are *exceptional*. A simple topological condition equivalent to exceptionality is that $\mathbf{m}$ is not a point, a circle, or a component of M.

PROPOSITION 4.4. *Let $\mathbf{m} \subset M$ be a one-dimensional minimal set.*

(a) *There is a connected one-dimensional Lie subgroup $R \subset G$ such that $\mathbf{m}$ is a minimal set for the induced local action of R on M.*

(b) *Every $p \in \mathbf{m}$ has the same isotropy group $G_p = G_{\mathbf{m}}$, which is a codimension-one closed normal subgroup.*

Proof. (a) The action of G on $\mathbf{m}$ factors through the canonical surjection $\pi : G \to G/G_{\mathbf{m}}$. As the latter group is connected and one-dimensional, it equals $\pi(R)$ with $R \subset G$ being a connected one-dimensional subgroup. Every orbit of G in $\mathbf{m}$ is also an orbit of R, which implies the conclusion.

(b) This follows from Proposition 4.3(c). □

THEOREM 4.5. *Let $M_0 \subset M$ be a compact surface.*

(i) *If M_0 contains $r \geq 1$ exceptional minimal sets, then $\mathsf{g}(M_0) \geq r$.*

(ii) *Every exceptional minimal set has a neighborhood $M_0 \subset M$ containing no other minimal set.*

Proof. For local flows, this is the same as Theorem 2.7 and therefore the general case follows from Proposition 4.4(b). □

PROPOSITION 4.6. *If $M_0 \subset M$ is a compact surface containing no fixed point, then the union of the circle orbits in M_0 is compact*†.

Proof. Let $\Gamma \subset M_0$ denote the union in question. Evidently, $\overline{\Gamma}$ is a compact invariant subset of M_0. We need to show that all orbits in the compact invariant set $\mathsf{Fr}(\Gamma) \subset M_0$ are circles.

As these orbits are compact, it suffices to show that their closures, which are connected, compact, one-dimensional invariant sets, are circles.

Each component of the compact invariant set $\overline{\Gamma} \cap \partial M$ is a circle orbit contained in ∂M_0. Therefore, it suffices to prove that

$$p \in \mathsf{Fr}(\Gamma) \backslash \partial M \implies p \in \Gamma.$$

The orbit closure $\overline{G \cdot p}$, a compact invariant subset of $\mathsf{Fr}(\Gamma) \subset M_0$, contains a minimal set $\mathbf{m}$ but no fixed point. Hence $0 \leq \mathsf{dim}\ (\mathbf{m}) \leq \mathsf{dim}\ \mathsf{Fr}(\Gamma) \leq 1$. Consequently, $\mathsf{dim}\ (\mathbf{m}) = 1$ and so, by Theorem 4.5(ii), $\mathbf{m}$ is a circle orbit. This proves that

$$\textit{there is a circle orbit } C \subset \overline{G} \cdot p, \tag{14}$$

† Cf. Plante [**28**, Lemma 2.4].

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Fixing C, we note that Proposition 4.4(a) implies that C is a cycle for the local flow X^α induced by a one-parameter subgroup $X \in \mathfrak{g}$. Therefore C lies in the orbit closure of p under X^α.

Fix an open neighborhood $N \subset M_0$ of C, that is either an annulus or a Möbius band. Consider the dynamics of X^α in N. Apply the Poincaré–Bendixson theorem (see [**12**]†; also [**5**, **22**]). Then there is a relatively open neighborhood $N \subset M_0$, homeomorphic to an annulus or a Möbius band, such that for every $x \in N$ one of the following holds.

(a) x is cyclic for X^α.

(b) x is not cyclic for X^α, and one of the semitrajectories of x under X^α spirals towards its limit set, a cycle in N.

Moreover, we also have the following.

(c) If (b) holds, it holds for all points in a neighborhood of x.

Choose a point $x \in (G \cdot p) \cap A$. Since $\overline{\Gamma}$ is invariant under G, every neighborhood of x meets a cycle for X^α. Therefore (c) shows that (a) holds for x, and hence it holds for p. □

PROPOSITION 4.7. *Let $M_0 \subset M$ be a compact connected surface with empty boundary, containing no fixed point. There is a compact invariant surface $P \subset M_0$ such that:*

(i) *each component of P is either an annulus or a Möbius band; and*

(ii) *$M_0 \backslash P$ contains a finite number of minimal sets.*

Proof. Let $\Lambda \subset M_0$ be the union of those circle orbits $C \subset M_0$ that are *non-isolated* in M_0, meaning that every neighborhood of C in M_0 contains another circle orbit. Λ is compact by Proposition 4.6. An argument based on Poincaré maps shows that each circle orbit $C \subset \Lambda$ lies in a compact invariant surface $P(C) \subset M_0$ satisfying (ii), whose boundary components are circle orbits.

Compactness of Λ implies the existence of $m \geq 1$ non-isolated circle orbits $C_j \subset M_0$ such that

$$\Lambda \subset P(C_1) \cup \cdots \cup P(C_m).$$

Set $P := \bigcup_i P(C_i)$. Induction on m shows that P is a compact invariant surface satisfying (i). Because M_0 is compact and every non-isolated circle orbit in M_0 lies in P, only a finite number of circle orbits lie in $M_0 \backslash P$. Since M_0 contains no fixed points and the number of exceptional minimal sets on M_0 is finite by Theorem 4.5(i), P satisfies (ii). □

5. *Proof of Theorem 1.1*

Recall the following hypotheses: M is a surface, G is a non-trivial, connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, (G, M, α) is an effective local action, and $K \subset M$ is an essential block of fixed points for the induced local flow of a one-parameter subgroup $X \in \mathfrak{g}$. The theorem states that $\mathsf{Fix}(G) \cap K \neq \varnothing$.

If $\dim G = 1$, the conclusion is trivial.

Induction hypothesis. $\dim (G) > 1$ and the conclusion holds for groups of lower dimension.

† Hartman's careful proof is stated for differential equations, but uses only local flows.

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Every neighborhood of K in M contains an isolating neighborhood U for (X, K) with $\overline{U}$ a compact surface. Essentiality of K implies $\mathsf{i}(X, U) \neq 0$. To complete the induction we will prove that

$$\mathsf{Fix}(G) \cap \overline{U} \neq \varnothing. \tag{15}$$

LEMMA 5.1. *If U contains only a finite number of minimal sets, then equation (15) holds.*

Proof. Since $\dim(\mathfrak{g}) > 1$ and $\mathfrak{g}$ is covered by codimension-one ideals (Lemma 4.2), Proposition 3.4 implies that there is an infinite sequence in $\mathfrak{g}$, $\{Y_j\}$, that converges to X, and a sequence $\{\mathfrak{h}_j\}$ of pairwise distinct, codimension-one ideals such that

$$Y_j \in \mathfrak{h}_j, \quad U \text{ is isolating for } Y_j, \quad \mathsf{i}(Y_j, U) \neq 0.$$

As the set $K_j := \mathsf{Fix}(\mathfrak{h}_j) \cap U$ is compact, non-empty by the induction hypothesis, and invariant by Proposition 2.2, there is a minimal set $L_j \subset K_j \subset U$. The hypothesis of the lemma implies that there exist indices i, j such that $\mathfrak{h}_i \neq \mathfrak{h}_j$ and $L_i = L_j$. Equation (15) now follows from Proposition 4.1. □

By Proposition 4.7 there is a non-empty, compact, invariant surface $P \subset \overline{U}$ such that

$$\chi(P) = 0 \text{ and } \overline{U} \setminus P \text{ contains only a finite number of minimal sets.} \tag{16}$$

If $\mathsf{Fix}(G) \cap P \neq \varnothing$ for all choices of U and P, then equation (15) holds by compactness of K. Henceforth, we assume that

$$\mathsf{Fix}(G) \cap P = \varnothing. \tag{17}$$

LEMMA 5.2. *There exists $Z \in \mathfrak{g}$ such that:*

(a) *U is isolating for Z;*
(b) *$\mathsf{i}(Z, U) = \mathsf{i}(X, U) \neq 0$; and*
(c) *$\mathsf{Fix}(Z) \cap \partial P = \varnothing$.*

Proof. Each of the finite number of components C_i of ∂P is a circle orbit because it contains no fixed point (equation (17)). Proposition 4.3 shows that its stabilizer is an ideal $\mathfrak{h}_i \subset \mathfrak{g}$. Therefore (c) holds for all Z in the dense open set $\mathfrak{g} \setminus \bigcup \mathfrak{h}_i$, while (a) and (b) hold for Z in the non-empty open set $\mathfrak{N}(X, U, \mathfrak{g})$ (defined in Proposition 3.4). Thus, the Lemma is satisfied by all Z in the non-empty set $\mathfrak{N}(X, U, \mathfrak{g}) \setminus \bigcup \mathfrak{h}_i$. □

Fix Z as in Lemma 5.2, so that

$$\mathsf{i}(Z, U) \neq 0. \tag{18}$$

Both $U \setminus P$ and $P \setminus \partial P$ are isolating for Z because $\mathsf{Fix}(Z) \cap \partial P = \varnothing$ and hence

$$\mathsf{i}(Z, U) = \mathsf{i}(Z, U \setminus P) + \mathsf{i}(Z, P \setminus \partial P) \tag{19}$$

by (D3) in §3.

Now $\mathsf{i}(Z, P \setminus \partial P) = \chi(P) = 0$ because $\mathsf{Fix}(Z) \cap \partial P = \varnothing$ (Proposition 3.4(b)). Consequently, (18) and (19) imply that

$$\mathsf{i}(Z, U \setminus P) \neq 0. \tag{20}$$

Equation (16) shows that $U \setminus P$ contains only a finite number of minimal sets. Therefore, by equation (20) and Lemma 5.1,

$$\mathsf{Fix}(G) \cap (U \setminus P) \neq \varnothing.$$

As this implies (15), the proof of Theorem 1.1 is complete.

https://doi.org/10.1017/etds.2015.73 Published online by Cambridge University Press

Acknowledgement. I thank Joseph Plante for helpful advice.

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