THE POSTULATION OF A MULTIPLE SURFACE

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1. In the present note we obtain the postulation of a multiple surface of S_4 for primals of sufficiently high order; the method is that previously used by Campedelli* to determine the postulation of a multiple curve of S_3.

We suppose, for simplicity, that the surface F is without improper nodes, and that the primals V^n in question contain F as i-ple surface. We denote by $P(n, i; l)$ the number of further conditions which must be imposed on primals V^n which already contain the i-ple surface F, in order that they should touch along F a given primal V^l containing the surface; and by $P(n, i)$ the required postulation of V^n, for sufficiently large values of n. If through F we draw $i+1$ arbitrary primals V^l, V^{l_1}, ..., V^{l_i}, the primals V^n which touch these along F will be forced to have F as $(i+1)$-ple surface; hence, by computing the number of further conditions so imposed, we obtain a relation between $P(n, i)$ and $P(n, i+1)$ from which the postulation may be determined.

2. We have first to establish the following proposition:

The primals V^n of sufficiently high order, which contain the i-ple surface F, meet V^l in a system of residual surfaces which intersect F in a complete, regular system of curves $|K|$.

This result is easily established by induction, using a lemma due to B. Segre† which is the analogue of a theorem of Castelnuovo's. Suppose that $i = 1$, and let F be the partial intersection of the primals V^l, V^{l_1}, the second of which can vary in a system of freedom greater than zero. If n is sufficiently large, the primals V^n passing through F can break up into V^{l_1} and a system of primals V^{n-l_1}. It is well known that the latter system cuts out on F a regular, complete system of curves $|C|$, provided that $n-l_1$ is sufficiently large; and if D is a curve in which F meets the residual intersection of V^l and a primal V^{l_1}, we can suppose that the series cut out on D by $|C|$ is non-special, by choosing $n-l_1$ sufficiently large; hence, by Segre's lemma, the system $|C+D|$ is also complete and regular. Since this is the system $|K|$ referred to in the enunciation, our proposition is proved

* Campedelli, *Rend. Palermo*, 55 (1931), 198.

† B. Segre, *Rend. Lincei* (6), 22 (1934), 488. The lemma states that, *if $|C|$ is a regular system of curves of dimension greater than zero, and D an irreducible curve, not common to all the curves C, and variable in a continuous system, then, if the sum of the series cut out on D by $|C|$ and the characteristic series on D is non-special, the system $|C+D|$ is regular.*

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in the case $i = 1$; the case $i = 2$ is now established by repetition of the argument and by use of the result for $i = 1$. And so in general.

3. We now determine the virtual dimension of the system $|K|$ which, by § 2, will be the effective dimension for sufficiently large values of n. For this purpose we proceed to calculate the order, rank and grade of $|K|$.

Suppose that F has ranks μ_0, μ_1, μ_2 and *ceto* ν_2; since by hypothesis it is without improper nodes, we have* $$\mu_0^2 = \mu_0 + \mu_1 + \mu_2. \tag{1}$$

If V^l passes simply through F, it will have δ nodes on F, where*

$$\delta = \mu_0(l-1)^2 - \mu_1(l-1) + \nu_2. \tag{2}$$

These points will evidently be i-ple on the curve K common to F and the residual intersection F' of V^l and V^n. Suppose that K has order ϵ_0 and rank ϵ_1, with classes of immersion ζ, ζ' in F and F' respectively. If Z is the class of the developable of tangent primes to V^l and V^n at points of K, then

$$\zeta + \zeta' = Z + \epsilon_1. \tag{3}$$

To find Z we consider the intersection of K with the first polar of an arbitrary point with respect to V^l; thus

$$Z + i\delta = \epsilon_0(l-1). \tag{4}$$

The tangent planes to F which meet a given line g have a curve Γ of contact, of order μ_1. At any point P of Γ draw the tangent primes to V^n and suppose that these meet a second line g' in a set of points (Q); let the tangent prime to V^l at P meet g' in Q'. The points Q, Q' define on g' a correspondence whose indices are the classes of immersion of Γ in V^l and V^n, namely $i(l-1)\mu_1$ and $(n-i)\mu_1$. The coincidences of the correspondence arise from the ζ tangent planes to F, at points of K, which meet g', and the tangent planes to F (in number $\mu_2 + \nu_2$) which meet g and g', the latter counted i times. Thus

$$\zeta + i(\mu_2 + \nu_2) = \mu_1\{n - i + i(l-1)\}. \tag{5}$$

The order of K is determined by a formula due to Severi†; we have

$$\epsilon_0 = \mu_0\{n - i + i(l-1)\} - i\mu_1. \tag{6}$$

Next, introduce the characters λ, λ', defined as the number of lines passing through a given point and meeting K and F or F' again elsewhere. By projecting F on to S_3 we obtain $$\lambda + \zeta = \epsilon_0(\mu_0 - 1). \tag{7}$$

If F' is of order $\mu_0' = ln - i\mu_0$, we have

$$\lambda' + \zeta' + (i-1)\,i\delta = \epsilon_0(\mu_0' - 1). \tag{8}$$

For, on the projection of F', the double curve contains the i-ple points on the projection of K, and these are $(i-1)$-ple on the first polars.

The cone projecting K from a point P meets V^l in a residual curve K' of order $\epsilon_0(l-1)$ whose intersections with V^n, other than those lying on K, give rise to

* Severi, *Memorie Acc. Torino* (2), 52 (1903), 61. † Severi, *op. cit.*

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lines through P which meet K and F or F' again. Now K' meets K at the δ nodes of V^l, which are i-ple on K and K', and in the Z points of contact of the tangent primes to V^l, in points of K, which pass through P; the latter count $i+1$ times, since K' has $(i+1)$-point contact there with V^n. We thus have

$$i\lambda + \lambda' + i^2\delta + (i+1)\,Z = \epsilon_0(l-1)\,n. \tag{9}$$

Eliminating λ, λ' from (7), (8), (9) and substituting for ζ' from (3), we have

$$\epsilon_1 - \epsilon_0(n-i-1) = iZ + i\delta + (1-i)\,\zeta.$$

Hence, from (4) and (5),

$$\epsilon_1 = \epsilon_0\{n-i-1+i(l-1)\} - (i-1)\,\mu_1\{n-i+i(l-1)\} + i(i-1)\,(\mu_2+\nu_2) - i(i-1)\,\delta.$$

Inserting the values of ϵ_0 and δ from (6) and (2) and writing

$$n-i+i(l-1) = \theta,$$

we obtain

$$\epsilon_1 = \mu_0\theta(\theta-1) - (2i-1)\,\mu_1\theta + i\mu_1 + i(i-1)\{\mu_2 + \mu_1(l-1) - \mu_0(l-1)^2\}. \tag{10}$$

4. If a second primal U^n is drawn to pass i times through F, it will define on F a second curve K_1 having i-ple points at the δ nodes of V^l, and meeting K in a number n_K of points elsewhere. To find this number, suppose that the tangent primes to U^n at a point P of K meet a given line g in the set of points (Q), and that the tangent prime to V^l at P meets g in the point Q'. The points Q, Q' define on g a correspondence whose indices are $i(l-1)\,\epsilon_0$ and $(n-i)\,\epsilon_0$, and the coincidences arise from the intersections of K and K_1, and the tangent planes to F, at points of K, which meet g, these counting i times. Hence

$$n_K + i^2\delta + i\zeta = \epsilon_0\{n-i+i(l-1)\}.$$

Thus, from (5) and (6), the grade of $|\,K\,|$ is given by

$$n_K = \mu_0\theta^2 - 2i\mu_1\theta + i^2\{\mu_2 + \mu_1(l-1) - \mu_0(l-1)^2\}. \tag{11}$$

Then the virtual dimension of $|\,K\,|$ is

$$r_K = n_K - \tfrac{1}{2}\epsilon_1 + \epsilon_0 + p_a,$$

and this, by § 2, is the effective dimension provided that n is sufficiently large. In these circumstances it follows that, in order to make V^n touch V^l along F, we have to subject it to r_K+1 further conditions. From (6), (10) and (11) we obtain for the number of these conditions the formula

$$P(n,i;l) = \tfrac{1}{2}\mu_0\{(n+il-2i)^2 - i(i+1)\,l(l-2) + 3(n+il-2i)\}$$
$$-\tfrac{1}{2}\mu_1\{(2i+1)\,(n+il-2i) - i(i+1)\,l + 3i\} + \tfrac{1}{2}i(i+1)\,(\mu_2-\mu_0-\mu_1) + p_a + 1. \tag{12}$$

5. If n is sufficiently large, the number of conditions which must be imposed on V^n in order that it should contain F as i-ple surface and touch V^l along F is

$$P(n,i) + P(n,i;l).$$

The primals V^n satisfying these conditions touch the primal V^{l_1} along the curve H in which F meets the residual intersection of V^l and V^{l_1}. They cut out on V^{l_1} a

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system of surfaces, passing through H, which defines a complete, regular system of curves on F, residual to H. This follows from § 2 by noting that among such primals there are those consisting of V^l and a primal V^{n-l} containing F as $(i-1)$-ple surface. The system of curves so defined on F therefore has dimension

$$P(n-l, i-1; l_1)-1;$$

and the number of conditions to be imposed on V^n in order that it should contain the i-ple surface F and touch V^l and V^{l_1} along F is

$$P(n, i)+P(n, i; l)+P(n-l, i-1; l_1).$$

Similarly, the number of conditions to be imposed on V^n in order that it should also touch V^{l_2} along F is

$$P(n, i)+P(n, i; l)+P(n-l, i-1; l_1)+P(n-l-l_1, i-2; l_2).$$

Proceeding in this manner we obtain the number of conditions to be imposed on primals V^n of sufficiently high order, in order that they should pass i times through F and touch V^l, V^{l_1}, ..., V^{l_i} along F. Since such primals contain F as $(i+1)$-ple surface we thus obtain the relation

$$\begin{aligned} P(n, i+1) = {} & P(n, i)+P(n, i; l)+P(n-l, i-1; l_1) \\ & +P(n-l-l_1, i-2; l_2)+\ldots+P(n-l-l_1 \ldots -l_{i-1}, 0; l_i). \end{aligned}$$

If in this relation the values of $P(n, i; l)$, $P(n-l, i-1; l_1)$, ... are substituted from (12) it will be found that the terms involving l, l_1, ..., l_{i-1} vanish. If, further, we replace p_a by the expression

$$\mu_0-1-\tfrac{2}{3}\mu_1+\frac{\mu_2}{6}+\frac{\nu_2}{12},$$

we derive

$$\begin{aligned} P(n, i+1)-P(n, i) = {} & \frac{(i+1)}{6}(3n^2+13n+13)\mu_0+\tfrac{1}{6}(3i^3-6i^2-16i-7)\mu_0 \\ & -\frac{(i+1)}{3}(3i+2)n\mu_0-\frac{(i+1)^2}{2}n\mu_1+\frac{1}{12}(6i^3+3i^2-11i-8)\mu_1 \\ & +\frac{(i+1)^3}{6}\mu_2+\frac{(i+1)}{12}\nu_2. \end{aligned} \tag{13}$$

From this recurrence relation and from the fact that $P(n, 0) = 0$, we obtain for the required postulation the formula

$$\begin{aligned} P(n, i) = {} & \frac{i(i+1)}{24}\{[6n^2-2(4i-13)n+3i^2-17i+26]\mu_0 \\ & +[3i^2-7i-4-2(2i+1)n]\mu_1+i(i+1)\mu_2+\nu_2\}. \end{aligned} \tag{14}$$

6. It may be of interest to give a verification of this result in the particular case where F is the complete intersection of two primals; for this purpose we follow the method given by Noether* for the analogous problem in three dimensions.

* Noether, *Annali di Mat.* (2), 5 (1871), 163.

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Suppose that F is the complete intersection of the primals, of orders p and q respectively, whose equations are $P = 0$, $Q = 0$. Then the equation of the primal V^n, which contains F as i-ple surface, is of the form

$$A_0 P^i + A_1 P^{i-1} Q + \ldots + A_i Q^i = 0,$$

where A_r is a polynomial of order $n - ip + r(p-q)$ in the coordinates. The equation of V^n can be written as

$$0 = V^n \equiv A_0 P^i + Q V^{n-q}, \tag{15}$$

where V^{n-q} has F for $(i-1)$-ple surface; and again, the equation of V^{n-q} can be written as

$$0 = V^{n-q} \equiv A_1 P^{i-1} + Q V^{n-2q},$$

where V^{n-2q} has F for $(i-2)$-ple surface; and so on. From these results we can obtain a recurrence relation for $P(n, i)$, assuming that n is sufficiently large. For, from (15), we have

$$P(n,i) = P(n-q, i-1) + \binom{n+4}{4} - \binom{n-ip+4}{4} - \binom{n-q+4}{4} + \binom{n-ip-q+4}{4}.$$

After reduction this relation becomes

$$P(n,i) - P(n-q, i-1) = ipq\{\tfrac{1}{2}n^2 - \tfrac{1}{2}(ip+q)\,n + \tfrac{5}{2}n + \tfrac{1}{6}(i^2p^2 + q^2) + \tfrac{1}{4}ipq - \tfrac{5}{4}(ip+q) + \tfrac{35}{12}\}. \tag{16}$$

If n is sufficiently large, the relation holds when n and i are changed successively into $n-2q$, $i-2$; $n-3q$, $i-3$, ..., until we reach the term $P(n-iq, 0)$, which is zero. Hence, from (16) we derive

$$\begin{aligned} P(n,i) = {} & \tfrac{1}{2}pq\left\{\frac{i(i+1)}{2}\,n^2 - \tfrac{1}{3}i(i-1)(i+1)\,nq + \tfrac{1}{12}i^2(i-1)(i+1)\,q^2\right\} \\ & - \tfrac{1}{2}p^2q\{\tfrac{1}{6}i(i+1)(2i+1)\,n - \tfrac{1}{6}i^2(i-1)(2i-1)\,q + \tfrac{1}{4}i^2(i-1)^2 q\} \\ & - \tfrac{1}{2}pq(q-5)\{\tfrac{1}{2}i(i+1)\,n - \tfrac{1}{6}i(i-1)(i+1)\,q\} + \tfrac{1}{24}p^3qi^2(i+1)^2 \\ & + \tfrac{1}{12}pq^3i(i+1) + \tfrac{35}{24}pqi(i+1) + \tfrac{1}{24}p^2q^2i(i+1)(2i+1) - \tfrac{5}{8}pq^2i(i+1) \\ & - \tfrac{5}{24}p^2qi(i+1)(2i+1). \end{aligned}$$

Thus

$$P(n,i) = \frac{i(i+1)}{24}\,pq\{6n^2 - 2(2i+1)(p+q)\,n + 30n + i(i+1)(p^2+q^2) + (i^2+i+1)pq - 5(2i+1)(p+q) + 35\}. \tag{17}$$

Now the characters of F are given by* the formulae

$$\mu_0 = pq, \quad \mu_1 = pq(p+q-2), \quad \mu_2 = pq\{(p-1)^2 + (p-1)(q-1) + (q-1)^2\},$$
$$\nu_2 = pq(p-1)(q-1).$$

If these are substituted in (17), the expression becomes identical with (14).

* Severi, *op. cit.*

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