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BOUNDS OF NON-REAL EIGENVALUES FOR EIGENPARAMETER-DEPENDENT STURM–LIOUVILLE PROBLEMS WITH INDEFINITE WEIGHT

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Abstract In this paper, the upper bounds of non-real eigenvalues of indefinite Sturm–Liouville (S-L) problems with boundary conditions depend on the eigenparameter are studied. The upper bounds of real parts, imaginary parts and absolute values of non-real eigenvalues are given under the condition that the coefficients are integrable.

Keywords: indefinite Sturm–Liouville problems; eigenparameter-dependent boundary condition; non-real eigenvalues

2010 Mathematics subject classification: 34L15; 34L05

1. Introduction

In this paper, we consider the following Sturm–Liouville (S-L) problem

$$\tau y := -(py')' + qy = \lambda wy,\ x \in [0,1], \tag{1.1}$$

with boundary conditions

$$\begin{aligned} B_1 y &:= (a\lambda + b)(py')(0) = (c\lambda + d)y(0), \\ B_2 y &:= \cos\alpha y(1) + \sin\alpha (py')(1) = 0, \end{aligned} \tag{1.2}$$

where a, b, c, d are all real numbers, $\alpha \in [0, \pi)$, p, q, w are real-valued functions and satisfy

$$p(x) > 0,\ w(x) \neq 0\ a.e.\ on\ x \in [0,1],\ \frac{1}{p}, q, w \in L^1[0,1]. \tag{1.3}$$

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λ is the spectral parameter, and w changes its sign on $[0,1]$ in the sense that

$$\text{mes}\{x \in [0,1] : w(x) > 0\} > 0, \ \text{mes}\{x \in [0,1] : w(x) < 0\} > 0.$$

$L^2_{|w|} := L^2_{|w|}[0,1]$ is the weighted Hilbert space of all Lebesgue measurable, complex-valued functions f on $[0,1]$ satisfying $\int_0^1 |w||f|^2 < \infty$ with the norm $\|f\|^2_{|w|} = \int_0^1 |w||f|^2$ and the inner product $\langle f, g\rangle_{|w|} = \int_0^1 |w| f\overline{g}$. Next, we define a Hilbert space $H = L^2_{|w|} \oplus \mathbb{C}$. The inner product in H is defined as

$$\langle F, G\rangle = \int_0^1 |w| f(x)\overline{g}(x)\mathrm{d}x + \frac{1}{\rho} f_1(x)\overline{g_1}(x),$$

where $\rho = bc - ad > 0$, $F := (f(x), f_1(x))^T$, $G := (g(x), g_1(x))^T \in H$.

Consider the following defined operator $\mathcal{T}$ in H:

$$\begin{aligned} D(\mathcal{T}) = \Big\{ & F \in H | f, pf' \in AC[0,1], \tau f \in L^2_{|w|}[0,1], \\ & f_1 = cy(0) - a(py')(0), \cos\alpha y(1) + \sin\alpha (py')(1) = 0 \Big\}, \end{aligned} \tag{1.4}$$

$$\mathcal{T}F = \begin{pmatrix} \frac{1}{|w|}\tau f \\ -(dy(0) - b(py')(0)) \end{pmatrix}, F \in D(\mathcal{T}), \tag{1.5}$$

then $\mathcal{T}$ is self-adjoint in H and it has only point spectrum which are all real-valued, bounded below and unbounded above ([16]). However, when the weighted function w changes its sign, this type of problem is called indefinite S-L problem, which has a sequence of real-valued eigenvalues unbounded below and above. Meanwhile, the problem may also exist non-real eigenvalues. It should be noted that weighted S-L problems appear in the context of some shallow water equations, and it is very useful in understanding the structure of the spectrum in fluid dynamics [14, 15].

The indefinite nature of the S-L problems had not been discussed until the works of Haupt, Richardson, and others([19, 25, 26]). In 1986, Mingarelli proposed some open questions about non-real eigenvalues of indefinite S-L problems, among which there are pre-estimates of the upper and lower bounds of non-real eigenvalues, respectively [21]. Until 2013, this open problem was solved by Xie and Qi for the regular indefinite S-L problem, they gave an upper bound estimation of the non-real eigenvalues, and obtained sufficient conditions for the existence and non-existence of the non-real eigenvalues under certain conditions [28]. Since then, such problems have been obtained various generalizations, for regular indefinite S-L problems with different boundary conditions and various conditions imposed on the coefficients, we refer to [20, 23, 24, 29]. For the singular indefinite S-L problems with limit circle or limit point endpoints, we refer to ([3–6, 27]).

In recent years, S-L problems with eigenparameter dependent boundary conditions have attracted more and more attention for its wide applications in mechanics and

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mathematical physics, such as vibration problems involving various types of loads, heat conduction through a liquid-solid interface, electric circuits problems, diffusion processes in probability theory, mechanical vibrations [1, 16, 17, 22]. In the last decades, a large amount of achievements were obtained for S-L problems with eigenparameter dependent boundary conditions, for example, the dependence of eigenvalues [30, 31], inverse spectral theory [11, 12, 13, 18], asymptotics of eigenvalues [2, 16], expansion of eigenfunctions [7, 8, 25], oscillation of eigenfunctions [9, 10] etc., for more details, see the references cited therein.

Note that most researchers estimate the non-real eigenvalues of second-order indefinite S-L problems under the classical boundary conditions (such as Dirichlet boundary conditions, S-L type separated boundary conditions, coupled boundary conditions, etc.), while there is little known about the bounds of non-real eigenvalues of indefinite S-L problems with an eigenparameter contained in the boundary conditions. Taking inspiration from the above results, in this paper, we will solve this problem, that is, we will give the priori upper bounds of the non-real eigenvalues of the regular indefinite S-L problems with an eigenparameter contained in the boundary conditions by assigning different constraint conditions to the weighted function.

An outline of this paper is as follows. In Section 2, we will provide upper bounds on non-real eigenvalues (Theorems 1 and 2). In Section 3, we give the proof of Theorems 1 and 2.

2. Main results

Consider the following indefinite S-L problem

$$-(py')' + qy = \lambda wy,\ x \in [0,1], \tag{2.1}$$

with boundary conditions

$$\begin{aligned} B_1 y &:= (a\lambda + b)(py')(0) = (c\lambda + d)y(0), \\ B_2 y &:= \cos\alpha y(1) + \sin\alpha (py')(1) = 0, \end{aligned} \tag{2.2}$$

where a, b, c, d are all real numbers, $\alpha \in [0, \pi)$, p, q, w are real-valued functions and satisfy

$$p(x) > 0, w(x) \neq 0 \ a.e.\ on\ x \in [0,1], \frac{1}{p}, q, w \in L^1[0,1]. \tag{2.3}$$

Let

$$\cot^* \alpha = \begin{cases} \cot\alpha,\ \alpha \in (0,\pi), \\ 0,\ \alpha = 0. \end{cases}$$

For the convenience of the following discussion, we use the following notations: $\|\cdot\|_1$ represents the norm of the $L^1[0,1]$, and $\|\cdot\|_\infty$ represents the maximum norm of the

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continuous function space $C[0,1]$. In addition, $q_{\pm} = \max\{\pm q, 0\}$ is also used to represent the positive and negative parts of function q. Let

$$\begin{aligned} &M = \frac{N + 2\sqrt{\rho\|w\|_1}}{|a|}, N = |b|\|w\|_1 + |c| + |a|(|\cot^* \alpha| + \|q\|_1), \rho = bc - ad > 0, \\ &\delta_p = \sup\left\{\min\left\{\delta, \frac{1}{2}\right\} : \int_{\widetilde{I}} \frac{1}{p} \le \frac{1}{4M}, \textit{for any interval } \widetilde{I} \subset [0,1] \textit{ with length } \delta\right\}, \\ &m = 1 + \max\{4, \lfloor 8M \rfloor\}, \|q\|_1 = \int_0^1 |q|, \|w\|_1 = \int_0^1 |w|, \end{aligned} \tag{2.4}$$

where $\lfloor\cdot\rfloor$ is the integer floor function. According to the definition of δ_p and $\frac{1}{p} \in L^1[0,1]$, it can be inferred that

$$\delta_p = \min\left\{\frac{1}{2}, \frac{1}{4M}\right\}, \textit{ and } \int_{I'} \frac{1}{p} \le \frac{1}{4M} \textit{ for any interval } I' \subset [0,1] \textit{ with length } \delta_p. \tag{2.5}$$

Let $x_t = \frac{t}{m}, t = 0, 1, 2, \cdots, m$. Then each interval $I_k := [x_{k-1}, x_k]$ has an equal length of $\frac{1}{m} < \frac{\delta_p}{2}$, $t = 0, 1, 2, \cdots, m$.

For any measurable set $E \subset [0,1]$, set

$$\nu(E) = \int_E w\mathrm{d}x, \ \mu(E) = \int_E |w|\mathrm{d}x,$$

then μ is a positive measure and ν is an indefinite measure, and since

$$\nu(E) = \int_E \operatorname{sgn} w|w|\mathrm{d}x = \int_E \operatorname{sgn} w\mathrm{d}\mu,$$

$\operatorname{sgn} w$ is the Rodon–Nikdym derivative of ν with respect to μ, there exists a measurable set $F \subset [0,1]$ with $mesF = 1$ that satisfies

$$\frac{d\nu}{d\mu}(\xi_t) = \operatorname{sgn}(w(\xi_t)) \neq 0, \forall\ \xi_t \in F.$$

If $\operatorname{sgn}(w(\xi_t)) = 1$, we can then find an $\eta_t > 0$ small enough such that for every $x' \in (\xi_t - \eta_t, \xi_t)$ and $x'' \in (\xi_t, \xi_t + \eta_t)$,

$$\int_{x'}^{\xi_t} w > \frac{9}{10}\int_{x'}^{\xi_t} |w| \quad \text{and} \quad \int_{\xi_t}^{x''} w > \frac{9}{10}\int_{\xi_t}^{x''} |w|$$

and hence

$$\int_{x'}^{x''} w > \frac{9}{10}\int_{x'}^{x''} |w|.$$

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Similarly, if $\operatorname{sgn}(w(\xi_t)) = -1$, we can get

$$\int_{x'}^{x''} w < -\frac{9}{10}\int_{x'}^{x''} |w|.$$

In a word, for every interval I_t there exists a $\xi_t \in F \cap I_t$ and an $\eta_t > 0$ such that $[\xi_t - \eta_t, \xi_t + \eta_t] \subset I_t$ and for every interval $I \subset [\xi_t - \eta_t, \xi_t + \eta_t]$ containing ξ_t we have

$$\left|\int_I w\right| \geq \frac{9}{10}\int_I |w|. \tag{2.6}$$

Define

$$\eta = \min\{\eta_t : t = 1, \cdots, m\}. \tag{2.7}$$

Let $a_t = \xi_t - \eta, b_t = \xi_t + \eta$, then $[a_t, b_t] \subset [\xi_t - \eta_t, \xi_t + \eta_t]$ and for every interval $I \subset [a_t, b_t]$ containing ξ_t, (2.6) holds. And set

$$w_t = \min\left\{\int_s^{s+t} |w| : s \in [0, 1-t]\right\}, \quad t \in (0,1), \tag{2.8}$$

$$p_t = \max\left\{\int_s^{s+t} p : s \in [0, 1-t]\right\}, \quad t \in (0,1). \tag{2.9}$$

Obviously, $w_t > 0$, $p_t > 0$ for $t > 0$. The following results hold.

Theorem 1. *For problem (2.1) and (2.2) with indefinite weight, let λ be the corresponding non-real eigenvalue, then the eigenvalue λ satisfies*

$$|Im\lambda| \leq \frac{32\sqrt{p_\eta M}}{3w_\eta \eta}, |\lambda| \leq \frac{32\sqrt{p_\eta M}}{3w_\eta \eta} + \frac{16}{3w_\eta}(M + \|q\|_1). \tag{2.10}$$

In the next theorem, since $w(x) \neq 0$ a.e., we can choose $\varepsilon > 0$ such that

$$\operatorname{mes}\Omega(\varepsilon) \leqslant \frac{1}{8}\delta_p, \quad \Omega(\varepsilon) = \{x \in [0,1] \colon w^2(x) \leqslant \varepsilon\}. \tag{2.11}$$

Theorem 2. *Suppose that $w \in AC[0,1]$ and $w' \in L_p^2[0,1]$. Then for any non-real eigenvalue λ of (2.1) and (2.2) it holds that*

$$\begin{aligned} |Re\lambda| &\leq \frac{4}{a\varepsilon\delta_p}\left(V + \sqrt{\frac{T + \sqrt{T^2 + W^2}}{2}}\right), |Im\lambda| \\ &\leq \frac{4}{a\varepsilon\delta_p}\left(|a|\|w'\|_{p,2}\sqrt{M} + \sqrt{\frac{-L + \sqrt{L^2 + W^2}}{2}}\right), \end{aligned} \tag{2.12}$$

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where $a \neq 0$, $w_0 = \{\max|w(x)| : x \in [0,1]\}$, $\|w'\|_{p,2} = (\int_0^1 p|w'|^2)^{\frac{1}{2}}$,

$$H = |d|w_0 + |b|w_0(|\cot^* \alpha| + M + \|q\|_1) + |b|\|w'\|_{p,2}\sqrt{M},$$

$$V = w_0(|b|w_0 + |c|) + |a|w_0(|\cot^* \alpha| + M + \|q\|_1) + |a|\|w'\|_{p,2}\sqrt{M},$$

$$L = -a^2\|w'\|_{p,2}^2 M - 4|a|w_0^2 H,\ W = 2|a|\|w'\|_{p,2}\sqrt{M}(V + 2|b|w_0^2),\ T = V^2 - L.$$

3. The proofs

In order to prove the main results (Theorems 1 and 2), we firstly introduce some lemmas.

Let λ and ϕ be a pair of eigenvalue and eigenfunction of the problem (2.1) and (2.2) on $[0,1]$, i.e., the problem

$$-(p\phi')' + q\phi = \lambda w\phi, x \in [0,1], \tag{3.1}$$

with the boundary condition

$$\begin{aligned} B_1\phi &:= (a\lambda + b)(p\phi')(0) = (c\lambda + d)\phi(0), \\ B_2\phi &:= \cos\alpha\phi(1) + \sin\alpha(p\phi')(1) = 0. \end{aligned} \tag{3.2}$$

Lemma 1. *Assume* λ *is a non-real eigenvalue of* (2.1) *and* (2.2) *and* ϕ *is the corresponding eigenfunction with* $\|\phi\|_\infty = 1$, *and the definition of* M, N, ρ *are given by* (2.4). *Then*

$$\int_0^1 p|\phi'|^2 \leq M. \tag{3.3}$$

Proof. Multiplying both sides of (3.1) by $\overline{\phi}$ and integrating by parts over the interval $[0,1]$, then according to $B_1\phi = 0, B_2\phi = 0$, we have

$$\begin{aligned} (c\lambda + d)|\phi(0)|^2 + (a\lambda + b)\cot^* \alpha|\phi(1)|^2 + (a\lambda + b) \\ \int_0^1 p|\phi'|^2 + (a\lambda + b)\int_0^1 q|\phi|^2 = \lambda(a\lambda + b)\int_0^1 w|\phi|^2, \end{aligned}$$

that is,

$$\begin{aligned} &\lambda^2 a\int_0^1 w|\phi|^2 + \lambda\Big[b\int_0^1 w|\phi|^2 - c|\phi(0)|^2 - a\cot^* \alpha|\phi(1)|^2 - a\int_0^1 p|\phi'|^2 - a\int_0^1 q|\phi|^2\Big] \\ &- d|\phi(0)|^2 - b\cot^* \alpha|\phi(1)|^2 - b\int_0^1 p|\phi'|^2 - b\int_0^1 q|\phi|^2 = 0. \end{aligned} \tag{3.4}$$

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Then we have

$$a_1\lambda^2 + b_1\lambda + c_1 = 0,$$

where

$$a_1 = a\int_0^1 w|\phi|^2,$$

$$b_1 = b\int_0^1 w|\phi|^2 - c|\phi(0)|^2 - a\cot^*\alpha|\phi(1)|^2 - a\int_0^1 p|\phi'|^2 - a\int_0^1 q|\phi|^2,$$

$$c_1 = -d|\phi(0)|^2 - b\cot^*\alpha|\phi(1)|^2 - b\int_0^1 p|\phi'|^2 - b\int_0^1 q|\phi|^2.$$

Let $R = \int_0^1 p|\phi'|^2$, since $Im\lambda \neq 0$, we have

$$\begin{aligned}
&a^2R^2 + R\Big(2ab\int_0^1 w|\phi|^2 + 2ac|\phi(0)|^2 + 2a^2\cot^*\alpha|\phi(1)|^2 + 2a^2\int_0^1 q|\phi|^2\Big)\\
&+ b^2\Big(\int_0^1 w|\phi|^2\Big)^2 - 2bc|\phi(0)|^2\int_0^1 w|\phi|^2 + 2ab\cot^*\alpha|\phi(1)|^2\int_0^1 w|\phi|^2\\
&+ 2ab\int_0^1 q|\phi|^2\int_0^1 w|\phi|^2 + c^2|\phi(0)|^4 + 2ac\cot^*\alpha|\phi(0)|^2|\phi(1)|^2\\
&+ 2ac|\phi(0)|^2\int_0^1 q|\phi|^2 + a^2(\cot^*\alpha)^2|\phi(1)|^4 + 2a^2\cot^*\alpha|\phi(1)|^2\\
&\int_0^1 q|\phi|^2 + a^2\Big(\int_0^1 q|\phi|^2\Big)^2 + 4ad|\phi(0)|^2\int_0^1 w|\phi|^2 < 0.
\end{aligned} \tag{3.5}$$

It can be seen that the quadratic function of R has an upward opening and a part below the real-axis, therefore,

$$R \leq \frac{-b_2 + \sqrt{b_2^2 - 4a_2c_2}}{2a_2}, \tag{3.6}$$

where

$$\begin{aligned}
&a_2 = a^2,\\
&b_2 = 2ab\int_0^1 w|\phi|^2 + 2ac|\phi(0)|^2 + 2a^2\cot^*\alpha|\phi(1)|^2 + 2a^2\int_0^1 q|\phi|^2,
\end{aligned}$$

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$$c_2 = b^2\Big(\int_0^1 w|\phi|^2\Big)^2 - 2bc|\phi(0)|^2\int_0^1 w|\phi|^2 + 2ab\cot^*\alpha|\phi(1)|^2\int_0^1 w|\phi|^2 + 2ab\int_0^1 q|\phi|^2 \int_0^1 w|\phi|^2 + c^2|\phi(0)|^4 + 2ac\cot^*\alpha|\phi(0)|^2|\phi(1)|^2 + 2ac|\phi(0)|^2\int_0^1 q|\phi|^2 + a^2(\cot^*\alpha)^2|\phi(1)|^4 + 2a^2\cot^*\alpha|\phi(1)|^2\int_0^1 q|\phi|^2 + a^2\Big(\int_0^1 q|\phi|^2\Big)^2 + 4ad|\phi(0)|^2\int_0^1 w|\phi|^2. \tag{3.7}$$

Substitute a_2, b_2 and c_2 into $\sqrt{b_2^2 - 4a_2c_2}$, we have

$$\sqrt{b_2^2 - 4a_2c_2} \le 4|a|\sqrt{\|w\|_1(bc-ad)}, \tag{3.8}$$

and

$$-b_2 = -(2ab\int_0^1 w|\phi|^2 + 2ac|\phi(0)|^2 + 2a^2\cot^*\alpha|\phi(1)|^2 + 2a^2\int_0^1 q|\phi|^2) \le 2|a||b|\|w\|_1 + 2|a||c| + 2a^2|\cot^*\alpha| + 2a^2\|q\|_1. \tag{3.9}$$

Then, from (3.6)–(3.9), we have

$$R \le \frac{N + 2\sqrt{\rho\|w\|_1}}{|a|} = M. \tag{3.10}$$

Therefore, we have the inequalities in (3.3) by (3.10). The proof is completed. □

From the definition of δ_p in (2.5), we know $\delta_p \in (0, \frac{1}{2}]$ and $\int_{I'}\frac{1}{p} \le \frac{1}{4M}$ for any interval $I' \subset [0,1]$ with length δ_p.

Lemma 2. *Let ϕ and δ_{p} be defined as above with $\|\phi\|_\infty = 1$, then there exists an interval $I_\phi \subset [0,1]$, with length $\delta_p = \min\{\frac{1}{2}, \frac{1}{4M}\}$, such that*

$$|\phi(x)| \ge \frac{1}{2} \text{ and } \int_0^1 |\phi|^2 \ge \frac{1}{4}\delta_p. \tag{3.11}$$

Proof. For any interval $I_\phi \subset [0,1]$ of a length δ_p, by Cauchy-Schwarz inequality, the definition of δ_p and Lemma 1, we have

$$\left(\int_{I_\phi}|\phi'|\right)^2 \le \int_{I_\phi}\frac{1}{p}\int_{I_\phi}p|\phi'|^2 \le \int_{I_\phi}\frac{1}{p}\int_0^1 p|\phi'|^2 \le \frac{1}{4}. \tag{3.12}$$

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Let $x_0 \in [0,1]$ be such that $|\phi(x_0)| = \|\phi\|_\infty = 1$. Then for $x \in [0,1]$ and $|x - x_0| \le \delta_p$,

$$||\phi(x)| - 1| = |\phi(x) - \phi(x_0)| = \left| \int_{x_0}^{x} \phi' \right| \le \frac{1}{2}$$

by (3.12), and hence $|\phi(x)| \ge \frac{1}{2}$ on $I_\phi = [x_0 - \delta_p, x_0]$ or $[x_0, x_0 + \delta_p]$. And from $\delta_p \in (0, \frac{1}{2}]$ one can see that [0,1] contains at least one such interval I_ϕ. Then

$$\int_0^1 |\phi|^2 > \int_{I_\phi} |\phi|^2 \ge \frac{1}{4}\delta_p.$$

□

Lemma 3. *Let $w = w_+ - w_-$ and $|w| = w_+ + w_-$, then for any $x \in [a, \xi]$,*

$$\int_x^{x+\eta} w_+ \ge 19 \int_x^{x+\eta} w_-, \quad \int_x^{x+\eta} w_+ \ge \frac{19}{20} \int_x^{x+\eta} |w|. \tag{3.13}$$

Proof. According to the segmentation on interval $[0,1]$ and Lemma 2, there exists a subinterval $I_t \subset I_\phi$, and hence, we can select $\xi_t \in [a_t, b_t] \subset I_t \subset I_\phi$. For convenience, cancel the subscript t, then $2\xi = a + b, 2\eta = b - a$ and from (2.6) we have

$$\left| \operatorname{sgn} w(\xi) \int_x^{x+\eta} w \right| \ge \frac{9}{10} \int_x^{x+\eta} |w|, \quad x \in [a, \xi].$$

We now assume $\operatorname{sgn} w(\xi) = 1$. The case of $\operatorname{sgn} w(\xi) = -1$ can be dealt with in a similar way. Since

$$\begin{aligned} &\int_x^{x+\eta} w_+ \mathrm{d}t = \int_x^{x+\eta} (w + w_-) = \int_x^{x+\eta} w + \int_x^{x+\eta} w_- \ge \frac{9}{10} \int_x^{x+\eta} |w| \mathrm{d}t \\ &+ \int_x^{x+\eta} (|w| - w_+) \ge \frac{9}{10} \int_x^{x+\eta} |w| + \int_x^{x+\eta} |w| - \int_x^{x+\eta} w_+. \end{aligned} \tag{3.14}$$

From (3.14), we have

$$\int_x^{x+\eta} w_+ \mathrm{d}t \ge \frac{19}{20} \int_x^{x+\eta} |w| \mathrm{d}t. \tag{3.15}$$

Since $|w| = w_+ + w_-$, $\int_x^{x+\eta} w_+ \mathrm{d}t \ge \frac{19}{20} \int_x^{x+\eta} |w| \mathrm{d}t$, we have

$$\int_x^{x+\eta} w_- \mathrm{d}t \le \frac{1}{20} \int_x^{x+\eta} |w| \mathrm{d}t.$$

And hence,

$$\int_x^{x+\eta} w_+ \ge 19 \int_x^{x+\eta} w_-.$$

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□

Lemma 4. *For any* $\hat{a}, \hat{b} \in \mathbb{R}$, *set* $\sqrt{\hat{a}+\hat{b}i} = \hat{c}+\hat{d}i$, *then*

$$|\hat{c}| = \sqrt{\frac{\sqrt{\hat{a}^2+\hat{b}^2}+\hat{a}}{2}}, |\hat{d}| = \sqrt{\frac{\sqrt{\hat{a}^2+\hat{b}^2}-\hat{a}}{2}}. \tag{3.16}$$

Proof. According to the Lemma 4, it can be inferred that

$$(\hat{c}+\hat{d}i)(\hat{c}+\hat{d}i) = \hat{a}+\hat{b}i,$$

$$\hat{c}^2 - \hat{d}^2 + 2\hat{c}\hat{d}i = \hat{a}+\hat{b}i,$$

$$\begin{cases} \hat{c}^2 - \hat{d}^2 = \hat{a}, \\ 2\hat{c}\hat{d} = \hat{b}. \end{cases}$$

Since

$$4\hat{c}^2\hat{d}^2 = \hat{b}^2, \hat{d}^2 = \hat{c}^2 - \hat{a},$$

$$4\hat{c}^2(\hat{c}^2 - \hat{a}) = \hat{b}^2.$$

Adding $\hat{a}^2$ on both sides simultaneously,

$$(2\hat{c}^2 - \hat{a})^2 = \hat{a}^2 + \hat{b}^2.$$

Since $\hat{c}^2 > 0$, $\hat{c}^2 = \frac{\sqrt{\hat{a}^2+\hat{b}^2}+\hat{a}}{2}$, we have

$$|\hat{c}| = \sqrt{\frac{\sqrt{\hat{a}^2+\hat{b}^2}+\hat{a}}{2}}.$$

Similarly, it can be inferred that

$$|\hat{d}| = \sqrt{\frac{\sqrt{\hat{a}^2+\hat{b}^2}-\hat{a}}{2}}.$$

□

Using these lemmas, we give the proof of Theorems 1 and 2.

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The proof of Theorem 1. Suppose that λ corresponds to an eigenfunction ϕ of problem (3.1) and (3.2) with $\|\phi\|_\infty = 1$. From the conclusion of Lemma 3, $\frac{1}{2} \le |\phi| \le 1$ on I_ϕ and the definition of w_t in (2.8), we estimate

$$\begin{aligned}\left|\int_x^{x+\eta} w|\phi|^2 \mathrm{d}t\right| &= \left|\int_x^{x+\eta} (w_+ - w_-)|\phi|^2 \mathrm{d}t\right| \ge \int_x^{x+\eta} \frac{1}{4} w_+ \mathrm{d}t - \int_x^{x+\eta} w_- \mathrm{d}t \\ &\ge (\frac{1}{4} - \frac{1}{19})\frac{19}{20}\int_x^{x+\eta} |w| \mathrm{d}t = \frac{3}{16}\int_x^{x+\eta} |w| \mathrm{d}t \ge \frac{3}{16} w_\eta.\end{aligned} \tag{3.17}$$

Multiplying both sides of (3.1) by $\overline{\phi}$ and integrating by parts on $[x, x+\eta]$ we get

$$-p\phi'\overline{\phi}\Big|_x^{x+\eta} + \int_x^{x+\eta} p|\phi'|^2 + \int_x^{x+\eta} q|\phi|^2 = \lambda \int_x^{x+\eta} w|\phi|^2. \tag{3.18}$$

Taking absolute values from both sides and considering $\|\phi\|_\infty = 1$, (3.3) and (3.17) yields

$$\frac{3}{16}|\lambda| w_\eta \le |p\phi'(x)\overline{\phi}(x)| + |p\phi'(x+\eta)\overline{\phi(x+\eta)}| + M + \|q\|_1 \le |p\phi'(x)| + |p\phi'(x+\eta)| + M + \|q\|_1, \tag{3.19}$$

$$\frac{3}{16}|Im\lambda| w_\eta \le |p\phi'(x)\overline{\phi}(x)| + |p\phi'(x+\eta)\overline{\phi(x+\eta)}| \le |p\phi'(x)| + |p\phi'(x+\eta)|, x \in [a, \xi]. \tag{3.20}$$

Integrating with respect to x over $[a, \xi]$, by the definition of p_t in (2.9) and using the Cauchy-Schwarz inequality, we obtain that, for $t = 0$ or $t = \eta$,

$$\left(\int_a^\xi p|\phi'(x+t)| \mathrm{d}x\right)^2 = \left(\int_a^\xi p^{\frac{1}{2}} p^{\frac{1}{2}} |\phi'(x+t)| \mathrm{d}x\right)^2 \le \int_a^\xi p \int_a^\xi p|\phi'(x+t)|^2 \le p_\eta M. \tag{3.21}$$

That is,

$$\int_a^\xi p|\phi'(x+t)| \mathrm{d}x \le \sqrt{p_\eta M}. \tag{3.22}$$

Then, from (3.19) and (3.20), we have

$$\frac{3}{16}|\lambda| \eta w_\eta \le 2\sqrt{p_\eta M} + \eta M + \eta \|q\|_1, \tag{3.23}$$

$$\frac{3}{16}|Im\lambda| \eta w_\eta \le 2\sqrt{p_\eta M}, \tag{3.24}$$

which imply the inequalities in (2.10).

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The proof of Theorem 2. Multiplying both sides of (3.1) by $w\overline{\phi}$ and integrating by parts on $[0,1]$ we get

$$(c\lambda+d)w(0)|\phi(0)|^2+(a\lambda+b)\cot^*\alpha w(1)|\phi(1)|^2+(a\lambda+b)\int_0^1 p\phi' w'\overline{\phi}+(a\lambda+b)\int_0^1 pw|\phi'|^2+(a\lambda+b)\int_0^1 qw|\phi|^2=\lambda(a\lambda+b)\int_0^1 w^2|\phi|^2, \tag{3.25}$$

that is,

$$\lambda^2 a\int_0^1 w^2|\phi|^2+\lambda\Big[b\int_0^1 w^2|\phi|^2-cw(0)|\phi(0)|^2-a\cot^*\alpha w(1)|\phi(1)|^2-a\int_0^1 p\phi' w'\overline{\phi}-a\int_0^1 pw|\phi'|^2-a\int_0^1 qw|\phi|^2\Big]-dw(0)|\phi(0)|^2-b\cot^*\alpha w(1)|\phi(1)|^2-b\int_0^1 p\phi' w'\overline{\phi}-b\int_0^1 pw|\phi'|^2-b\int_0^1 qw|\phi|^2=0. \tag{3.26}$$

Let

$$A=a\int_0^1 w^2|\phi|^2,$$

$$B=b\int_0^1 w^2|\phi|^2-cw(0)|\phi(0)|^2-a\cot^*\alpha w(1)|\phi(1)|^2-a\int_0^1 p\phi' w'\overline{\phi}-a\int_0^1 pw|\phi'|^2-a\int_0^1 qw|\phi|^2,$$

$$C=-dw(0)|\phi(0)|^2-b\cot^*\alpha w(1)|\phi(1)|^2-b\int_0^1 p\phi' w'\overline{\phi}-b\int_0^1 pw|\phi'|^2-b\int_0^1 qw|\phi|^2.$$

Therefore, (3.26) becomes in the form of $A\lambda^2+B\lambda+C=0$. Then we have

$$\lambda=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$

Hence, we can get

$$|Im\lambda|\le\frac{1}{2A}\left(\left|ImB\right|+\left|Im\sqrt{B^2-4AC}\right|\right). \tag{3.27}$$

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Next, let's estimate $\left|Im\sqrt{B^2-4AC}\right|$. Let $B=B_1+iB_2, C=C_1+iC_2$, then

$$\begin{aligned} & B^2-4AC \\ =&(B_1+iB_2)^2-4A(C_1+iC_2) \\ =&B_1^2-B_2^2+2B_1B_2i-4AC_1-4AC_2i \\ =&(B_1^2-B_2^2-4AC_1)+i(2B_1B_2-4AC_2). \end{aligned} \tag{3.28}$$

Set $B_1^2-B_2^2-4AC_1=E, 2B_1B_2-4AC_2=K, \sqrt{B^2-4AC}=m+in$, that is, $\sqrt{E+iK}=m+in$. From (3.16) in Lemma 4, we know that

$$n^2=\frac{\sqrt{E^2+K^2}-E}{2}.$$

Let K be fixed and take the derivative of E, we can get

$$\left(\frac{\sqrt{E^2+K^2}-E}{2}\right)'=\frac{E}{2\sqrt{E^2+K^2}}-\frac{1}{2}.$$

Since $E\le\sqrt{E^2+K^2}$, $\left(\frac{\sqrt{E^2+K^2}-E}{2}\right)'\le 0$. Therefore, as E becomes smaller, n^2 becomes larger, and $|n|$ becomes larger. One can see that

$$E=B_1^2-B_2^2-4AC_1\ge -B_2^2-4|A||C_1|,$$

$$K=2B_1B_2-4AC_2\le 2|B_1||B_2|+4|A||C_2|.$$

Now let's find the maximum values of $|A|,|B_1|,|B_2|,|C_1|,|C_2|$. Using $\|\phi\|_\infty=1$, $w_0=\{\max|w(x)|:x\in[0,1]\}$, $\|w'\|_{p,2}=(\int_0^1 p|w'|^2)^{\frac{1}{2}}$, (3.3) in Lemma 1, and Cauchy-Schwarz inequality, we get

$$\int_0^1 p\phi' w'\overline{\phi}\le\left(\int_0^1 p|\phi'|^2\right)^{\frac{1}{2}}\left(\int_0^1 p|w'|^2\right)^{\frac{1}{2}}\le\|w'\|_{p,2}\sqrt{M}, \tag{3.29}$$

$$\int_0^1 pw|\phi'|^2\le w_0\int_0^1 p|\phi'|^2\le w_0M. \tag{3.30}$$

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It is obtained from the above formulas that

$$
\begin{aligned}
|A| &= \left| a\int_0^1 w^2|\phi|^2 \right| \le |a|w_0^2,\\
|B_1| &= \left| b\int_0^1 w^2|\phi|^2 - cw(0)|\phi(0)|^2 - a\cot^*\alpha w(1)|\phi(1)|^2 - Re\left(a\int_0^1 p\phi' w'\overline{\phi}\right)\right.\\
&\quad \left. - a\int_0^1 pw|\phi'|^2 - a\int_0^1 qw|\phi|^2 \right| \le |b|w_0^2 + |c|w_0\\
&\quad + |a||\cot^*\alpha|w_0 + |a|\|w'\|_{p,2}\sqrt{M} + |a|w_0 M + |a|w_0\|q\|_1,\\
|B_2| &= \left| Im\left(a\int_0^1 p\phi' w'\overline{\phi}\right)\right| \le |a|\|w'\|_{p,2}\sqrt{M},\\
|C_1| &= \left| - dw(0)|\phi(0)|^2 - b\cot^*\alpha w(1)|\phi(1)|^2\right.\\
&\quad \left. - Re\left(b\int_0^1 p\phi' w'\overline{\phi}\right) - b\int_0^1 pw|\phi'|^2 - b\int_0^1 qw|\phi|^2\right|\\
&\quad \le |d|w_0 + |b||\cot^*\alpha|w_0 + |b|\|w'\|_{p,2}\sqrt{M} + |b|w_0(M + \|q\|_1),\\
|C_2| &= \left| Im\left(b\int_0^1 p\phi' w'\overline{\phi}\right)\right| \le |b|\|w'\|_{p,2}\sqrt{M}.
\end{aligned}
\tag{3.31}
$$

Therefore, from the definitions of H, L, V, and W in Theorem 2, it can be concluded that

$$E \ge -B_2^2 - 4|A||C_1| \ge -a^2\|w'\|_{p,2}^2 M - 4|a|w_0^2 H = L, \tag{3.32}$$

$$K \le 2|B_1||B_2| + 4|A||C_2| \le 2|a|\|w'\|_{p,2}\sqrt{M}V + 4|a||b|w_0^2\|w'\|_{p,2}\sqrt{M} = W. \tag{3.33}$$

We have

$$\left| Im\sqrt{B^2 - 4AC}\right| = |n| \le \sqrt{\frac{-L + \sqrt{L^2 + W^2}}{2}}. \tag{3.34}$$

Let ε be defined as in (2.11), from (3.11) in Lemma 2, we get

$$a\int_0^1 w^2|\phi|^2 \ge a\varepsilon\int_{[0,1]\setminus\Omega(\varepsilon)}|\phi|^2 = a\varepsilon\left(\int_{[0,1]}|\phi|^2 - \int_{\Omega(\varepsilon)}|\phi|^2\right) \ge a\varepsilon(\frac{1}{4}\delta_p - \frac{1}{8}\delta_p) = \frac{1}{8}a\varepsilon\delta_p. \tag{3.35}$$

So we can know that

$$2A = 2a\int_0^1 w^2|\phi|^2 \ge \frac{1}{4}a\varepsilon\delta_p. \tag{3.36}$$

Since

$$|ImB| = |B_2| \le |a|\|w'\|_{p,2}\sqrt{M}. \tag{3.37}$$

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(3.27), (3.34), (3.36) and (3.37) leads us to

$$|Im\lambda| \leq \frac{4}{a\varepsilon\delta_p}\left(|a|\|w'\|_{p,2}\sqrt{M} + \sqrt{\frac{-L+\sqrt{L^2+W^2}}{2}}\right),\ a \neq 0. \tag{3.38}$$

Since

$$|Re\lambda| \leq \frac{1}{2A}\left(\left|ReB\right| + \left|Re\sqrt{B^2-4AC}\right|\right). \tag{3.39}$$

Next, let's estimate $\left|Re\sqrt{B^2-4AC}\right|$. From the previous assumptions, we know that

$$\left|Re\sqrt{B^2-4AC}\right| = |m|.$$

From (3.16) in Lemma 4,

$$|m| = \sqrt{\frac{\sqrt{E^2+K^2}+E}{2}}.$$

Obviously, the larger E and K, the larger $|m|$. We know that

$$\begin{aligned} E = B_1^2 - B_2^2 - 4AC_1 \leq |B_1|^2 + |B_2|^2 + 4|A||C_1| \leq \Big(|b|w_0^2 + |c|w_0 + |a||\cot^*\alpha|w_0 \\ + |a|\|w'\|_{p,2}\sqrt{M} + |a|w_0M + |a|w_0\|q\|_1\Big)^2 + a^2\|w'\|_{p,2}^2M + 4|a|w_0^2\Big(|d|w_0 + |b| \\ |\cot^*\alpha|w_0 + |b|\|w'\|_{p,2}\sqrt{M} + |b|w_0(M+\|q\|_1)\Big). \end{aligned} \tag{3.40}$$

Derived from the definitions of H, V, L, W, T and (3.33)

$$E \leq T, K \leq W. \tag{3.41}$$

So

$$\left|Re\sqrt{B^2-4AC}\right| = |m| \leq \sqrt{\frac{T+\sqrt{T^2+W^2}}{2}}. \tag{3.42}$$

Since

$$|ReB| = |B_1| \leq |b|w_0^2 + |c|w_0 + |a||\cot^*\alpha|w_0 + |a|\|w'\|_{p,2}\sqrt{M} + |a|w_0M + |a|w_0\|q\|_1 = V. \tag{3.43}$$

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Then

$$|Re\lambda| \leq \frac{4}{a\varepsilon\delta_p}\left(V+\sqrt{\frac{T+\sqrt{T^2+W^2}}{2}}\right),\ a\neq 0 \tag{3.44}$$

by (3.36), (3.39), (3.42), (3.43).

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