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DICHOTOMY RESULT FOR INDEPENDENCE-FRIENDLY PREFIXES OF GENERALIZED QUANTIFIERS

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Abstract. We study the expressive power of independence-friendly quantifier prefixes composed of universal $(\forall x/X)$, existential $(\exists x/X)$, and majority quantifiers $(\mathsf{M}x/X)$. We provide four quantifier prefixes that can express NP hard properties and show that all quantifier prefixes capable of expressing NP-hard properties embed at least one of these four quantifier prefixes. As for the quantifier prefixes that do not embed any of these four quantifier prefixes, we show that they are equivalent to a first-order quantifier prefix composed of $\forall x$, $\exists x$, and $\mathsf{M}x$. In unison, our results imply a dichotomy result: every independence-friendly quantifier prefix is either decidable in LOGSPACE or NP hard.

§1. Introduction. In a formula of first-order logic, if a quantifier Qx appears superordinate to another quantifier $Q'y$, then the quantification of y in $Q'y$ depends on x. Branching quantifiers [11] and later independence-friendly logic [12,18] sought to dissect the syntactical relationship between quantifiers from the dependence relationship they impose on their variables. Technically speaking, such *logics of imperfect information* impose a partial order on the quantifiers of their expressions. For instance, in the following branching quantifier, also known as the *Henkin quantifier*, the existentially quantified variables y_i only depend on the universally quantified variables x_i:

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix}. \tag{1}$$

In independence-friendly (IF) logic, the independence pattern of (1) is captured by the following string of quantifiers:

$$\forall x_1 \exists y_1 \big(\forall x_2/\{x_1, y_1\}\big)\big(\exists y_2/\{x_1, y_1\}\big). \tag{2}$$

In (2), the set of variables following the slash indicates that the quantification of y_2 in $\big(\exists y_2/\{x_1, y_1\}\big)$, although subordinate to the quantifiers $\forall x_1$ and $\exists y_1$, is independent of their variables x_1 and y_1.

The expressive power of these logics with imperfect information is known to be stronger than that of first-order logic. For instance, it was shown by Caicedo and Krynicki [5] that the IF formula

$$\forall x \exists y \big(\exists z/\{x\}\big)(z = x \wedge y \neq c) \tag{3}$$

expresses Dedekind infinity on structures that interpret the constant symbol c.

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In the area of descriptive complexity, Blass and Gurevich [3] showed that the Henkin quantifier (1) appended with the quantifier-free formula

$$\big(x_1 = x_2 \to y_1 = y_2\big) \wedge \big(y_1 = 0 \vee y_1 = 1 \vee y_1 = 2\big) \wedge \big(E(x_1, x_2) \to y_1 \neq y_2\big) \quad (4)$$

expresses 3-COLORABILITY on graphs with edge relation E and three distinct constants 0,1, and 2.

In linguistics, Barwise [2] argued that the sentence "most boys and most girls love each other" is captured by a branching majority quantifier:

$$\begin{pmatrix} \mathsf{M}_{\mathrm{BOY}}\, x \\ \mathsf{M}_{\mathrm{GIRL}}\, y \end{pmatrix} \mathrm{LOVE}(x, y), \quad (5)$$

In this expression, the constituent quantifier $\mathsf{M}_{\mathrm{BOY}}\, x$ asserts that more than half of the boys x have a particular property. In the context of a structure with finite universe M, Barwise proposed a semantics for branching quantifiers according to which (5) is equivalent to the second-order expression

$$\exists X \exists Y \Big(2|X| \geq |\mathrm{BOY}| \ \wedge \ 2|Y| \geq |\mathrm{GIRL}| \ \wedge \\ \forall x \forall y \big[\big(X(x) \wedge Y(y)\big) \to \big(\mathrm{BOY}(x) \wedge \mathrm{GIRL}(y) \wedge \mathrm{LOVE}(x, y)\big) \big] \Big). \quad (6)$$

More systematically, it is known that IF logic and first-order logic prefixed by branching quantifiers are as expressive as existential second-order logic [6, 20]. On the other hand, fragments of IF logic are known that have genuine imperfect information yet first-order expressive power. One such fragment is the class of IF formulas that produce extensive games with perfect recall [16, 19].

No systematic analysis has been undertaken to characterize the patterns of quantifier dependence and independence that cause the complexity jump from first-order to existential second-order logic (ESO). This will be the central topic of the present paper. We shall work in the context of IF quantifier prefixes, collected in IF(M), that consist of the customary IF quantifiers $(\forall x/X)$ and $(\exists x/X)$, and the IF majority quantifier $(\mathsf{M}x/X)$.

We will take a computational view on ESO, which by Fagin's theorem [8] coincides with the complexity class NP: on finite structures every NP-computable problem is definable by an ESO sentence and vice versa. It follows that NP-computable problems are the hardest problems that can be defined in ESO: every NP-hard problem definable in ESO is NP complete. Gottlob, Koliatis and Schwentick [10] showed that every ESO prefix is either NP complete or tractable, i.e., computable in polynomial time P. This *dichotomy result* bears relevance to Ladner's Theorem [15] according to which there are problems that are neither NP complete nor tractable, if $\mathrm{P} \neq \mathrm{NP}$.

In this paper, we characterize the IF quantifier prefixes that contribute to the characteristic ESO expressive power of IF logic. We characterize these prefixes in terms of a set of quantifier (in)dependence patterns. The described set will be such that an IF(M) prefix embeds at least one of its patterns if, and only if, the prefix defines an NP-hard problem. We shall further show that every IF(M) prefix that does not embed any of the set's patterns is equivalent to a first-order prefix, i.e., a prefix that consists of the "independence-unfriendly" quantifiers $\forall x$, $\exists x$, and $\mathsf{M}x$. It follows from a result in descriptive complexity that such prefixes are computable

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in LOGSPACE $\subseteq$ P [14]. Thus we derive a dichotomy result that is the IF analog of the dichotomy result in [10]: every IF(M) prefix is either NP hard or tractable.

In Section 2, we collect definitions and preliminary results. In this section we also introduce a set of four dependence patterns. A prefix that does not have any of the four patterns is called *primary*. In Section 3, it is shown that each primary prefix is equivalent to a first-order prefix. In Section 4, it is shown that nonprimary prefixes are capable of defining NP-hard problems. The final section, Section 5, ties the results together and relates it to previous research, including the literature on perfect recall.

§2. Preliminaries. Independence-friendly quantifiers $(Q_i u_i / U_i)$ carry a set U_i that contains the variables on which it depends. The set U_i is called a *slash set*.

DEFINITION 2.1. *An* IF(M) prefix *is a series* $(Q_1 u_1 / U_1) \dots (Q_n u_n / U_n)$ *of quantifiers* $(Q_i u_i / U_i)$ *where* $Q_i \in \{\forall, \exists, \mathsf{M}\}$, u_i *a variable and* U_i *a finite set of variables disjoint from* $\{u_i, \dots, u_n\}$. *We require that all quantified variables* u_i *in an* IF(M) *prefix be distinct*.

An FOL(M) prefix *is an* IF(M) *prefix in which every slash set is empty.*

An IF (*respectively*, FOL) prefix *is an* IF(M) (*respectively*, FOL(M)) *prefix that does not contain the majority quantifier symbol* M.

If Π and Π' are two quantifier prefixes, we write $\Pi\Pi'$ for their concatenation. For two quantifiers $(Q_i u_i / U_i)$ and $(Q_j u_j / U_j)$ in a prefix with $i < j$, we say that $(Q_j u_j / U_j)$ *depends on* $(Q_i u_i / U_i)$ if $u_i \notin U_j$. A quantifier (Qu/U) is written as Qu whenever its slash set U is empty. We shall mostly omit the curly brackets inside quantifiers. In this manner, we write $(Qu/v, w)$ instead of $\big(Qu/\{v, w\}\big)$ and $(Qu/U \cup v)$ instead of $\big(Qu/U \cup \{v\}\big)$.

An *atom* is an expression $R^n(x_1, \dots, x_n)$ in which R^n is an n-ary relation symbol, and $x_1, \dots, x_n$ are variables. We shall mostly omit to mention the arity of a relation symbol and write R instead of R^n.

The *quantifier-free formulas* of concern are obtained by closing the set of possibly negated atoms $(\neg)R(x_1, \dots, x_n)$ under taking conjunctions $\wedge$ and disjunctions $\vee$. We shall usually write α for quantifier-free formulas. An IF(M) *formula* is an IF(M) prefix that is followed by a quantifier-free formula.

A prefix $\Pi = (Q_1 u_1 / U_1) \dots (Q_n u_n / U_n)$ is *sentential* if $U_i \subseteq \{u_1, \dots, u_{i-1}\}$ for every $1 \leq i \leq n$. Dually, we say that Π is *nonsentential* if there is an i and a variable x for which $x \in \big(U_i \cap \{u_1, \dots, u_{i-1}\}\big)$. We say that such a variable x is *free* in Π.

Recall the proviso in Definition 2.1 according to which every slash set U_i in a prefix Π is disjoint from the variables $\{u_i, \dots, u_n\}$ that were quantified to the left of it. It can be seen that if Π is nonsentential, then there is a prefix Σ such that $\Sigma\Pi$ is sentential.

An IF(M) formula $\Pi\alpha$ is a *sentence* if Π is sentential and all variables appearing in α are contained in the set of variables quantified in Π. If $\Pi\alpha$ is not a sentence, Π is nonsentential and/or there is a variable x in α that is not quantified in Π. We say that such a variable x is *free* in $\Pi\alpha$.

If $\phi = (Q_1 u_1 / U_1) \dots (Q_n u_n / U_n)\alpha$ is an IF(M) formula and u is a variable, then

$$\phi / u = (Q_1 u_1 / U_1 \cup u) \dots (Q_n u_n / U_n \cup u)\alpha. \tag{7}$$

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A structure $\mathbb{M} = (M, R_1, R_2, \ldots, c_1, c_2, \ldots)$ has a *universe* M and interprets relation symbols R_i and constants c_j as usual. In our notation we shall not distinguish between a relation symbol and the interpretation it receives on a given structure, and likewise for constants. The expansion of a structure $\mathbb{M}$ by a relation R is written as $(\mathbb{M}, R)$.

An *assignment s in $\mathbb{M}$* is a function that assigns to every variable in its domain $dom(s)$ an object from M. Whenever two assignments s, s' have the same domain V, we write $s \approx_W s'$ if s and s' agree on every variable in $V - W$. A set of assignments X is called an *assignment team in $\mathbb{M}$*, if all of its assignments have the same domain, written $dom(X)$, and are themselves assignments in $\mathbb{M}$. We shall omit to mention the structure wherein an assignment or assignment team operates whenever it is clear from the context.

Let X be an assignment team, let $W \subseteq dom(X)$ and let f be a function with domain X. We say f is *W-uniform* or *uniform in W*, if $s \approx_W s'$ implies $f(s) = f(s')$, for all assignments $s, s' \in X$. For a function $f : X \to \wp(M)$, $X[x, f]$ denotes

$$\{s/(x, a) \colon s \in X, a \in f(s)\}, \tag{8}$$

where $s/(x, a)$ is the assignment that assigns a to x and is otherwise like s. If X only contains the empty assignment $\varnothing$, we write $[x, f]$ instead of $\{\varnothing\}[x, f]$.

For a set of objects M, define[1]

$$\exists^M = \{A \subseteq M \colon |A| = 1\} \tag{9}$$

$$\forall^M = \{M\} \tag{10}$$

$$\mathsf{M}^M = \{A \subseteq M \colon 2|A| \geq |M|\}. \tag{11}$$

We now introduce the semantics for IF(M) following Hodges' so-called trump semantics [13]. Unlike Hintikka and Sandu's initial interpretation for independence-friendly logic, which had a game-theoretic nature, this interpretation is compositional. We relate our results with the game-theoretic view on independence-friendliness in Section 5.

DEFINITION 2.2. *Let $\mathbb{M}$ be a structure and X an assignment team in $\mathbb{M}$. Then, whenever $dom(X)$ contains all variables that are free in the formula at hand,*

$$\mathbb{M}, X \models R(x_1, \ldots, x_n) \quad \textit{iff} \quad \big(s(x_1), \ldots, s(x_n)\big) \in R \textit{ for every } s \in X \tag{12}$$

$$\mathbb{M}, X \models \neg R(x_1, \ldots, x_n) \quad \textit{iff} \quad \big(s(x_1), \ldots, s(x_n)\big) \notin R \textit{ for every } s \in X \tag{13}$$

$$\mathbb{M}, X \models \phi \vee \phi' \quad \textit{iff} \quad \mathbb{M}, Y \models \phi \textit{ and } \mathbb{M}, Y' \models \phi', \textit{ for some partitioning } Y, Y' \textit{ of } X \tag{14}$$

$$\mathbb{M}, X \models \phi \wedge \phi' \quad \textit{iff} \quad \mathbb{M}, X \models \phi \textit{ and } \mathbb{M}, X \models \phi' \tag{15}$$

$$\mathbb{M}, X \models (Qu/U)\phi \quad \textit{iff} \quad \mathbb{M}, X[x, f] \models \phi, \textit{ for some } U\textit{-uniform function } f : X \to Q^M. \tag{16}$$

The quantifier clause is consistent with Engström's definition of monotone slashed generalized quantifiers [7, Definition 2.5].

We write $\mathbb{M} \models \phi$ for $\mathbb{M}, \{\varnothing\} \models \phi$, in which case ϕ is a sentence.

[1] Alternatively, $\exists^M$ is defined as $\{A \subseteq M \colon A \neq \varnothing\}$. The two definitions are interchangeable on account of the downward monotonicity principle (Proposition 2.5).

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EXAMPLE 2.3. *As an example of the compositional semantics, consider the* IF(M) *sentence*

$$\phi \quad = \quad \mathsf{M}_{\mathrm{BOY}}\, x\, (\mathsf{M}_{\mathrm{GIRL}}\, y/x)\ \mathrm{LOVE}(x,y), \tag{17}$$

that is essentially the serialization of (5). *To show that the proposed semantics of slashed majority quantifiers coincides with Barwise's interpretation* (6), *we stretch the semantics of* M *to its relativized version* M_P. *For a predicate* P, *i.e., a unary relation, let*

$$\mathsf{M}_P^M = \{A \subseteq P\colon 2|A| \geq |P|\}. \tag{18}$$

If $\mathbb{M} \models \phi$, *there exists a function* $f : \{\varnothing\} \to \mathsf{M}^M_{\mathrm{BOY}}$ *such that*

$$\mathbb{M}, [x, f] \ \models\ (\mathsf{M}_{\mathrm{GIRL}}\, y/x)\ \mathrm{LOVE}(x,y) \tag{19}$$

and an $\{x\}$*-uniform function* $f' : [x, f] \to \mathsf{M}^M_{\mathrm{GIRL}}$ *such that*

$$\mathbb{M}, [x, f][y, f'] \ \models\ \mathrm{LOVE}(x,y). \tag{20}$$

Let $B = f(\{\varnothing\})$. *Since* $B \in \mathsf{M}^M_{\mathrm{BOY}}$, *it contains at least half of the boys in* $\mathbb{M}$. *The function* f' *is* $\{x\}$*-uniform. Therefore it assigns the same set* G *to each assignment in* $[x, f]$. *Since* $G \in \mathsf{M}^M_{\mathrm{GIRL}}$, *it contains at least half of the girls in* $\mathbb{M}$. *Clause* (20) *guarantees that for every* $b \in B$ *and* $g \in G$, $(b,g) \in$ LOVE. *Thus we see that condition* (6) *is satisfied.*

The converse is similar. If the second-order reading (6) *holds on a structure, there exist sets* B *and* G *of the appropriate sizes such that for every* $b \in B$ *and* $g \in G$, $(b,g) \in$ LOVE. *Let* f *be the function that assigns* B *to* $\{\varnothing\}$ *and* f' *the* $\{x\}$*-uniform function that assigns* G *to each assignment in* $[x, f]$. *It follows that*

$$\mathbb{M}, [x, f][y, f'] \ \models\ \mathrm{LOVE}(x,y). \tag{21}$$

We define a notion of equivalence between quantifier prefixes that is based on equivalence between IF(M) formulas.

DEFINITION 2.4. *Let* ϕ *and* ϕ' *be* IF(M) *formulas. Let* V *be the set of free variables in* ϕ *and* ϕ'. *Then,* ϕ *and* ϕ' *are* equivalent, *if for every suitable structure* $\mathbb{M}$ *and assignment team* X *in* $\mathbb{M}$ *with* $V \subseteq dom(X)$,

$$\mathbb{M}, X \models \phi \quad \textit{if, and only if,} \quad \mathbb{M}, X \models \phi'. \tag{22}$$

Let Π *and* Π' *be* IF(M) *prefixes. Then,* Π *and* Π' *are* equivalent *if* $\Pi\alpha$ *and* $\Pi'\alpha$ *are equivalent for every quantifier-free formula* α.

We will now inspect a number of equivalence principles that are known in the literature or quite straightforward. The first principle is known as the *downward monotonicity principle*.

PROPOSITION 2.5 ([13]). *For every* IF(M) *formula* ϕ *and assignment teams* $Y \subseteq X$, *if* $\mathbb{M}, X \models \phi$ *then* $\mathbb{M}, Y \models \phi$.

Let X and Y be two assignment teams with $dom(X) \subseteq dom(Y)$. We say that Y *extends* X if for every $t \in Y$ there is an $s \in X$ that agrees with t on all variables in $dom(X)$.

PROPOSITION 2.6 ([4]). *Let* X *and* Y *be two assignment teams such that* Y *extends* X *and* $dom(X) \cup \{u\} = dom(Y)$ *for some variable* u. *Then, for every* IF(M) *formula* ϕ *in which* u *does not occur,* $\mathbb{M}, X \models \phi$ *if, and only if,* $\mathbb{M}, Y \models \phi/u$.

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The following principle is an immediate consequence of the fact that every U-uniform function is uniform in U' whenever $U' \subseteq U$.

PROPOSITION 2.7. *For every* IF(M) *formula* $(Qu/U)\phi$ *and* $U' \subseteq U$, *if* $\mathbb{M}, X \models (Qu/U)\phi$ *then* $\mathbb{M}, X \models (Qu/U')\phi$.

For an assignment team X and variable $u \in dom(X)$, we say X is *constant in* u, if $s(u) = s'(u)$ for every two assignments $s, s' \in X$. If an assignment team X is constant in a variable u, every function with domain X is uniform in u. It follows that the choice function for any quantifier evaluated at X is uniform in u, regardless of whether the quantifier depends on u. This reasoning suffices to prove the next result.

PROPOSITION 2.8. *For every* IF(M) *formula* ϕ, *if* X *is constant in the variable* u, *then*

$$\mathbb{M}, X \models (Qv/V)\phi \quad \textit{if, and only if,} \quad \mathbb{M}, X \models (Qv/V \cup u)\phi. \tag{23}$$

PROOF. Suppose $\mathbb{M}, X \models (Qv/V)\phi$. Then, there exists a V-uniform function $f : X \to Q^M$, such that $\mathbb{M}, X[v, f] \models \phi$. Since all assignments in X agree on the value of u, f is automatically uniform in $V \cup \{u\}$. It follows that $\mathbb{M}, X \models (Qv/V \cup u)\phi$.
The converse is immediate from Proposition 2.7. ⊣

The following principle shows that adding identical quantifiers to the left of equivalent prefixes preserves equivalence. It can be used when proving the prefix equivalences that follow.

PROPOSITION 2.9. *Let* $\Sigma\Pi$ *and* $\Sigma'\Pi'$ *be* IF(M) *prefixes for which* $\Sigma = \Sigma'$. *If* Π *and* Π' *are equivalent*, *so are* $\Sigma\Pi$ *and* $\Sigma'\Pi'$.

PROOF. By induction on the number of quantifiers in Σ. Suppose $\mathbb{M}, X \models (Qu/U)\Sigma\Pi$. Then, there exists a U-uniform function $f : X \to Q^M$ for which $\mathbb{M}, X[u, f] \models \Sigma\Pi$. By the induction hypothesis, $\mathbb{M}, X[u, f] \models \Sigma\Pi'$, and therefore, $\mathbb{M}, X \models (Qu/U)\Sigma\Pi'$. ⊣

We can *swap* two consecutive quantifiers if the right-hand quantifier is independent of the left-hand quantifier: although the syntactical ordering is obviously affected by a swap of two quantifiers, the (in)dependence relation between any two quantifiers in the prefix persists.

PROPOSITION 2.10. *Every two* IF(M) *prefixes*

$$(Qu/U)(Qv/V \cup u)\Pi \quad \textit{and} \quad (Qv/V - u)(Qu/U \cup v)\Pi$$

are equivalent.

In the context of the semantic relation $\models$, the slash sets of universal quantifiers turn out to be immaterial.

PROPOSITION 2.11. *Every two* IF(M) *prefixes*

$$(\forall v/V)\Pi \quad \textit{and} \quad \forall v\Pi$$

are equivalent.

PROOF. Every function with co-domain $\forall^M$ is uniform in every set of variables V. Whence, $\mathbb{M}, X \models \forall v\Pi$ implies $\mathbb{M}, X \models (\forall v/V)\Pi$. The converse is immediate from Proposition 2.7. ⊣

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We shall apply Proposition 2.11 tacitly, for instance, when we swap a universal quantifier $\forall x$ with another quantifier (Qu/U) in any prefix in which the former appears immediately to the left of the latter. If $x \in U$, namely, Proposition 2.10 grants us that the IF(M) prefix

$$\Pi \forall x (Qu/U) \Pi' \tag{24}$$

is equivalent to

$$\Pi (Qu/U - x)(\forall x/u) \Pi', \tag{25}$$

which by Proposition 2.11 is equivalent to

$$\Pi (Qu/U - x) \forall x \Pi'. \tag{26}$$

Under special conditions we may also remove the slash sets of existential and majority quantifiers. The result presented here is a special case of [1, Theorem 7.9], appropriately extended to include the majority quantifier.

LEMMA 2.12 ([1]). *Every two sentential* IF(M) *prefixes*

$$\Pi (Qv/V) \Pi' \quad \textit{and} \quad \Pi Qv \Pi'$$

are equivalent, whenever V *contains only variables that are quantified by existential quantifiers in* Π.

PROOF. Let X be an assignment team with domain equal to the set of variables in Π. Let g be a function with domain X. Suppose u is a variable that is quantified by an existential quantifier $(\exists u/U)$ in Π. Then, $X = Y[u, f]$, for some function $f : Y \to M$, since $\Pi (Qv/V) \Pi'$ is sentential. Let s and s' be two assignments from X that agree on all variables in the domain of X except, possibly, u, that is, $s \approx_{\{u\}} s'$. Then, $s(u) = f(t)$ and $s'(u) = f(t')$, where t is the assignment in Y that is extended by s and t' is the assignment that is extended by s'. Since $s \approx_{\{u\}} s'$ and the domain of s is equal to domain of t plus $\{u\}$, we derive that $t = t'$, that $f(t) = f(t')$ and that $s(u) = s'(u)$. It follows that $s = s'$ and that $g(s) = g(s')$. We conclude that the function g is uniform in $\{u\}$.

The argument can be extended to show that g is uniform in any set of variables that is quantified by existential quantifiers. Thus, if $\mathbb{M}, X \models Qv \Pi' \alpha$, then $\mathbb{M}, X \models (Qv/V) \Pi' \alpha$ whenever V only contains existentially quantified variables in Π. The converse direction is immediate from Proposition 2.7. ⊣

The following definition introduces the dependence patterns of IF prefixes that shall have our interest.

DEFINITION 2.13. *Let* Π *be an* IF(M) *prefix.* Π *is* signaling *if it contains three quantifiers ordered* (Qu/U), $(Q'v/V)$, $(Q''w/W)$ *with* $Q \in \{\forall, \mathsf{M}\}$ *and* $Q', Q'' \in \{\exists, \mathsf{M}\}$ *such that*

- $(Q'v/V)$ *depends on* u;
- $(Q''w/W)$ *depends on* v *and not on* u.

Π *is* Henkin *if it contains four quantifiers* $\forall x, \forall y, (\exists u/U), (\exists v/V)$ *in any order such that*

- $(\exists u/U)$ *depends on* x *and not on* y *and* v;
- $(\exists v/V)$ *depends on* y *and not on* x *and* u.

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Π *is* branching *if it contains two distinct quantifiers ordered* $(\mathsf{M}u/U), (\mathsf{M}v/V)$ *such that*

- $(\mathsf{M}v/V)$ *does not depend on* u.

Π *is* $\forall\exists/\mathsf{M}$ *if it contains three quantifiers* $\forall x, (\exists u/U), (\mathsf{M}v/V)$ *in any order such that*

- $(\exists u/U)$ *depends on* x *and not on* v;
- $(\mathsf{M}v/V)$ *does not depend on* x *and* u.

Π *is* primary *if it has none of the above four properties.*

Trivially, no IF prefix is branching or $\forall\exists/\mathsf{M}$, as the majority quantifier is foreign to the syntax of IF.

We encountered three of the above patterns: sentence (3) has a signaling quantifier prefix, (2) has a Henkin quantifier prefix, and (17) has a branching quantifier prefix.

The following lemma shows that being primary is preserved under three syntactical operations on IF prefixes that were introduced above.

LEMMA 2.14. *Let* Π *be a* $\mathrm{IF}(\mathsf{M})$ *prefix and let* Π' *be the* $\mathrm{IF}(\mathsf{M})$ *prefix that is the result of*

- *swapping two quantifiers in* Π *(as in Proposition* 2.10*); or*
- *emptying the slash set of a universal quantifier in* Π *(as in Proposition* 2.11*); or*
- *emptying the slash set of a quantifier in* Π *that contains only existentially quantified variables (as in Proposition* 2.12*).*

Then, Π' *is primary, whenever* Π *is.*

§3. Primary quantifier prefixes. In this section we characterize the sentential $\mathrm{IF}(\mathsf{M})$ prefixes that have equivalents in $\mathrm{FOL}(\mathsf{M})$. The heavy lifting will be done by two lemmas showing that the slash sets of quantifiers (Qu/U) can be emptied in primary sentential prefixes, for $Q = \exists$ respectively $Q = \mathsf{M}$. As a corollary, we obtain a characterization of sentential IF prefixes that have equivalents in FOL.

LEMMA 3.1. *Every primary, sentential* $\mathrm{IF}(\mathsf{M})$ *prefix* $\Pi(\exists v/V)\Pi'$ *in which* Π *is an* $\mathrm{FOL}(\mathsf{M})$ *prefix is equivalent to a primary, sentential* $\mathrm{IF}(\mathsf{M})$ *prefix* $\Sigma\Pi'$ *in which* Σ *is an* $\mathrm{FOL}(\mathsf{M})$ *prefix.*

PROOF. Let Qx be the quantifier in Π with $Q \in \{\forall, \mathsf{M}\}$ that is closest to $(\exists v/V)$, i.e., $\Pi(\exists v/V)\Pi'$ is of the form

$$\ldots Qx \underbrace{\qquad\qquad}_{\exists} (\exists v/V)\Pi', \tag{27}$$

with the brace indicating that in between Qx and $(\exists v/V)$ only existential quantifiers occur. If no such Qx exists, there are only existential quantifiers to the left of $(\exists v/V)$. Then, since $\Pi(\exists v/V)\Pi'$ is sentential, all variables in V are existentially quantified, and $\Pi(\exists v/V)\Pi'$ is equivalent to $\Pi\exists v\Pi'$ on account of Lemma 2.12. Henceforth we shall assume that such a Qx exists to the left of $(\exists v/V)$.

We apply the transformations detailed in the cases below until V is empty, i.e., until $\Pi(\exists v/V)$ is an $\mathrm{FOL}(\mathsf{M})$ prefix.

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We distinguish two cases.

Case 1: Suppose $x \notin V$. We distinguish two subcases.

- Case 1.a: Suppose there is a quantifier $Q'y$ with $Q' \in \{\forall, \mathsf{M}\}$ to the left of Qx, such that $(\exists v/V)$ does not depend on y, that is, $y \in V$. We have that $Q = \forall$, for otherwise the prefix would be signaling. We further have that $Q' = \forall$, for otherwise the prefix would be $\forall\exists/\mathsf{M}$. So we have the following situation:

$$\ldots \forall y \ldots \forall x \underbrace{\qquad}_{\exists} (\exists v/V)\Pi'. \tag{28}$$

  Assume that a quantifier Qu appears in between $\forall y$ and $\forall x$. If $Q = \mathsf{M}$, the prefix is $\forall\exists/\mathsf{M}$ if $u \in V$, and signaling if $u \notin V$. If $Q = \exists$, the prefix is Henkin if $u \in V$, and signaling if $u \notin V$. Thus, if there is a quantifier in between $\forall y$ and $\forall x$, it is universal:

$$\ldots \forall y \underbrace{\qquad}_{\forall} \forall x \underbrace{\qquad}_{\exists} (\exists v/V)\Pi'. \tag{29}$$

  Let $\exists u$ be any existential quantifier between $\forall x$ and $(\exists v/V)$:

$$\ldots \forall y \underbrace{\qquad}_{\forall} \forall x \underbrace{\qquad}_{\exists} \exists u \underbrace{\qquad}_{\exists} (\exists v/V)\Pi'. \tag{30}$$

  Since the prefix is not signaling, $(\exists v/V)$ is independent of u. So we can swap $(\exists v/V)$ with all existential quantifiers to its left until we reach $\forall x$, yielding:

$$\ldots \forall y \underbrace{\qquad}_{\forall} \forall x(\exists v/V') \underbrace{\qquad}_{\exists} (\exists u/v) \underbrace{\qquad}_{\exists} \Pi'. \tag{31}$$

  where V' is the set of variables that remains after swapping. Since v is quantified by an existential quantifier, Proposition 2.12 allows us to empty the slash set of $(\exists u/v)$:

$$\ldots \forall y \underbrace{\qquad}_{\forall} \forall x(\exists v/V') \underbrace{\qquad}_{\exists} \exists u \underbrace{\qquad}_{\exists} \Pi'. \tag{32}$$

  It follows from Lemma 2.14 that prefix (32) is primary.

  We can swap the universal quantifiers in the block of universal quantifiers spanned by $\forall y$ and $\forall x$ until $\forall y$ appears immediately to the left of $(\exists v/V')$. Then we swap $\forall y$ and $(\exists v/V')$. The resulting prefix is of the form

$$\ldots \underbrace{\qquad}_{\forall} \forall x(\exists v/V'-y)\forall y \underbrace{\qquad}_{\exists} \exists u \underbrace{\qquad}_{\exists} \Pi'. \tag{33}$$

  Again, it follows from Lemma 2.14 that prefix (33) is primary.
- Case 1.b: Suppose $(\exists v/V)$ depends on all nonexistential quantifiers to the left of $\forall x$. Since $(\exists v/V)$ also depends on x it follows from Lemma 2.12 that the sentential prefix $\Pi(\exists v/V)\Pi'$ is equivalent to $\Pi\exists v\Pi'$. This prefix is primary, on account of Lemma 2.14.

Case 2: Suppose $x \in V$. Since $\Pi(\exists v/V)\Pi'$ is not signaling, $u \in V$ for every quantifier $\exists u$ that appears between Qx and $(\exists v/V)$. Recall that only existential quantifiers can appear between Qx and $(\exists v/V)$, see (27). We can consecutively

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swap $(\exists v/V)$ with every quantifier $\exists u$ to its left and ultimately with Qx. After doing so, $\Pi(\exists v/V)\Pi'$ is of the form

$$\dots(\exists v/V')(Qx/v)\dots(\exists u/v)\dots\Pi', \tag{34}$$

where V' is the set of variables that remains after swapping $(\exists v/V)$. Since the variable v is existentially quantified, it follows from Lemma 2.12 that (34) is equivalent to

$$\dots(\exists v/V')Qx\dots\exists u\dots\Pi'. \tag{35}$$

⊣

THEOREM 3.2. *Every primary, sentential* IF *prefix is equivalent to a sentential* FOL *prefix*.

PROOF. Immediate from Proposition 2.11 and Lemma 3.1. ⊣

LEMMA 3.3. *Every primary, sentential* IF(M) *prefix* $\Pi(\mathsf{M}v/V)\Pi'$ *in which* Π *is an* FOL(M) *prefix, is equivalent to a primary, sentential* IF(M) *prefix* $\Sigma\Pi'$ *in which* Σ *is an* FOL(M) *prefix*.

PROOF. Let Qx be the quantifier in Π with $Q \in \{\forall, \mathsf{M}\}$ that is closest to $(\mathsf{M}v/V)$, i.e., $\Pi(\mathsf{M}v/V)\Pi'$ is of the form

$$\dots Qx \underbrace{\qquad\qquad}_{\exists} (\mathsf{M}v/V)\Pi'. \tag{36}$$

If no such Qx exists, there are only existential quantifiers to the left of $(\mathsf{M}v/V)$. Then, since $\Pi(\mathsf{M}v/V)\Pi'$ is sentential, all variables in V are existentially quantified, and $\Pi(\mathsf{M}v/V)\Pi'$ is equivalent to $\Pi\mathsf{M}v\Pi'$ on account of Lemma 2.12. Henceforth we shall assume that such a Qx exists to the left of $(\mathsf{M}v/V)$.

We apply the transformations detailed in the cases below until V is empty, i.e., until $\Pi(\exists v/V)$ is an FOL(M) prefix.

We distinguish two cases.

Case 1: Suppose $x \notin V$. We distinguish two subcases.

- Case 1.a: Suppose there is a quantifier $Q'y$ with $Q' \in \{\forall, \mathsf{M}\}$ to the left of Qx, such that $(\mathsf{M}v/V)$ does not depend on y, that is, $y \in V$. We have that $Q = \forall$, for otherwise the prefix would be signaling. We further have that $Q' = \forall$, for otherwise the prefix would be $\forall\exists/\mathsf{M}$. That is, we have the following situation:

$$\dots\forall y\dots\forall x \underbrace{\qquad\qquad}_{\exists} (\mathsf{M}v/V)\Pi'. \tag{37}$$

  By the same reasoning as in the corresponding situation of Lemma 3.1, only universal quantifiers can occur between $\forall y$ and $\forall x$:

$$\dots\forall y \underbrace{\qquad\qquad}_{\forall} \forall x \underbrace{\qquad\qquad}_{\exists} (\mathsf{M}v/V)\Pi'. \tag{38}$$

  Suppose there were an existential quantifier $\exists u$ in between $\forall x$ and $(\mathsf{M}v/V)$. If $u \in V$, then the prefix is $\forall\exists/\mathsf{M}$; if $u \notin V$, the prefix is signaling. We conclude that no existential quantifier occurs between $\forall x$ and $(\mathsf{M}v/V)$:

$$\dots\forall y \underbrace{\qquad\qquad}_{\forall} \forall x(\mathsf{M}v/V)\Pi'. \tag{39}$$

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Swap $\forall y$ with the universal quantifier to the right of it, and continue doing so until $(\mathsf{M}v/V)$ is to the right of it. Then swap $\forall y$ and $(\mathsf{M}v/V)$ to obtain:

$$\dots \underbrace{\qquad\qquad}_{\forall} \forall x(\mathsf{M}v/V - y)\forall y\Pi'. \tag{40}$$

It follows from Lemma 2.14 that prefix (40) is primary.

- Case 1.b: Suppose $(\mathsf{M}v/V)$ depends on all nonexistential quantifiers to the left of $\forall x$. Since $(\mathsf{M}v/V)$ also depends on x it follows from Lemma 2.12 that $\Pi(\mathsf{M}v/V)\Pi$ is equivalent to the first-order prefix $\Pi\mathsf{M}y'\Pi'$. This prefix is primary, on account of Lemma 2.14.

Case 2: Suppose $x \in V$. We have that $Q = \forall$, for otherwise the prefix would be branching. Suppose there were an existential quantifier $\exists u$ in between $\forall x$ and $(\mathsf{M}v/V)$. If $u \in V$, the prefix is $\forall\exists/\mathsf{M}$; if $u \notin V$, the prefix is signaling. So either way, the prefix is not primary, which is a contradiction. We conclude that no quantifiers exist between $\forall x$ and $(\mathsf{M}v/V)$. We can thus swap these quantifiers and obtain:

$$\dots(\mathsf{M}v/V - x)\forall x\Pi', \tag{41}$$

which is primary on account of Lemma 2.14. ⊣

THEOREM 3.4. *Every primary, sentential* IF(M) *prefix is equivalent to a sentential* FOL(M) *prefix.*

PROOF. Immediate from Proposition 2.11 and Lemmas 3.1 and 3.3. ⊣

§4. Nonprimary quantifier prefixes. Each IF(M) sentence ϕ gives rise to the following decision problem: given a finite structure $\mathbb{M}$, does $\mathbb{M} \models \phi$ hold? We say that a prefix Π is *NP hard*, if there exists a quantifier-free formula α for which the decision problem defined by $\Pi\alpha$ is NP hard. We say that Π is *in LOGSPACE*, if the decision problem defined by $\Pi\alpha$ is computable in LOGSPACE for every quantifier-free formula α.

In this section we show that every nonprimary, sentential IF(M) prefix is NP hard.

DEFINITION 4.1. *Let* Π *and* Π' *be two* IF(M) *prefixes. We say that* Π' extends Π, *if*

- *there exists a function* μ *from quantifiers in* Π *to quantifiers in* Π' *that preserves quantifier symbols and variables, that is, for every* (Qu/U) *in* Π, *if* $\mu\big((Qu/U)\big) = (Q'u'/U')$ *then* $Q = Q'$ *and* $u = u'$; *and*
- *for every pair of quantifiers* (Qu/U) *and* (Qv/V) *in* Π, (Qu/U) *depends on* (Qv/V) *if, and only if,* $\mu\big((Qu/U)\big)$ *depends on* $\mu\big((Qv/V)\big)$.

The four properties in Definition 2.13 are obviously preserved under taking extensions, that is, if Π' extends Π and Π is signaling, then so is Π', and similarly for Henkin, branching and $\forall\exists/\mathsf{M}$ prefixes Π. We proceed to show that NP hardness is also preserved under taking extensions. This result allows us to focus on the "simplest" NP-hard prefixes.

LEMMA 4.2. *Every sentential* IF(M) *prefix that extends an NP-hard prefix is NP hard itself.*

PROOF. Let $\Sigma\Pi\alpha$ be an IF(M) sentence with

$$\Sigma = (Q_1u_1/U_1)\dots(Q_iu_i/U_i) \tag{42}$$

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and

$$\Pi = (Q_{i+1}u_{i+1}/U_{i+1})\dots(Q_nu_n/U_n). \tag{43}$$

Let v be a variable that is not in $\{u_1,\dots,u_n\}$. Consider any extension $\Sigma(Qv/V)\Pi'$ of $\Sigma\Pi$, such that

$$\Pi' = (Q_{i+1}u_{i+1}/U_{i+1}\cup W_{i+1})\dots(Q_nu_n/U_n\cup W_n). \tag{44}$$

where $V\subseteq\{u_1,\dots,u_i\}$ and $W_j\subseteq\{v\}$ for every $i+1\le j\le n$. It suffices to show that there is an α' such that the decision problem defined by $\Sigma\Pi\alpha$ can be reduced to that defined by $\Sigma(Qv/V)\Pi'\alpha'$. For a structure $\mathbb{M}$ we introduce the predicate RNG^1, such that $\mathrm{RNG}\in Q^M$ and the constant c such that $c\in\mathrm{RNG}$. If $Q=\forall$, we let $\mathrm{RNG}=M$. If $Q=\exists$, we let $\mathrm{RNG}=\{c\}$. Let $R(\mathbb{M})=(\mathbb{M},\mathrm{RNG},c)$.

The remainder of the proof falls apart in two.

Claim 4.3. *For every assignment team X with domain $\{u_1,\dots,u_i\}$,*

$$\mathbb{M},X\models\phi \quad \textit{if, and only if,} \quad R(\mathbb{M}),X\models(Qv/V)\big(\mathrm{RNG}(v)\wedge(v=c\to\phi')\big), \tag{45}$$

where

$$\phi = (Q_{i+1}u_{i+1}/U_{i+1})\dots(Q_nu_n/U_n)\alpha \tag{46}$$

$$\phi' = (Q_{i+1}u_{i+1}/U_{i+1}\cup W_{i+1})\dots(Q_nu_n/U_n\cup W_n)\alpha. \tag{47}$$

We can add Σ to the left of ϕ and ϕ' on account of Proposition 2.9, preserving equivalence.

Note that $\Sigma(Qv/V)\Pi'$ is not an IF(M) sentence, as it is not in Prenex normal form. The semantics in Definition 2.2 are sufficiently general to assign a truth value to this expression, though. Claim 4.4 addresses the issue of transforming it into an equivalent IF(M) sentence.

Claim 4.4. *There is an efficient (polynomial-time) and truth-preserving transformation from $\Sigma(Qv/V)\phi'$ into an* IF(M) *sentence with prefix $\Sigma(Qv/V)\Pi'$.*

Proof of Claim 4.3. Suppose $\mathbb{M},X\models\phi$. Let $f:X\to Q^M$ be the function that assigns RNG to every assignment in X, and write Y for $X[v,f]$. Evidently, $(\mathbb{M},\mathrm{RNG},c),Y\models\mathrm{RNG}(v)$. Let Y_1,Y_2 be the partition of Y, such that $Y_1=\{s\in X: s(v)=c\}$. Clearly we have $\mathbb{M},Y_2\models v\neq c$. ⊣

The assignment team Y_1 is constant in v. By induction, we can show that for $i+1\le j\le n$, the following are equivalent statements:

$$\mathbb{M},X\models(Q_ju_j/U_j)\dots(Q_nu_n/U_n)\alpha \tag{48}$$

$$\mathbb{M},Y_1\models(Q_ju_j/U_j\cup W_j)\dots(Q_nu_n/U_n\cup W_n)\alpha \tag{49}$$

$$\mathbb{M},Y_1\models(Q_ju_j/U_j\cup v)\dots(Q_nu_n/U_n\cup v)\alpha \tag{50}$$

$$\mathbb{M},Y_1\models\phi/v. \tag{51}$$

The case for $j=n+1$ is immediate, since the variable v does not appear in α. The inductive step follows from Proposition 2.8.

Statement (49) implies $\mathbb{M},Y_1\models\phi'$. Therefore,

$$(\mathbb{M},\mathrm{RNG},c),X\models(Qv/V)\big(\mathrm{RNG}(v)\wedge(v=c\to\phi')\big). \tag{52}$$

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For the converse direction, suppose

$$(\mathbb{M}, \mathrm{RNG}, c), X \models (Qv/V)\big(\mathrm{RNG}(v) \wedge (v = c \to \phi')\big). \tag{53}$$

There can be only one function $f : X \to Q^M$ for which

$$(\mathbb{M}, \mathrm{RNG}, c), X[v, f] \models \mathrm{RNG}(v) \wedge (v = c \to \phi'), \tag{54}$$

namely the function f that assigns RNG to every assignment in X. Write Y_1 for the same team of assignments as above. From (54) we derive that $\mathbb{M}, Y_1 \models \phi'$. We have that $\mathbb{M}, Y_1 \models \phi/v$ due to equivalences leading to (51). Since v does not occur in ϕ and Y_1 extends X, we have that $\mathbb{M}, Y_1 \models \phi/v$ if, and only if, $\mathbb{M}, X \models \phi$ on account of Proposition 2.6.

Proof of Claim 4.4. The sentence $\Sigma\phi'$ is not in the desired Prenix normal form. We can apply the quantifier extraction rule to pull all quantifiers to the left:

$$(Q_j u_j/U_j)\chi \circ \chi' \quad \Rightarrow \quad (Q_j u_j/U_j)(\chi \circ \chi'), \tag{55}$$

for any connective $\circ \in \{\vee, \wedge\}$. This manipulation is truth preserving (see [4]), since every variable in $(Qv/V)\phi'$ is quantified once and χ' does not contain the variable u_j. Indeed, each such χ' is a quantifier-free formula consisting of $\mathrm{RNG}(v)$ and $v = c$ as only building blocks. It can be seen that the resulting sentence has quantifier prefix $(Qv/V)\Pi'$. ⊣

The quantifier extraction procedure a number of steps that is linear in the number of quantifiers in ϕ'. ⊣

The dependence patterns in Definition 2.13 allow some quantifiers to assume the symbol $\forall$ or M and others to assume the symbol $\exists$ or M. The following lemma shows that, computationally speaking, the quantifier symbol options $\forall$ and $\exists$ are as hard as the majority option M.

Lemma 4.5. *Every sentential* IF(M) *prefixes that is NP-hard remains NP hard after replacing any quantifier symbol* $Q \in \{\exists, \forall\}$ *by* M.

Proof. Let $\phi = \Pi\alpha$ be an IF(M) sentence with $\Pi = (Q_1 u_1/U_1)\dots(Q_n u_n/U_n)$. We write ϕ_j for

$$(Q_j u_j/U_j)\dots(Q_n u_n/U_n)\alpha \tag{56}$$

and ϕ_{n+1} for α. Let Π' be the result of replacing Q_i in Π by M.

Let UNIV^1 and RNG^1 be new predicates and let c be a new constant symbol. Define $R(\phi)$ as follows:

$$R(\alpha) = \alpha, \tag{57}$$

$$R\big((\exists u_j/U_i)\phi_{j+1}\big) = \begin{cases} (\exists u_j/U_j)\ \big(\mathrm{UNIV}(u_j) \wedge R(\phi_{j+1})\big) & \text{if } j \neq i, \\ (\mathsf{M}u_j/U_j)\big(\big[\mathrm{UNIV}(u_j) \to R(\phi_{j+1})\big] \wedge u_j \neq c\big) & \text{if } j = i, \end{cases} \tag{58}$$

$$R\big((\forall u_j/U_i)\phi_{j+1}\big) = \begin{cases} (\forall u_j/U_j)\ \big(\mathrm{UNIV}(u_j) \to R(\phi_{j+1})\big) & \text{if } j \neq i, \\ (\mathsf{M}u_j/U_j)\big(\mathrm{UNIV}(u_j) \wedge R(\phi_{j+1})\big) & \text{if } j = i, \end{cases} \tag{59}$$

$$R\big((\mathsf{M}u_j/U_i)\phi_{j+1}\big) = (\mathsf{M}u_j/U_i)\psi_{j+1} \tag{60}$$

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where ψ_{j+1} is defined as

$$\big(\mathrm{UNIV}(u_j) \to R(\phi_{j+1})\big) \wedge \big(\neg\mathrm{UNIV}(u_j) \to \mathrm{RNG}(u_j)\big). \tag{61}$$

For a structure $\mathbb{M}$ with finite universe M let N be a set of fresh objects of size equal to the size of M. By $R(\mathbb{M})$ we denote the structure that is exactly like $\mathbb{M}$ except that its universe is $M \cup N$ and that it interprets the vocabulary of $R(\phi)$ that is foreign to ϕ in such a way that $c \in N$ and

$$\mathrm{UNIV} = M, \tag{62}$$

$$\mathrm{RNG} = \text{any subset of } N \text{ with } \lfloor m/2 \rfloor \text{ elements}, \tag{63}$$

where $m = |M|$.

The following two claims suffice to establish the result.

Claim 4.6. *For every structure* $\mathbb{M}$ *and assignment team* X *in* $\mathbb{M}$,

$$\mathbb{M}, X \models \phi_j \quad \textit{if, and only if,} \quad R(\mathbb{M}), X \models R(\phi_j). \tag{64}$$

Claim 4.7. *There is an efficient* (*polynomial-time*) *and truth-preserving transformation from* $R(\phi)$ *into an* $\mathrm{IF}(\mathsf{M})$ *sentence with prefix* Π'.

Proof of Claim 4.6. By induction on $1 \leq j \leq n+1$. The base case $j = n+1$ is immediate. ⊣

Case $Q_j = \mathsf{M}$. Assume

$$\mathbb{M}, X \models (\mathsf{M}u_j/U_j)\phi_{j+1}. \tag{65}$$

Then, for some U_j-uniform function $f : X \to \mathsf{M}^M$,

$$\mathbb{M}, Y \models \phi_{j+1}, \tag{66}$$

where $Y = X[u_j, f]$. Since Y is an assignment team in $\mathbb{M}$, by the induction hypothesis (64),

$$R(\mathbb{M}), Y \models R(\phi_{j+1}). \tag{67}$$

Obviously, by definition, $f(s) \in M = \mathrm{UNIV}$, for every $s \in X$. Therefore,

$$R(\mathbb{M}), Y \models \mathrm{UNIV}(u_j). \tag{68}$$

Let g be the function that sends every assignment $s \in X$ to $f(s) \cup \mathrm{RNG}$. Write Y' for $X[u_j, g] - Y$. Observe that $t(u_j) \in \mathrm{RNG}$ and $t(u_j) \notin \mathrm{UNIV}$, for every $t \in Y'$. Thus,

$$R(\mathbb{M}), Y' \models \mathrm{RNG}(u_j), \tag{69}$$

$$R(\mathbb{M}), Y' \models \neg\mathrm{UNIV}(u_j). \tag{70}$$

By (67) and (70),

$$R(\mathbb{M}), Y \cup Y' \models \neg\mathrm{UNIV}(u_j) \vee R(\phi_{j+1}). \tag{71}$$

By (68) and (69),

$$R(\mathbb{M}), Y \cup Y' \models \mathrm{UNIV}(u_j) \vee \mathrm{RNG}(u_j). \tag{72}$$

From (71) and (72), and the fact that $X[u_j, g] = Y \cup Y'$,

$$R(\mathbb{M}), X[u_j, g] \models \psi_{j+1}. \tag{73}$$

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Since $f(s) \in \mathsf{M}^M$, $|f(s)| \geq \lceil m/2 \rceil$ for every $s \in X$. In addition, we have that $f(s)$ and RNG are disjoint for every $s \in X$. Therefore, again for every $s \in X$,

$$|g(s)| = |f(s)| + |\mathrm{RNG}| \geq \lceil m/2 \rceil + \lfloor m/2 \rfloor = m. \tag{74}$$

It follows that $g(s) \in \mathsf{M}^{M \cup N}$. It is easy to see that U_j-uniformity of f implies U_j-uniformity of g.

Now that we have established that g is a U_j-uniform function of type $X \to \mathsf{M}^{M \cup N}$, it follows from (73) that

$$R(\mathbb{M}), X \models (\mathsf{M}u_j / U_j)\psi_{j+1}. \tag{75}$$

The converse direction is similar.

Case $Q_j = \exists$. The subcase $j \neq i$ is straightforward. We prove one direction. Assume

$$\mathbb{M}, X \models (\exists u_j / U_i)\phi_{j+1}, \tag{76}$$

in which case there exists a U_j-uniform function $f : X \to \exists^M$ such that

$$\mathbb{M}, Y \models \phi_{j+1}, \tag{77}$$

where $Y = X[u_j, f]$. By the induction hypothesis (64),

$$R(\mathbb{M}), Y \models R(\phi_{j+1}). \tag{78}$$

Define g as the function with domain X for which $g(s) = f(s) \cup \big(N - \{c\}\big)$. Let $Y' = X[u_j, g] - Y$. If $t \in Y'$, then $t(u_j) \notin M = \mathrm{UNIV}$. Therefore,

$$R(\mathbb{M}), Y' \models \neg \mathrm{UNIV}(u_j). \tag{79}$$

From (78) and (79) follows

$$R(\mathbb{M}), X[u_j, g] \models \neg \mathrm{UNIV}(u_j) \vee R(\phi_{j+1}). \tag{80}$$

Obviously, by definition of g,

$$R(\mathbb{M}), X[u_j, g] \models u_j \neq c. \tag{81}$$

Combining (80) and (81) yields

$$R(\mathbb{M}), X[u_j, g] \models \big(\neg \mathrm{UNIV}(u_j) \vee R(\phi_{j+1})\big) \wedge u_j \neq c. \tag{82}$$

By the fact that $f(s)$ is a singleton disjoint from N, $|g(s)|$ contains precisely m objects. Therefore, its co-domain is $\mathsf{M}^{M \cup N}$. It is easy to see that U_j-uniformity of f implies U_j-uniformity of g. Whence,

$$R(\mathbb{M}), X \models (\mathsf{M}u_j / U_j)\big[\big(\neg \mathrm{UNIV}(u_j) \vee R(\phi_{j+1})\big) \wedge u_j \neq c\big]. \tag{83}$$

Case $Q_j = \forall$. The subcase $j \neq i$ is straightforward. We prove one direction. Assume

$$R(\mathbb{M}), X \models (\mathsf{M}u_j / U_j)\big(\mathrm{UNIV}(u_j) \wedge R(\phi_{j+1})\big). \tag{84}$$

Then, for an appropriate function f,

$$R(\mathbb{M}), X[u_j, f] \models \mathrm{UNIV}(u_j) \wedge R(\phi_{j+1}). \tag{85}$$

The first conjunct enforces that $f(s) \subseteq \mathrm{UNIV} = M$, for every $s \in X$. It follows that $X[u_j, f]$ is an assignment team in $\mathbb{M}$. Then, by the second conjunct and the induction hypothesis (64),

$$\mathbb{M}, X[u_j, f] \models \phi_{j+1}. \tag{86}$$

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Since f's co-domain is $\mathsf{M}^{M\cup N}$, $|f(s)| \geq m$. We derived that $f(s) \subseteq M$. Since M contains m elements, it must be the case that $f(s) = M$, for every $s \in X$. So, f's co-domain can also be written as $\forall^M$, and therefore,

$$\mathbb{M}, X \models (\forall u_j / U_j)\phi_{j+1}. \tag{87}$$

Proof of Claim 4.7. Similar to proof of Claim 4.4 of Lemma 4.2. ⊣

The NP-hardness results are obtained by reduction from decision problems from graph theory, set theory and propositional logic. A *graph* is a structure (V, E) with vertices V and edges $E \subseteq V \times V$. We will introduce the decision problems before each claim, in the following format.

Problem: 3-COLORABILITY
Instance: A symmetric graph (V, E) without self loops.
Question: Does the graph admit a 3-coloring of its vertices such that no two adjacent vertices have the same color?

Theorem 4.8. *Every sentential Henkin* IF(M) *prefix is NP hard.*

Proof. It was shown that the IF sentence with prefix $\Pi = (2)$ and quantifier-free formula (4) expresses the NP-hard decision problem of 3-COLORABILITY.

The prefix Π is evidently equivalent to

$$\forall x_1 \exists y_1 (\forall x_2 / X_2)(\exists y_2 / x_1, y_1) \tag{88}$$
$$\forall x_1 \forall x_2 (\exists y_1 / x_2)(\exists y_2 / x_1, y_1) \tag{89}$$
$$\forall x_1 (\forall x_2 / x_1)(\exists y_1 / x_2)(\exists y_2 / x_1, y_1). \tag{90}$$

It can be shown that, modulo renaming of variables, every sentential Henkin prefix extends either prefix Π or one of the prefixes (88) with $X_2 \subseteq \{x_1, y_1\}$, (89) or (90). The claim follows from Lemma 4.2. ⊣

We use the following set-theoretical problem, see [9, p. 221].

Problem: EXACT COVER BY 3-SETS
Instance: A set U with $|U| = 3k$ and a collection C of 3-element subsets of U.
Question: Does C admit a subset of size k whose union coincides with U?

Theorem 4.9. *Every sentential signaling* IF(M) *prefix is NP hard.*

Proof. We will exploit the fact that $C \in$ EXACT COVER BY 3-SETS if, and only if, there are two functions $f : U \to \{1, \ldots, k\}$ and $f' : \{1, \ldots, k\} \to C$ such that $a \in f'\big(f(a)\big)$, for all $a \in U$.

For a given instance C of EXACT COVER BY 3-SETS, construct the structure $\mathbb{M}$ with universe $M = U \cup C$ that interprets the predicate U^1 as itself and furthermore:

$$\mathrm{CNT}^1 = \text{any subset of } M \text{ of size } k, \tag{91}$$
$$\mathrm{CONTAINS}^2 = \big\{(S, a) \colon S \in C, a \in S\big\}. \tag{92}$$

We assume the elements in CNT are named $\{1, \ldots, k\}$. Let ϕ be the IF sentence $\forall x \exists y (\exists z / x)\alpha$, where

$$\alpha = \Big(U(x) \to \big[\mathrm{CNT}(y) \wedge \mathrm{CONTAINS}(z, x)\big]\Big). \tag{93}$$

We show that $C \in$ EXACT COVER BY 3-SETS if, and only if, $\mathbb{M} \models \phi$. Suppose there are functions $f : U \to \{1, \ldots, k\}$ and $f' : \{1, \ldots, k\} \to C$ such that

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$a \in f'\big(f(a)\big)$, for all $a \in U$. Denote $[x, h]$ by X, where h is the function of type $\{\varnothing\} \to \forall^M$. Let g be a function $g : X \to \mathrm{CNT}$ such that for $s \in X$, $g(s) = f\big(s(x)\big)$. Further, let $g' : X[y, g] \to C$ be a function such that for $t \in X[y, g]$, $g'(s) = f'\big(t(y)\big)$. It is clear that g' is uniform in $\{x\}$. It can be checked that $X[y, g][z, g'] \models \alpha$.

For the converse direction, suppose $\mathbb{M} \models \phi$. As above, let h be the function of type $\{\varnothing\} \to \forall^M$ and write X for $[x, h]$. Let g and g' be functions such that $\mathbb{M}, Z \models \alpha$, where $Z = X[y, g][z, g']$ for some $g : X \to \exists^M$ and $\{x\}$-uniform $g' : X[y, f] \to \exists^M$.

We partition Z in two sets Z_1, Z_2, where Z_1 contains the assignments $s \in Z$ such that $s(x) \in U$. Then we must have

$$\mathbb{M}, Z_1 \models \mathrm{CNT}(y) \wedge \mathrm{CONTAINS}(z, x). \tag{94}$$

Let $f : U \to \{1, \ldots, k\}$ be the function such that $f(a) = g\big(s(x)\big)$, where s is the assignment that assigns a to x. The range of f is indeed $\{1, \ldots, k\}$ since all assignments in $s \in Z_1$ assign to y value $s(y) \in \mathrm{CNT}$. Further, we let $f' : \{1, \ldots, k\} \to C$ be the function such that $f'(i) = g'\big(t(y)\big)$, where t is the assignment that assigns i to y. The function f' is well-defined since g' is uniform in $\{x\}$. By construction of the binary predicate CONTAINS it follows that $a \in f'\big(f(a)\big)$, for all objects a that are assigned to x by the assignments in Z_1. Since $\{s(x) \colon s \in Z_1\}$ coincides with M, the claim follows. ⊣

A graph (V, E) is *bipartite* if we can partition its vertices V in two disjoint sets V_1 and V_2 such that $E \subseteq V_1 \times V_2$. The following graph problem is NP hard [9, p. 196].

Problem: BALANCED COMPLETE BIPARTITE GRAPH (BCBG)
Instance: An integer k and a bipartite graph (V, E).
Question: Are there two sets $V_1, V_2 \subseteq V$ both of size at least k such that $V_1 \times V_2 \subseteq E$?

We shall be interested in the following variant of BCBG.

Problem: HALF SEMI-CLIQUE
Instance: A graph (V, E).
Question: Are there two sets $V_1, V_2 \subseteq V$ both of size at least $|V|/2$ such that $V_1 \times V_2 \subseteq E$?

PROPOSITION 4.10. *HALF SEMI-CLIQUE is NP hard.*

PROOF. Consider a bipartite graph $G = (V, E)$ with $|V| = m$ vertices. Let V_1, V_2 be a partitioning of V such that $E \subseteq V_1 \times V_2$. We let $\hat{G} = (\hat{V}, \hat{E})$ be the graph with $\hat{V} = V \cup U$ and

$$U = \text{a set containing } d = m - 2k \text{ fresh elements}, \tag{95}$$

$$\hat{E} = E \cup (V_1 \times U) \cup (U \times U) \cup (U \times V_2). \tag{96}$$

Suppose $(k, G) \in \mathrm{BCBG}$. There are two sets $W_1, W_2 \subseteq V$ both of size at least k such that $W_1 \times W_2 \subseteq E$. Consider the sets $X_1 = W_1 \cup U$ and $X_2 = W_2 \cup U$. By construction of $\hat{E}$, $X_1 \times X_2 \subseteq \hat{E}$. X_1 and X_2 each contain at least $k + d = m - k$ elements. Since $\hat{G}$ contains $m + d = 2m - 2k$ elements, $\hat{G}$ sits in HALF SEMI-CLIQUE.

For the converse direction, suppose $\hat{G}$ has two sets of vertices $\hat{V}_1, \hat{V}_2$ both of size at least $m - k$ such that $\hat{V}_1 \times \hat{V}_2 \subseteq \hat{E}$. Let $W_1 = \hat{V}_1 - U$ and $W_2 = \hat{V}_2 - U$. Since

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$|U| = d$, W_1 and W_2 each contain at least $(m - k) - d = k$ elements. Clearly, $W_1 \times W_2 \subseteq E$ and the result follows. ⊣

The following result, showing that branching majority quantifiers are NP hard, appeared earlier in the author's PhD thesis [19, Theorem 5.6.2].

THEOREM 4.11 ([19]). *Every sentential branching* IF(M) *prefix is NP hard.*

PROOF. Let ϕ be the IF(M) sentence $\mathsf{M}x(\mathsf{M}y/x)E(x, y)$. It is clear that $G \in$ HALF SEMI-CLIQUE if, and only if, $G \models \phi$. ⊣

A (*propositional*) *atom* is a Boolean variable, which takes the value 0 or 1. A *literal* is an atom with zero or more negations $\neg$ in front of it. A *clause* is a set of literals. An *assignment* T is a function that assigns to each atom 0 or 1. We extend T to literals by defining $T(\neg L) = 1 - T(L)$ for every literal L. A clause is *satisfied* (respectively, *validated*) by an assignment if it assigns 1 to one (respectively, all) of its literals.

The following problem is NP complete even for $n = 2$, see [9, pp. 259–260].

Problem: MAX-n-SAT
Instance: An integer k and a collection C of k or more clauses each containing n literals.
Question: Does C admit an assignment of its variables that satisfies at least k of its clauses?

We will use the following variant.

Problem: MAX-n/n-SAT
Instance: An integer k and a collection C of k or more clauses each containing n literals.
Question: Does C admit an assignment of its variables that validates at least k of its clauses?

PROPOSITION 4.12. *MAX-2/2-SAT is NP complete.*

PROOF. For a clause $\alpha = \{L, L'\}$, let $\hat{\alpha}$ denote $\{\alpha, \{\neg L, L'\}, \{L, \neg L'\}\}$. For a set of clauses C let $\hat{C}$ denote $\bigcup_{\alpha \in C} \hat{\alpha}$.

It suffices to prove that $(k, C) \in$ MAX-2-SAT if, and only if, $(k, \hat{C}) \in$ MAX-2/2-SAT. The from-left-to-right direction is immediate from the observation that every assignment that satisfies α validates one of the clauses in $\hat{\alpha}$. As for the converse direction, observe that every assignment that validates one the clauses in $\hat{\alpha}$, satisfies α. In addition, we note that there can be no assignment that validates more than one clause in $\hat{\alpha}$. Hence, if an assignment validates at least k clauses in $\hat{C}$, there are at least k sets $\hat{\alpha}$ in which it validates exactly one clause. But then, the assignment satisfies at least k clauses α in C. ⊣

THEOREM 4.13. *Every sentential* IF(M) *prefix that is* $\forall\exists/\mathsf{M}$ *is NP hard.*

PROOF. Let (k, C) be an instance of MAX-2/2-SAT with m clauses and n atoms $\{A_1, \ldots, A_n\}$. We can assume without loss of generality that $n \geq m$ and that $n + m$ is even. Let $\mathbb{M}$ be a structure with universe $M = \mathrm{CL} \cup \mathrm{ATOM}$ that interprets the constant symbols 0 and 1 as distinct objects and furthermore interprets the

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following symbols:

$$\mathrm{CL}^1 = \text{the set of clauses, } C \tag{97}$$

$$\mathrm{ATOM}^1 = \text{the set of atoms in } C, \{A_1, \ldots, A_n\} \tag{98}$$

$$\mathrm{RNG}^1 = \text{any set such that } \mathrm{CL} \subseteq \mathrm{RNG} \subseteq M \text{ and } |\mathrm{RNG} \cap \mathrm{ATOM}| = (m+n)/2 - k \tag{99}$$

$$\mathrm{POSITIVE}^2 = \{(\alpha, A) \colon A \in \alpha\} \tag{100}$$

$$\mathrm{NEGATIVE}^2 = \{(\alpha, A) \colon \neg A \in \alpha\} \tag{101}$$

Construction of RNG requires that $(m+n)/2 - k \leq n$. In the extreme case that $k = 0$, we must thus have $n \geq m$, which can be assumed without loss of generality.

Let ϕ denote $\mathsf{M}x\forall y(\exists z/x)\alpha$, where

$$\alpha = \Big(\mathrm{RNG}(x) \wedge \big[\big(\mathrm{CL}(x) \wedge \mathrm{ATOM}(y)\big) \rightarrow \big(\mathrm{POSITIVE}(x,y) \rightarrow z = 1 \wedge \mathrm{NEGATIVE}(x,y) \rightarrow z = 0\big)\big]\Big). \tag{102}$$

We show that $(k, C) \in$ MAX-2/2-SAT if, and only if, $\mathbb{M} \models \phi$.

Suppose there is an assignment T that validates at least k clauses in C. Let $B \subseteq C$ be the set of clauses that is validated by T and suppose it contains $k' \geq k$ clauses. We add elements to B from $\mathrm{RNG} \cap \mathrm{ATOM}$ until it contains $(m+n)/2$ elements. There are sufficient elements in $\mathrm{RNG} \cap \mathrm{ATOM}$ to do so, since $|\mathrm{RNG} \cap \mathrm{ATOM}| = (m+n)/2 - k$.

Let f be the function with domain $\{\varnothing\}$ such that $f(\varnothing) = B$. By construction, B contains precisely half of the objects in M. Therefore, f is of type $\{\varnothing\} \rightarrow \mathsf{M}^M$.

Write X for $\{\varnothing\}[x, f]$ and Y for $X[y, g]$, where g is the function of type $X \rightarrow \forall^M$. Consider the function $h : Y \rightarrow \{0, 1\}$ defined by

$$h(s) = \begin{cases} T\big(s(y)\big) & \text{if } s(y) \text{ is an atom} \\ 0 & \text{otherwise.} \end{cases} \tag{103}$$

The object 0 is arbitrary and can be replaced by any other object from M. Observe that h is uniform in $\{x\}$.

Write Z for $Y[z, h]$. We proceed to show that $\mathbb{M}, Z \models \alpha$. Since $f(\varnothing) \subseteq \mathrm{RNG}$, the left-hand conjunct in α is satisfied. We split Z in two: Z_1 contains all assignments s that assign a nonclause object to x or a nonvariable to y; Z_2 contains the remaining assignments, that is, $Z_2 = Z - Z_1$. It is evident that

$$\mathbb{M}, Z_2 \models \mathrm{CL}(x) \wedge \mathrm{ATOM}(y). \tag{104}$$

The claim thus follows if we can show that

$$\mathbb{M}, Z_2 \models \mathrm{POSITIVE}(x,y) \rightarrow z = 1 \tag{105}$$

and

$$\mathbb{M}, Z_2 \models \mathrm{NEGATIVE}(x,y) \rightarrow z = 0. \tag{106}$$

As for (105), suppose s is an assignment in Z_2 such that $\big(s(x), s(y)\big) \in$ POSITIVE, that is, $s(x)$ is a clause α and $s(y)$ is an atom A such that $A \in \alpha$. All objects in B

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that are clauses are validated by T. Hence, $s(x)$ is a clause that is validated by T. It follows that $T(A) = 1$, and therefore, by construction of h, $s(z) = 1$.

The case for (106) follows similarly.

Suppose $\mathbb{M} \models \phi$. Let f, g, h be appropriate functions such that $\mathbb{M}, [x, f][y, g][z, h] \models \alpha$, that is, f is a function of type $\{\varnothing\} \to \mathsf{M}^M$, g is a function of type $X \to \forall^M$ with domain $X = [x, f]$, and h is a function of type $Y \to \exists^M$ with domain $Y = X[y, g]$. Write Z for $Y[z, h]$.

As above, we designate $f(\varnothing)$ by B. Since $\mathbb{M}, Z \models \alpha$, $B \subseteq \mathrm{RNG}$. Also, since f has range M^M, B contains at least $m + n/2$ elements. The intersection of RNG and ATOM contains $(m + n)/2 - k$ elements. It follows that the intersection of B and CL contains at least k elements.

From the fact that

$$\mathbb{M}, Z \models \big(\mathrm{CL}(x) \wedge \mathrm{ATOM}(y)\big) \to \big(\mathrm{POSITIVE}(x, y) \to z = 1 \wedge \mathrm{NEGATIVE}(x, y) \to z = 0\big) \tag{107}$$

it follows that there is a partition Z_1, Z_2 of Z for which

$$\mathbb{M}, Z_1 \models \neg\mathrm{CL}(x) \vee \neg\mathrm{ATOM}(y) \tag{108}$$

and

$$\mathbb{M}, Z_2 \models \mathrm{POSITIVE}(x, y) \to z = 1 \wedge \mathrm{NEGATIVE}(x, y) \to z = 0. \tag{109}$$

The partitioning Z_1, Z_2 is identical to the one above: Z_1 contains all assignments s that assign a nonclause object to x or a nonvariable to y; Z_2 contains the remaining assignments.

Define the assignment T as the function that sends every atom A to $s(z)$, where s is any assignment in Z_2 with $s(y) = A$. Since h is uniform in $\{x\}$, it assigns the same value to every pair of assignments s and s' in Z_2 that agree with each other on y. Thus we see that T is well defined.

Consider any two assignments $s, s' \in Z_2$ that assign to x a clause $s(x) = s'(x) = \alpha$ and that assign distinct atoms to y. Suppose $s(y) = A$ and $s'(y) = A'$ appear unnegated in α. Then, by the left-hand conjunct in (109) we derive that $s(z) = s'(z) = 1$, implying that T assigns 1 to both A and A'. It follows that T validates α. In a similar vein it can be shown that T validates α also if $\alpha \in \big\{\{A, \neg A'\}, \{\neg A, A'\}, \{\neg A, \neg A'\}\big\}$.

There are at least k distinct clauses α referred to by the assignments in Z_2. It follows that T validates at least k clauses. ⊣

The previous complexity results can be combined to prove this section's main result.

THEOREM 4.14. *Every nonprimary* IF(M) *prefix is NP hard.*

PROOF. A nonprimary IF prefix is either Henkin, signaling, branching or $\forall\exists$/M. By respectively Theorems 4.8, 4.9, 4.11, and 4.13 any such prefix is NP hard. ⊣

§5. Conclusion. The following dichotomy, which divides IF(M) in "hard" and "easy" prefixes, is this paper's main result.

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THEOREM 5.1. *Every sentential* IF(M) *prefix is either NP hard or equivalent to a sentential* FOL(M) *prefix* (*and therefore in LOGSPACE*).

PROOF. Immediate from Theorems 3.4 and 4.14. It is known that every first-order expression with majority quantifiers can be computed in LOGSPACE [14]. It follows that every FOL(M) prefix is computable in LOGSPACE. ⊣

The analogous result by Gottlob, Koliatis and Schwentick [10] showed that on graphs every quantifier prefix of existential second-order logic is either NP hard or computable in P. Our results are established on arbitrary structures. It is conceivable that on graphs there are nonprimary IF(M) prefixes that are neither NP hard nor have an equivalent in FOL(M).

Although we did not use this observation in the technical sections, it may be of interest to note that the majority quantifier, which is not definable in first-order logic, seems to behave like a combined universal-existential quantifier. If we replace, namely, in the Henkin quantifier (1) a block of quantifiers $\forall x_i \exists y_i$ by $\mathsf{M} z_i$ the resulting prefix is either $\forall\exists/\mathsf{M}$:

$$\begin{pmatrix} \forall x_1 \exists y_1 \\ \mathsf{M} z_2 \end{pmatrix}, \tag{110}$$

or branching:

$$\begin{pmatrix} \mathsf{M} z_1 \\ \mathsf{M} z_2 \end{pmatrix}. \tag{111}$$

These two constructions define NP-hard IF(M) prefixes.

Games of imperfect information are one of the prime motivations for independence-friendly logic [12, 16]. According to this interpretation, the existential and universal quantifiers are "controlled" or "played" by the players Eloise and Abelard, respectively. In the literature, it is customary to ascribe epistemic states to these players at each quantifier decision point they control, in terms of the variables they "know." For instance in the signaling prefix

$$\forall x \exists y (\exists z / x) \tag{112}$$

Eloise knows x at the control point associated with $\exists y$, and knows y at the control point associated with $(\exists z/x)$ but not x.

In the game-theoretic literature [17], the notion of perfect recall is used to describe games in which no player forgets information he or she once knew (knowledge memory) and in which each player memorizes his or her own moves (action recall). In the context of IF prefixes, information available to a player equates to the variables he knows when playing one of his or her quantifiers. Thus, prefix (112) has action recall but not knowledge memory.

As another example, the prefix

$$\forall x_1 \exists y_1 (\forall x_2 / X_2)(\exists y_2 / x_1, y_1) \tag{113}$$

appearing earlier as (88), violates both action recall and knowledge memory. At $(\exists y_2/x_1, y_1)$, Eloise has forgotten her own move y_1 as well as Abelard's move x_1, which she knew at $\exists y_1$.

A result established for arbitrary IF logic [16, 19] holds that every sentential IF prefix with perfect recall is equivalent to a sentential FOL prefix. It was later shown

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by Barbero [1, Theorem 7.10] that knowledge memory is the critical criterion, that is, again confined to prefixes, every sentential IF prefix with knowledge memory is equivalent to a sentential FOL prefix.

Knowledge memory does not characterize the IF prefixes that have an equivalent FOL prefixes though. There are IF prefixes without knowledge memory that are still equivalent to FOL prefixes, such as, for instance, the prefix

$$\forall x \exists y (\exists z / x, y), \tag{114}$$

which also violates action recall. Swapping the slashed quantifier twice yields an equivalent FOL prefix

$$\exists z \forall x \exists y. \tag{115}$$

It is readily seen that if an IF prefix has knowledge memory, it cannot be signaling or Henkin. The converse does not hold: (114) does not have knowledge memory, but it is neither signaling nor Henkin. In fact, it follows from our results that being signaling and/or Henkin is a necessary and sufficient set of conditions for having an equivalent FOL prefix.

THEOREM 5.2. *Every sentential* IF *prefix is primary if, and only if, it is equivalent to a* FOL *prefix*.

PROOF. The if-direction is Theorem 3.2. If an IF prefix is not primary, then it is either Henkin or signaling. From Theorems 4.8 and 4.9 it follows that the prefix can express an NP hard property, which cannot be done in first-order logic [14]. ⊣

For the theory of independence-friendly logic, it may be interesting to research if Theorem 5.2 can be extended to the general case, i.e., to the logical language that is the result of closing IF prefixes under connectives.

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