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A PROOF OF XIN–ZHANG'S TRIDIAGONAL DETERMINANT CONJECTURE

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Abstract

We confirm a recent conjecture of Xin and Zhang ['A variation of the Morris constant term', Preprint, 2024, arXiv:2409.14356v2], which establishes a simple product formula for the characteristic polynomial of an $(n-1)\times(n-1)$ tridiagonal matrix. This characteristic polynomial arises from a recurrence relation that enumerates $n\times n$ nonnegative integer matrices with all row and column sums equal to t, also called the Ehrhart polynomial of the nth Birkhoff polytope.

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1. Introduction

Our main objective is to prove a conjecture of Xin and Zhang [13], which gives a product formula for the characteristic polynomial of a certain tridiagonal matrix.

CONJECTURE 1.1 [13, Conjecture 4.8]. Let n be a positive integer and let C be the $(n-1)\times(n-1)$ matrix defined by

$$C_{i,j}=\begin{cases} t-(n-2i+2)(n-1-i)-1 & \text{if } i=j,\\ (i-1)(n-i) & \text{if } i=j+1,\\ -(n-1-i)(n-i) & \text{if } i=j-1,\\ 0 & \text{otherwise.}\end{cases}$$

Then, $\det C$ is equal to

$$\begin{cases}(t-n+1)\prod_{i=0}^{r-2}(t-n+1-r(r-1)+i(i+1))^2 & \text{if } n=2r,\\ (t-n+1)(t-n+1-(r-1)^2)\prod_{i=1}^{r-2}(t-n+1-(r-1)^2+i^2)^2 & \text{if } n=2r-1.\end{cases}$$

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This conjecture arises in the study of the Ehrhart polynomial $H_n(t)$ of the nth Birkhoff polytope, where $H_n(t)$ counts the number of $n \times n$ nonnegative integer matrices with all row and column sums equal to t. The computation of $H_n(t)$ has been a subject of extensive research (see, for example, [1, 5, 6, 8–11, 14]). So far, $H_n(t)$ is known only for $n \le 9$, with $H_9(t)$ computed by Beck and Pixton [3] using residue techniques.

From [3, Corollary 4],

$$H_n(t) = \sum_{\boldsymbol{m}} \binom{n}{\boldsymbol{m}} \operatorname*{CT}_{\boldsymbol{x}} \mathcal{H}^{\boldsymbol{m}}(t),$$

where the sum is over all nonnegative integer sequences $\boldsymbol{m} = (m_1, m_2, \ldots, m_n)$ such that $\sum_{i=1}^{n} m_i = n$ and $\sum_{i=1}^{r} m_i > r$ for $1 \le r < n$. Here, $\operatorname*{CT}_{\boldsymbol{x}} f$ denotes the constant term of the Laurent series f in $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and

$$\mathcal{H}^{\boldsymbol{m}}(t) := \prod_{i=1}^{n} \frac{x_i^{(m_i-1)t}}{\prod_{j=1, j\neq i}^{n} (1 - x_j/x_i)^{m_i}}.$$

In particular, for $\boldsymbol{m} = (2, 1^{n-2}, 0)$, we define

$$\begin{aligned} h_n(t) &:= (-1)^{((n-1)(n-2))/2} \operatorname*{CT}_{\boldsymbol{x}} \mathcal{H}^{(2,1^{n-2},0)}(t) \\ &= \operatorname*{CT}_{\boldsymbol{x}} \frac{(x_n/x_1)^{-t} \prod_{i=2}^{n-1} (x_i/x_1)^{2i-n} \prod_{i=1}^{n-1} (1 - x_n/x_i)}{\prod_{i=2}^{n} (1 - x_i/x_1)^3 \prod_{2 \le i < j \le n} (1 - x_j/x_i)^2}. \end{aligned}$$

Noting the similarity with the Morris constant term identity studied in [2, 12], Xin and Zhang [13] derived a recurrence for $h_n(t)$ of the form

$$h_n(t) = -\frac{1}{c_0(t+n-1)} \sum_{i=1}^{n-1} c_i(t+n-1) h_n(t-i), \tag{1.1}$$

in which $c_0(t) = \det C$. Therefore, if Conjecture 1.1 holds, then (1.1) indeed yields a recurrence when

$$t > \left\lfloor \frac{(n-1)^2}{4} \right\rfloor = \begin{cases} r(r-1) & \text{if } n = 2r, \\ (r-1)^2 & \text{if } n = 2r-1. \end{cases}$$

The ideas in [13] also apply to $\operatorname*{CT}_{\boldsymbol{x}} \mathcal{H}^{(k,1^{n-k},0^{k-1})}(t)$ for $3 \le k \le n-1$. Moreover, the determinant $\det C$ arises in a similar but more complicated recurrence. These aspects will be elaborated on in an upcoming paper [15].

Further developments motivate us to consider the equivalent determinant of

$$tI - \widetilde{C}^{(n-1)} = C + (n-1)I,$$

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where I is the identity matrix and $\widetilde{C}^{(n)}$ is an $n \times n$ tridiagonal matrix with entries

$$(\widetilde{C}^{(n)})_{i,j} = \begin{cases} (n+1-i)(n+1-2i) & \text{if } i = j, \\ (n+1-i)(1-i) & \text{if } i = j+1, \\ (n+1-i)(n-i) & \text{if } i = j-1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

It is hard to determine the characteristic polynomial of $\widetilde{C}^{(n)}$: none of the minors of $tI - \widetilde{C}^{(n)}$ admits a simple factorisation. This prevents us from proving the conjecture by traditional techniques such as diagonalisation, LU decompositions, continued fractions and so on. Ultimately, we discovered that $\widetilde{C}^{(n)}$ is similar to a lower triangular matrix. It is noteworthy that the similar matrix is a variation of Pascal's triangle. We state this result as the following theorem.

THEOREM 1.2. *For a positive integer n, let $\widetilde{C}^{(n)}$ be the matrix defined in (1.2). There exists an $n \times n$ nonsingular upper triangular matrix U with entries*

$$U_{i,j} = \binom{n-i}{n-j}, \tag{1.3}$$

such that $U\widetilde{C}^{(n)}U^{-1}$ is a lower triangular matrix whose entries are given by

$$(U\widetilde{C}^{(n)}U^{-1})_{i,j} = \begin{cases} i(n-i) & \text{if } i = j, \\ (n+1-i)(1-i) & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, Conjecture 1.1 holds and we have the nice formula

$$\det C = \prod_{i=1}^{n-1} (t - n + 1 - i(n-1-i)).$$

For instance,

$$\widetilde{C}^{(4)} = \begin{pmatrix} 12 & 12 & 0 & 0 \\ -3 & 3 & 6 & 0 \\ 0 & -4 & -2 & 2 \\ 0 & 0 & -3 & -3 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U\widetilde{C}^{(4)}U^{-1} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ 0 & -4 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}.$$

In Section 2, we prove Theorem 1.2 using an explicit formula for U^{-1}, together with several combinatorial identities. Section 3 contains concluding remarks.

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2. The proof of Theorem 1.2

Our proof of Theorem 1.2 is based on the following well-known combinatorial identities (the proofs are omitted):

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \tag{2.1}$$

$$n\binom{n-1}{k-1} = k\binom{n}{k} = (n-k+1)\binom{n}{k-1}, \tag{2.2}$$

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}, \tag{2.3}$$

where $n \geq k \geq j$ are nonnegative integers.

PROOF OF THEOREM 1.2. By [4, Theorem 1], the entries of U^{-1} are given by

$$(U^{-1})_{i,j} = (-1)^{i+j}U_{i,j} = (-1)^{i+j}\binom{n-i}{n-j}.$$

Consider the (i,j)-entry of $U\widetilde{C}^{(n)}U^{-1}$:

$$\begin{aligned}(U\widetilde{C}^{(n)}U^{-1})_{i,j} &= \sum_{1\leq l,m\leq n} U_{i,l}(\widetilde{C}^{(n)})_{l,m}(U^{-1})_{m,j} \\ &= \sum_{l=1}^{n} U_{i,l}(\widetilde{C}^{(n)})_{l,l}(U^{-1})_{l,j} + \sum_{l=2}^{n} U_{i,l}(\widetilde{C}^{(n)})_{l,l-1}(U^{-1})_{l-1,j} \\ &\quad + \sum_{l=1}^{n-1} U_{i,l}(\widetilde{C}^{(n)})_{l,l+1}(U^{-1})_{l+1,j},\end{aligned}$$

where the second equality holds because $(\widetilde{C}^{(n)})_{l,m} = 0$ when $|l-m| > 1$. Observe that $(\widetilde{C}^{(n)})_{l,l} = (\widetilde{C}^{(n)})_{l,l-1} + (\widetilde{C}^{(n)})_{l,l+1}$, and the formulae for $(\widetilde{C}^{(n)})_{l,l-1}$ and $(\widetilde{C}^{(n)})_{l,l+1}$ vanish when $l = 1$ and $l = n$, respectively. The computation is naturally divided into two parts, denoted by S_1 and S_2. In detail,

$$\begin{aligned}S_1 &= \sum_{l=1}^{n} U_{i,l}(\widetilde{C}^{(n)})_{l,l-1}(U^{-1})_{l,j} + \sum_{l=2}^{n} U_{i,l}(\widetilde{C}^{(n)})_{l,l-1}(U^{-1})_{l-1,j} \\ &= \sum_{l=1}^{n} U_{i,l}(\widetilde{C}^{(n)})_{l,l-1}((U^{-1})_{l,j} + (U^{-1})_{l-1,j}) \\ &= \sum_{l=1}^{n} U_{i,l}(\widetilde{C}^{(n)})_{l,l-1}(-1)^{j+l-1}\left(\binom{n-l+1}{n-j} - \binom{n-l}{n-j}\right) \\ &= \sum_{l=1}^{n} U_{i,l}(\widetilde{C}^{(n)})_{l,l-1}(-1)^{j+l-1}\binom{n-l}{n-j-1} \qquad \text{(by (2.1))}.\end{aligned}$$

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Similarly,

$$\begin{aligned} S_2 &= \sum_{l=1}^{n} U_{i,l}(\widetilde{C}^{(n)})_{l,l+1}(U^{-1})_{l,j} + \sum_{l=1}^{n-1} U_{i,l}(\widetilde{C}^{(n)})_{l,l+1}(U^{-1})_{l+1,j} \\ &= \sum_{l=1}^{n} U_{i,l}(\widetilde{C}^{(n)})_{l,l+1}(-1)^{j+l}\binom{n-l-1}{n-j-1}. \end{aligned}$$

Using (2.2), we can further compute S_2:

$$\begin{aligned} S_2 &= \sum_{l=1}^{n} U_{i,l}(-1)^{j+l}(n+1-l)(n-l)\binom{n-l-1}{n-j-1} \\ &= \sum_{l=1}^{n} U_{i,l}(-1)^{j+l}(n+1-l)(j-l+1)\binom{n-l}{n-j-1}. \end{aligned}$$

Then,

$$\begin{aligned} (U\widetilde{C}^{(n)}U^{-1})_{i,j} = S_1 + S_2 &= j\sum_{l=1}^{n}(-1)^{j+l}\binom{n-i}{n-l}\binom{n-l}{n-j-1}(n+1-l) \\ &= j\sum_{l=1}^{n}(-1)^{j+l}\binom{n-i}{n-j-1}\binom{j+1-i}{j+1-l}(n+1-l) \qquad \text{(by (2.3))}. \end{aligned}$$

It is clear that the above formula equals 0 when $i > j+1$ and equals $(1-i)(n+1-i)$ when $i = j+1$. When $i \le j$, we can further write

$$\begin{aligned} (U\widetilde{C}^{(n)}U^{-1})_{i,j} &= j\sum_{l=i}^{j+1}(-1)^{j+l}\binom{n-i}{n-j-1}\binom{j+1-i}{j+1-l}((n-j)+(j+1-l)) \\ &= -j(n-j)\binom{n-i}{n-j-1}\sum_{l=i}^{j+1}\binom{j+1-i}{j+1-l}(-1)^{j+1-l} \\ &\quad + j(j+1-i)\binom{n-i}{n-j-1}\sum_{l=i}^{j}\binom{j-i}{j-l}(-1)^{j-l} \qquad \text{(by (2.2))} \\ &= \binom{n-i}{n-j-1}(-j(n-j)(1+(-1))^{j+1-i} + j(j+1-i)(1+(-1))^{j-i}) \\ &= \begin{cases} i(n-i) & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases} \end{aligned}$$

This completes the proof. □

As a byproduct, we have the following result.

PROPOSITION 2.1. *Let U be the matrix with entries as in (1.3) and M be a matrix satisfying $M_{i,j} = 0$ whenever $i - j \ge 2$. Then, $(UMU^{-1})_{i,j} = M_{i,j}$ whenever $i - j \ge 1$.*

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Proposition 2.1 can be visualised as

$$U\begin{pmatrix} * & * & * & \cdots & * \\ M_{2,1} & * & * & \cdots & * \\ 0 & M_{3,2} & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{n,n-1} & * \end{pmatrix} U^{-1} = \begin{pmatrix} \triangle & \triangle & \triangle & \cdots & \triangle \\ M_{2,1} & \triangle & \triangle & \cdots & \triangle \\ 0 & M_{3,2} & \triangle & \cdots & \triangle \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{n,n-1} & \triangle \end{pmatrix},$$

where '$*$' and '$\triangle$' stand for unspecified entries.

PROOF OF PROPOSITION 2.1. Consider the (i,j)-entry of UMU^{-1}:

$$(UMU^{-1})_{i,j} = \sum_{1\le l,m\le n} U_{i,l}M_{l,m}(U^{-1})_{m,j} = \sum_{1\le l,m\le n} (-1)^{m+j}\binom{n-i}{n-l}\binom{n-m}{n-j}M_{l,m}.$$

For fixed i and j, we have $\binom{n-i}{n-l}\binom{n-m}{n-j} = 0$ unless $l \ge i$ and $m \le j$.

In the case where $i - j \ge 2$, the condition $l \ge i$ and $m \le j$ implies that $l - m \ge i - j \ge 2$. Hence, $M_{l,m} = 0$ and so $(UMU^{-1})_{i,j} = 0 = M_{i,j}$.

In the case where $i - j = 1$,

$$(-1)^{m+i-1}\binom{n-i}{n-l}\binom{n-m}{n-i+1}M_{l,m} = \begin{cases} M_{i,i-1} & \text{if } l = i \text{ and } m = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $(UMU^{-1})_{i,i-1} = M_{i,i-1}$. This completes the proof. □

3. Concluding remarks

The observation $(\widetilde{C}^{(n)})_{i,i} = (\widetilde{C}^{(n)})_{i,i-1} + (\widetilde{C}^{(n)})_{i,i+1}$ leads us to consider a matrix A satisfying $A_{i,i} = r + A_{i,i-1} + A_{i,i+1}$, where r is a parameter. Further exploration gives a generalisation of Theorem 1.2, which requires a cumbersome condition. We elaborate on this generalisation in the extended version [7, Section 3].

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