Math. Proc. Camb. Phil. Soc. (1999), **127**, 55

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A splitting for the stable mapping class group

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(*Received* 23 *December* 1997; *revised* 23 *March* 1998)

Abstract

The main result of [**15**] is that the classifying space of the stable mapping class group after plus construction $B\Gamma_\infty^+$ is an infinite loop space. This result is used to show that, localized away from two, a connected component of the stable homotopy groups of spheres QS^0 splits off $B\Gamma_\infty^+$. The splitting is a splitting of infinite loop spaces. It follows immediately that the homology with coefficients in $\mathbb{Z}[\frac{1}{2}]$ of the infinite symmetric group is a direct summand of the homology of the stable mapping class group.

Introduction

Let $F_{g,1}$ be a surface of genus g with one boundary component. By definition, the mapping class group $\Gamma_{g,1}$ is the group of connected components $\pi_0 \mathrm{Diff}^+(F_{g,1}; \partial)$, where $\mathrm{Diff}^+(F_{g,1}; \partial)$ is the topological group of orientation preserving diffeomorphisms that fix the boundary pointwise. Its classifying space $B\Gamma_{g,1}$ is homotopy equivalent to the moduli space of Riemann surfaces of topological type $F_{g,1}$. By attaching a torus with two boundary components to $F_{g,1}$ and extending diffeomorphisms by the identity, one can define a homomorphism $\Gamma_{g,1} \to \Gamma_{g+1,1}$. A theorem of Harer [**7**] as improved by Ivanov [**8**] says that this map induces an isomorphism of homology groups in dimensions less than $g/2$.

Here we study the classifying space of the stable mapping class group $\Gamma_\infty := \lim_{g\to\infty} \Gamma_{g,1}$. More precisely, we will consider the group completion of its classifying space, $B\Gamma_\infty^+$, which has the same homology as the stable mapping class group Γ_∞ and a rich topological structure. In particular, it is an infinite loop space [**15**]. The main result to be proved here is the following theorem. Let $QS^0 := \lim_{n\to\infty} \Omega^n S^n$ be the infinite loop space representing stable homotopy and let $QS^0_{(\frac{1}{2})}$ denote its localization away from two.

Theorem C. *There is a splitting of infinite loop spaces* $\mathbb{Z} \times B\Gamma_\infty^+ \simeq QS^0_{(\frac{1}{2})} \times Z$.

By the solution of the Adam's conjecture, there is a splitting of spaces $QS^0 \simeq \mathrm{Im}\, J \times \mathrm{Coker}\, J$ where $\mathrm{Im}\, J$ denotes the image of the classical J-homomorphism from the space of orthogonal real matrices to QS^0. The analogue of Theorem C for $\mathrm{Im}\, J$ was proved in [**15**] as a corollary of the stable splitting theorem of Charney and Cohen [**2**]. The proof here is more direct.

Recall that by the theorem of Barratt-Priddy, Quillen and Segal, a connected

† Partially supported by the DFG.

https://doi.org/10.1017/S0305004199003485 Published online by Cambridge University Press

component of QS^0 has the same homology as the infinite symmetric group Σ_∞. The corollary below is thus an immediate consequence of Theorem C. The homology classes corresponding to the factor $\operatorname{Im} J$ are well-known already by work of Charney and Lee [**3**] while those corresponding to $\operatorname{Coker} J$ seem to be torsion classes that have not been detected before.

COROLLARY. *$H_*(\Sigma_\infty; \mathbb{Z}[\frac{1}{2}])$ is a direct summand of $H_*(\Gamma_\infty; \mathbb{Z}[\frac{1}{2}])$.*

The main step in the proof of Theorem C is the construction of a map of infinite loop spaces $\alpha: \mathbb{Z} \times B\Gamma_\infty^+ \to A(*)$ to the Waldhausen K-theory of a point. $A(*)$ can be thought of as the algebraic K-theory of the ring up to homotopy QS^0. The ring homomorphism $\pi_0: QS^0 \to \mathbb{Z}$ defines a map $L: A(*) \to K(\mathbb{Z})$ to the algebraic K-theory of the integers. On the other hand, the classical representation of the mapping class groups on the first homology of the underlying surfaces induces a well-known map $\tau: \mathbb{Z} \times B\Gamma_\infty^+ \to K(\mathbb{Z})$ which is a map of infinite loop spaces [**15**]; α is a natural lift of this map up to a sign:

$$\begin{array}{ccc} & \overset{(-1)\alpha}{\nearrow} & A(*) \\ \mathbb{Z} \times B\Gamma_\infty^+ & & \downarrow L \\ & \underset{\tau}{\searrow} & K(\mathbb{Z}). \end{array}$$

By a fundamental theorem of Waldhausen, $A(*)$ splits as the product of QS^0 and the smooth Whitehead space $\mathrm{Wh}(*)$ of a point. In particular, the splitting map $s: A(*) \to QS^0$ is a map of infinite loop spaces and hence so is $s \circ \alpha: \mathbb{Z} \times B\Gamma_\infty^+ \to QS^0$. Theorem C now follows from the general theory of infinite loop spaces because α on components maps 1 to 2. Recent work of Dwyer, Weiss and Williams [**4**] implies that α actually factors through QS^0.

In Section 1, we reprove the main results of [**15**]. We do this because we need a description of the cobordism category $\mathscr{S}$ for the definition of α. Furthermore, it is important that we consider the diffeomorphism groups rather than the mapping class groups of surfaces. The construction of $\mathscr{S}$ here is thus more general but also simpler than that in [**15**]. Section 2 reviews the two definitions of Waldhausen K-theory via the S-construction and via matrices. In Section 3 the map α is defined as an infinite loop map and shown to fit into the above homotopy commutative diagram. In Section 4 we deduce Theorem C and conclude with some remarks as well as a description of α in terms of Becker–Gottlieb transfer maps.

1. *The cobordism category $\mathscr{S}$ and its classifying space*

The cobordism category $\mathscr{S}$ is a 2-category, that is a category in which the morphisms are not sets but categories and the associative composition is given by functors [**10**]. Roughly speaking, the set of objects in $\mathscr{S}$, also called 0-morphisms, is the set of closed 1-manifolds; the set of objects in the morphism categories, also called 1-morphisms, are cobordisms between them; and the morphisms in the morphism categories are the diffeomorphisms for the cobordisms relative to the boundary. $\mathscr{S}$ has an obvious symmetric monoidal structure induced by disjoint union. We will work with the following concrete model for $\mathscr{S}$.

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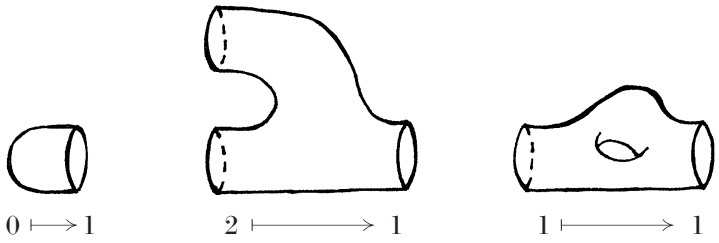


Fig. 1. Atomic surfaces.

The set of objects in $\mathscr{S}$ are the natural numbers $0, 1, 2, \ldots$. The monoidal structure is given by addition.

For the definition of the objects in the morphism categories $\mathscr{S}(n, m)$ consider the following three *atomic surfaces*:

(i) a fixed smooth disc as a morphism from 0 to 1;

(ii) a fixed smooth pair of pants as a morphism from 2 to 1;

(iii) a fixed smooth torus with two discs removed as a morphism from 1 to 1.

Each comes equipped with a fixed smooth collar, that is a smooth diffeomorphism of $S^1 \times [0, \epsilon)$ to a neighbourhood of each of its boundary circles. The objects in $\mathscr{S}(n, m)$ will be those cobordisms from n to m copies of a circle obtained from these atomic surfaces by a finite number of applications of the following three operations:

(I) gluing of two cobordisms along the target boundary circles of one and the source boundary circles of the other using the parametrization induced by the collars;

(II) disjoint union of cobordisms;

(III) relabelling of source and target boundary components.

Note that a cobordism in $\mathscr{S}(n, m)$ has exactly m connected components, each having exactly one target boundary component. In particular, $\mathscr{S}(n, 0)$ is empty. Composition of two composable cobordisms F_1 and F_2 is defined by gluing them along the target boundary circles of F_1 and source boundary circles of F_2 according to their labelling and parametrization induced by the collars. The resulting cobordism will be denoted by $F_1 \circ F_2$. The monoidal product for two cobordisms $F_1 \in \mathscr{S}(n_1, m_1)$ and $F_2 \in \mathscr{S}(n_2, m_2)$ is given by their disjoint union $F_1 \cup F_2 \in \mathscr{S}(n_1 + n_2, m_1 + m_2)$ where the labels of the source and target boundary components of F_2 are shifted by n_1 and m_1 respectively. We remark here that the symmetric group on n letters is contained in $\mathscr{S}(n, n)$ for each n. These are best thought of as cylinders of zero length with appropriate labelling.

The set of morphisms between two cobordisms F and F' in $\mathscr{S}(n, m)$ is empty, unless F and F' are diffeomorphic. In that case it is the space $\mathrm{Diff}^+(F, F'; \partial)$ of orientation preserving diffeomorphisms that leave the collar of the boundary pointwise fixed; more precisely, the diffeomorphisms take a collar to a collar and preserve the labels of the different boundary components. In the case when F is just a permutation, then $\mathrm{Diff}^+(F, F'; \partial)$ is understood to be empty unless $F = F'$ in which case it contains just the identity morphism for F. These are the sets of 2-morphisms. 2-morphisms have two compositions. Composition of 2-morphisms as morphisms between 1-morphisms (vertical composition) is here defined as composition of diffeomorphisms. The second composition (horizontal composition) is induced by gluing. Because the collars are pointwise fixed, diffeomorphisms $\phi_1 : F_1 \to F_1'$ and

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$\phi_2 : F_2 \to F_2'$ define a diffeomorphism $\phi_1 \circ \phi_2 : F_1 \circ F_2 \to F_1' \circ F_2'$. Similarly, ϕ_1 and ϕ_2 define a diffeomorphism $\phi_1 \cup \phi_2 : F_1 \cup F_2 \to F_1' \cup F_2'$. Note that both compositions are strictly associative and hence $\mathscr{S}$ is a strict 2-category. We topologize it as follows. Objects and 1-morphisms are given the discrete topology, and each set $\mathrm{Diff}^+(F, F'; \partial)$ of the 2-morphisms is equipped with the usual compact-open topology. Clearly, both horizontal and vertical composition, all source and target maps as well as disjoint union are continuous with respect to these topologies.

Let $\mathscr{B}\mathscr{S}$ denote the topological category obtained from $\mathscr{S}$ by replacing the morphism categories by their classifying spaces, i.e. $(\mathscr{B}\mathscr{S})(n, m) := B(\mathscr{S}(n, m))$. We define the classifying space of the 2-category $\mathscr{S}$ to be the classifying space of $\mathscr{B}\mathscr{S}$,

$$B\mathscr{S} := B(\mathscr{B}\mathscr{S}).$$

Let $F_{g,n}$ denote a fixed compact, orientable, smooth surface of genus g with n boundary components, and let $\Gamma_{g,n} := \pi_0(\mathrm{Diff}^+(F_{g,n}; \partial))$ be its mapping class group. Define the stable mapping class group to be $\Gamma_{\infty,n+1} := \lim_{g \to \infty} \Gamma_{g,n+1}$, where the limit is taken by gluing one of the boundary components of $F_{1,1+1}$ to one of the boundary components of $F_{g,n+1}$. By a theorem of Powell [**12**], $\Gamma_\infty := \Gamma_{\infty,1}$ is perfect so that Quillen's plus construction may be applied to its classifying space $B\Gamma_\infty$. For a topological space X, let ΩX denote its space of based loops. Theorem 3·1 of [**15**] has the following analogue.

THEOREM 1·1. $\Omega B\mathscr{S} \simeq \mathbb{Z} \times B\Gamma_\infty^+$.

Proof. The proof of theorem 3·1 in [**15**] can be adapted to the setting here. In outline the argument goes as follows. For each object n, it is easy to see that the morphism category $\mathscr{S}(n, 1)$ is equivalent to a full subcategory that has exactly one surface of topological type $F_{g,n+1}$. Hence, there is a homotopy equivalence

$$\mathscr{B}\mathscr{S}(n, 1) \simeq \bigcup_{g \geq 0} B\,\mathrm{Diff}^+(F_{g,n+1}; \partial).$$

Let $\mathscr{B}\mathscr{S}_1(n)$ be this space and let $\mathscr{B}\mathscr{S}_\infty(n)$ be its telescope with respect to right translation by the atomic surface of genus 1 in $\mathscr{B}\mathscr{S}(1, 1)$. As projection to connected components induces a homotopy equivalence $\mathrm{Diff}^+(F_{g,n+1}; \partial) \simeq \Gamma_{g,n+1}$ [**5**, **6**], we have

$$\mathscr{B}\mathscr{S}_\infty(S) \simeq \mathbb{Z} \times \lim_{g \to \infty} B\,\mathrm{Diff}^+(F_{g,n+1}; \partial) \simeq \mathbb{Z} \times B\Gamma_{\infty,n+1}.$$

Both $\mathscr{B}\mathscr{S}_1 := \bigcup_{n \geq 0} \mathscr{B}\mathscr{S}_1(n)$ and $\mathscr{B}\mathscr{S}_\infty := \bigcup_{n \geq 0} \mathscr{B}\mathscr{S}_\infty(n)$ are $\mathscr{B}\mathscr{S}$-diagrams via composition on the left. By Harer's stability theorem [**7**–**9**], every vertex in $\mathscr{B}\mathscr{S}$ acts on $\mathscr{B}\mathscr{S}_\infty$ by homology isomorphisms. Hence, by the generalized group completion theorem [**11**, **15**], the pullback diagram

$$\begin{array}{ccc} \mathscr{B}\mathscr{S}_\infty(n) & \longrightarrow & E_{\mathscr{B}\mathscr{S}}\mathscr{B}\mathscr{S}_\infty \\ \downarrow & & \downarrow \pi \\ n & \xrightarrow{\tau} & \mathscr{B}\mathscr{S} \end{array}$$

is a homology fibration for all n in the sense that the canonical map from $\mathscr{B}\mathscr{S}_\infty(n)$ to the homotopy fibre of π at n is a homology isomorphism. As $\mathscr{B}\mathscr{S}_\infty(n)$ is a telescope

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of $\mathscr{BS}_1(n)$ so is the Borel construction $E_{\mathscr{BS}}\mathscr{BS}_\infty$ a telescope of the Borel construction $E_{\mathscr{BS}}\mathscr{BS}_1$. As the latter is contractible (see for example lemma 3·3 [**15**]), hence so is the former. The homotopy fibre of π is therefore $\Omega B\mathscr{S}$ and so

$$\mathscr{BS}_\infty(n) \to \Omega B\mathscr{S}.$$

is a homology equivalence. Let n be equal to 0. By Powell's result, the mapping class group is perfect and hence, by the Whitehead theorem, the induced map of simply connected spaces $\mathscr{BS}_\infty(0)^+ \simeq \mathbb{Z}\times B\Gamma_\infty^+ \to \Omega B\mathscr{S}$ is a homotopy equivalence.

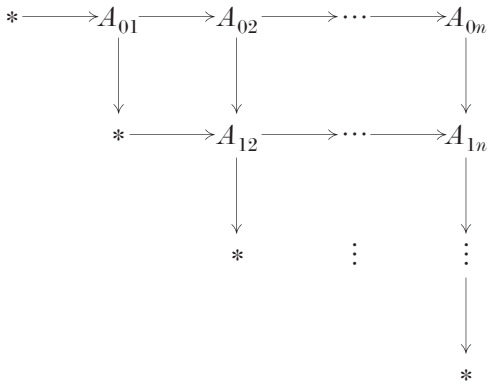
Corollary 1·2. *$\mathbb{Z}\times B\Gamma_\infty^+$ is an infinite loop space.*

Proof. The topological category $\mathscr{BS}$ has a strict monoidal structure induced by disjoint union. It is furthermore easy to see that this monoidal structure is permutative where the symmetry $\sigma_{n_1,n_2}\in\mathscr{BS}(n_1+n_2, n_2+n_1)$ is given by the permutation that swaps the first n_1 labels with the last n_2 labels. Hence, by the theory of infinite loop spaces [**13**], $B\mathscr{S} = B\mathscr{BS}$ is an infinite loop space because $B\mathscr{S}$ is also connected and therefore group-like; but then so is its loop space $\mathbb{Z}\times B\Gamma_\infty^+$.

2. *Waldhausen K-theory of a point*

There are several definitions of $A(*)$. We will mainly work with the definitions via the S-construction but will review the definition via matrices as well because this definition is conceptionally easier and more familiar from topological and algebraic K-theory.

Definition via the S-construction (see section 1 of [**17**] for details). Let $\mathscr{C}$ be a pointed category with cofibrations $\mathrm{co}(\mathscr{C})$ and weak equivalences $w\mathscr{C}$. Both cofibrations and weak equivalences are subcategories with the same objects as $\mathscr{C}$ satisfying various axioms. In particular, these axioms allow us to form subquotients B/A for any cofibration $A \hookrightarrow B$ and wedge products $A\vee B$. From $\mathscr{C}$ one can build the simplicial category $wS_\cdot\mathscr{C}$ in which the objects at the nth simplicial level are commutative diagrams A of the form



where all horizontal maps are cofibrations and $A_{ij}\simeq A_{0j}/A_{0i}$ is a choice of subquotient. A morphism between A and A' is given by weak equivalences $A_{ij}\to A'_{ij}$ such that all diagrams commute. The ith boundary of A is the diagram obtained from A by deleting the ith column and row. Face maps are given by inserting identity

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maps. The simplicial structure maps on the morphisms are defined similarly. The algebraic K-theory of the category $\mathscr{C}$ with weak equivalences $w\mathscr{C}$ is then defined as the loop space of the realization of the simplicial classifying space of $wS.\mathscr{C}$, denoted by

$$K(\mathscr{C}) \coloneqq \Omega |wS.\mathscr{C}|.$$

Wedge sum $(A, A') \to A \vee A'$ makes $wS.\mathscr{C}$ into a Γ-category [**13**]. Hence, as $|wS.\mathscr{C}|$ is connected, it is an infinite loop space and so is its deloop $K(\mathscr{C})$. There is a canonical inclusion of $w\mathscr{C}$ into the category at the first simplicial level of $wS.\mathscr{C}$. As the 0th simplicial category just defines a point, on realizations this gives a map from the suspension of $|w\mathscr{C}|$ to $|wS.\mathscr{C}|$ and its adjoint is the 'group completion' map

$$|w\mathscr{C}| \xrightarrow{\tilde{i}} \Omega |wS.\mathscr{C}|.$$

From now on let $\mathscr{P}$ denote the category of pointed spaces which are homotopic to a compact CW-space. Its cofibrations are the closed embeddings with the weak homotopy extension property and its weak equivalences are the weak homotopy equivalences. It follows from work of Strom [**14**] that this satisfies all of Waldhausen's axioms for a category with cofibrations and weak equivalences. (We make no attempt to downsize the category.)

Definition 2·1. $A(*) \coloneqq K(\mathscr{P}) = \Omega |wS.\mathscr{P}|$.

The morphism spaces of $|wS.\mathscr{P}|$ are a priori discrete. However, we would like to consider them with their natural compact-open topology. The resulting K-theory is the same up to homotopy equivalence. This follows by proposition 2·2·5 of [**17**].

Definition via matrices. Let $H_k^n \coloneqq \mathrm{Aut}_*(\bigvee_k S^n)$ denote the space of pointed homotopy equivalences of a k-fold wedge of n-spheres. H_k^n is an associative monoid under composition. Suspension and addition of the identity map define two stabilization maps

$$H_k^n \to H_k^{n+1}, \quad H_k^n \to H_{k+1}^n.$$

Let $H_k \coloneqq \lim_{n\to\infty} H_k^n$ and $H_\infty \coloneqq \lim_{k\to\infty} H_k$. There is a homotopy equivalence of H-spaces between H_k and $\widehat{\mathrm{GL}}_k QS^0$, the group of homotopy invertible $k \times k$-matrices over the ring up to homotopy $QS^0 \coloneqq \lim_{n\to\infty} \Omega^n S^n$. More precisely, $\widehat{\mathrm{GL}}_k QS^0$ is defined by the following pullback diagram

$$\begin{array}{ccc} \widehat{\mathrm{GL}}_k QS^0 & \longrightarrow & \mathrm{M}_k QS^0 \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ \mathrm{GL}_k \mathbb{Z} & \longrightarrow & \mathrm{M}_k \mathbb{Z} \end{array}$$

Then the classifying space BH_∞ has fundamental group $\mathrm{GL}_\infty \mathbb{Z}$. Hence, Quillen's plus construction may be applied to the maximal perfect subgroup. The Waldhausen K-theory of a point can then be defined as

$$A(*) \coloneqq \mathbb{Z} \times BH_\infty^+ \simeq \mathbb{Z} \times (\widehat{B\mathrm{GL}}_\infty QS^0)^+.$$

Direct sum of wedges of spheres induces an infinite loop space structure on $A(*)$.

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For any X in $\mathscr{P}$ let $w\mathscr{P}|_X$ denote the full subcategory of $w\mathscr{P}$ containing all objects weakly equivalent to X. We have the following variant of Waldhausen's theorem identifying the two definitions of $A(*)$ up to homotopy.

PROPOSITION 2·2. *There is a homotopy equivalence $\mathbb{Z}\times BH^+_\infty \to \Omega|wS.\mathscr{P}|$ of infinite loop spaces. On $BH^n_k \simeq |w\mathscr{P}|_{\vee_k S^n}|$ this map corresponds to the group completion map $\bar{\imath}$ up to the sign $(-1)^n$.*

Proof. Waldhausen [**17**; section 2·2] proves the analogue of this where he uses the category $R_f(*)$ of pointed homotopy finite CW-spaces and cellular maps. An application of the Approximation Theorem [**17**; theorem 1·6·7], however, shows that the inclusion $R_f(*)\to\mathscr{P}$ induces a homotopy equivalence of infinite loop spaces $|wS.R_f(*)|\to|wS.\mathscr{P}|$. The sign in the second statement is due to the fact that the group completion map commutes with suspension only up to a sign [17; proposition 1.6.2].

3. *The map to Waldhausen K-theory*

In this section we will define a map $\alpha\colon \mathbb{Z}\times B\Gamma^+_\infty\to A(*)$ as an infinite loop map. Indeed, it will be constructed in two ways, from a map of Γ-spaces where we think of $A(*)$ as defined in terms of the S-construction and as a lift of the usual homology representation where we think of $A(*)$ as defined via matrices.

In terms of the S-construction.[1] We will now think of the category of surfaces $\mathscr{S}$ as a simplicial category. The category at the 0th simplicial level is the totally disconnected discrete category with objects the closed 1-dimensional manifolds and their identity morphisms. At the nth simplicial level, for $n>0$, its objects are the composable n-tuples of cobordisms $\bar{F}=(F_1,F_2,\dots,F_n)$ with $F_i\colon T_{i-1}\to T_i$. A morphism between $\bar{F}$ and $\bar{F}'$ is an n-tuple $\phi=(\phi_1,\phi_2,\dots,\phi_n)$ of diffeomorphisms $\phi_i\in\mathrm{Diff}^+(F_i,F'_i;\partial)$. The ith simplicial boundary map is induced by gluing the cobordisms F_i and F_{i+1} along their common boundary when $0<i<n$. When $i=0$ or $i=n$, the boundary maps are given by dropping the first or last cobordisms of the n-tuple, which on the first simplicial level corresponds to assigning the source or target boundary 1-manifold. The boundary maps on morphisms are defined accordingly and the face maps are again induced by inserting identities.

We can now define a functor $\mathscr{F}:\mathscr{S}\to wS.\mathscr{P}$ by mapping $\bar{F}=(F_1,F_2,\dots,F_n)$ to the commutative diagram A with $A_{0j}=(F_1\circ F_2\circ\cdots\circ F_j)/T_0$ and $A_{ij}=(F_1\circ F_2\circ\cdots\circ F_j)/(F_1\circ\cdots\circ F_i)$ when $0<i\leqslant j$. Here we use the convention that $F/T=F\cup *$ is F with a disjoint basepoint if T is empty. Note that diffeomorphisms are in particular weak equivalences and the functor $\mathscr{F}$ on morphisms can be defined accordingly. More or less by definition, $\mathscr{F}$ is functorial with respect to the simplicial structure maps. Furthermore, disjoint union in $\mathscr{S}$ is taken to wedge product in $wS.\mathscr{P}$ so that $\mathscr{F}$ is indeed a map of Γ-categories defining an infinite loop map $|\mathscr{F}|\colon|\mathscr{S}|\to|wS.\mathscr{P}|$. Let α be the map induced by $|\mathscr{F}|$ on loop spaces. Of course, $|\mathscr{S}|$ is just the same as $B\mathscr{S}=B\mathscr{B}\mathscr{S}$. Thus in terms of Theorem 1·1 and Definition 2·1 we have proved

THEOREM 3·1. *$\alpha\colon\mathbb{Z}\times B\Gamma^+\to A(*)$ is a map of infinite loop spaces.*

In terms of matrices. The classical representation τ of the mapping class group $\Gamma_{g,1}$ in $\mathrm{GL}_{2g}\mathbb{Z}$ given by the action on the first homology of the surface $F_{g,1}$ has a homotopy

[1] This construction was developed out of a conversation with M. Weiss.

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lifting. To see this choose a homotopy $f\colon F_{g,1} \to \bigvee_{2g} S^1$ and a homotopy inverse f^{-1}. Then there is a map $\rho\colon \mathrm{Diff}^+(F_{g,1};\partial) \to H_{2g}$ taking a diffeomorphism ϕ to the stabilization of the homotopy equivalence $f \circ \phi \circ f^{-1}$. The representations ρ and τ are related by the commutative diagram

$$\begin{array}{ccc} \mathrm{Diff}^+(F_{g,1};\partial) & \xrightarrow{\rho} & H_{2g} \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ \Gamma_{g,1} & \xrightarrow{\tau} & \mathrm{GL}_{2g}\mathbb{Z} \end{array}$$

where the left vertical map is a homotopy equivalence [**6**]. By choosing f carefully it can be shown that ρ stabilizes with respect to the genus g. Though ρ is not a map of groups, it is a map of A_∞-spaces and hence still induces a map on classifying spaces. By the universal property of the plus construction, we have a map $\rho\colon \mathbb{Z} \times B\Gamma_\infty^+ \to \mathbb{Z} \times BH_\infty^+$. Note that on connected components this map corresponds to multiplication by 2. The commutativity of the diagram in the introduction is implied by the following result.

PROPOSITION 3·2. *There is a homotopy* $\alpha \simeq (-1)\rho\colon \mathbb{Z} \times B\Gamma_\infty^+ \to A(*)$.

Proof. Let $\mathrm{Aut}_*(F_{g,1})$ denote the based homotopy equivalences of $F_{g,1}$. Its classifying space is homotopic to $|w\mathscr{P}|_{F_{g,1}}|$ in the notation of Section 2. Similarly, $BH_{2g}^1 \simeq |w\mathscr{P}|_{\vee_{2g} S^1}|$. Consider the following diagram.

$$\begin{array}{ccccc} B\mathrm{Diff}^+(F_{g,1+1};\partial) & \longrightarrow & BH_*(F_{g,1}) & \xrightarrow{\tilde{i}} & \Omega\,|\,w\mathscr{S}.\mathscr{P}\,| \\ \downarrow{\scriptstyle \flat'} & & \downarrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \simeq (-1)} \\ B\mathrm{Diff}^+(F_{g,1};\partial) & \longrightarrow & BH_{2g}^1 & \xrightarrow{\tilde{i}} & \mathbb{Z} \times BH_\infty^+ \end{array}$$

The top left horizontal map is induced by mapping $F_{g,1+1}$ to $\mathscr{F}(F_{g,1+1})$, the bottom left map takes ϕ to $f \circ \phi \circ f^{-1}$, and $\flat'$ is induced by gluing a disc onto the incoming boundary component. The composition of the top line is just α while on the bottom it is ρ. The diagram commutes by Proposition 2.2.

4. *The main result and some remarks*

We now derive Theorem C of the introduction and conclude with some remarks including a description of α as a transfer map.

One of the fundamental results in Waldhausen K-theory is that $A(*)$ splits as a product of QS^0 and the Whitehead space $\mathrm{Wh}(*)$, that is

$$A(*) \simeq QS^0 \times \mathrm{Wh}(*).$$

A map from QS^0 to $A(*)$ is easily described. Recall that $QS^0 \simeq \mathbb{Z} \times B\Sigma_\infty^+$ where $\Sigma_\infty = \lim_{n\to\infty} \Sigma_n$ denotes the infinite symmetric group. The latter is the infinite loop space associated to the symmetric monoidal category of finite pointed sets and their isomorphisms. This category naturally includes into $\mathscr{P}$ inducing the required map. In terms of the matrix description of $A(*)$ this map is given by the inclusion of $\Sigma_\infty \to H_\infty$

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as permutations. Here, we need only the easier part of the splitting result, the fact that there is a map of infinite loop spaces

$$A(*) \xrightarrow{s} QS^0,$$

that is a homotopy inverse of the map $QS^0 \to A(*)$ described above. This was first proved by Waldhausen in [**16**]. Theorem C of the introduction is a corollary of the following result.

THEOREM 4·1. *There is a canonical map $\theta\colon QS^0 \to \mathbb{Z} \times B\Gamma_\infty^+$ of infinite loop spaces such that the following diagram is commutative up to homotopy.*

$$\begin{array}{ccc} QS^0 & \xrightarrow{-2} & QS^0 \\ {\scriptstyle\theta}\downarrow & & \uparrow{\scriptstyle s} \\ \mathbb{Z} \times B\Gamma_\infty^+ & \xrightarrow{\alpha} & A(*) \end{array}$$

Proof.[2] Map the zero dimensional sphere S^0 to $\mathbb{Z} \times B\Gamma_\infty^+$ such that the base point goes to the base point in the 0-component and the non-basepoint goes to the basepoint in the 1-component. Using the infinite loop space structure of $\mathbb{Z} \times B\Gamma_\infty^+$ this map can be extended to a map $\theta\colon QS^0 \to \mathbb{Z} \times B\Gamma_\infty^+$ of infinite loop spaces. Under the map $s \circ \alpha \simeq s \circ (-1)\rho$ the 1-component of $\mathbb{Z} \times B\Gamma_\infty^+$ clearly goes to the (-2)-component of QS^0. As $s \circ \alpha \circ \theta$ is a map of infinite loop spaces, this determines it as the multiplication by -2 on QS^0.

4.2. *At the prime two.* Note that Theorem C cannot hold without localization away from two because the fundamental group of $B\Sigma_\infty^+$ is $\mathbb{Z}/2\mathbb{Z}$ while the fundamental group of $B\Gamma_\infty^+$ is trivial. The above theorem, however, gives also information at the prime two. For example, as $\pi_3 QS^0$ is cyclic of order 24, it follows that $\pi_3 B\Gamma_\infty^+$ must contain a cyclic summand of order 2^s with $s \geqslant 2$.

4·3. *Relation to transfer maps.* As far as the splitting of $B\Gamma_\infty^+$ is concerned, the main purpose of constructing the map α to Waldhausen K-theory was to construct a map of infinite loop spaces

$$\mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow QS^0$$

on composition with Waldhausen's splitting s. One might wonder whether the map α detects more of $A(*)$. This is not the case as we will now explain.

Via the group completion map, $A(*)$ is the universal recipient of Euler characteristics and transfer maps. For let X be any pointed space in $\mathscr{P}$. Any fibration E over B with section and fibre X is classified up to homotopy by a map $f_E\colon B \to |w\mathscr{P}|_X|$. It is a folk theorem that the composition

$$B \xrightarrow{f_E} |w\mathscr{P}|_X| \xrightarrow{\tilde{i}} A(*) \xrightarrow{s} QS^0$$

is a pointed transfer tr_E for E. The latter map is by definition the adjoint of the stable Becker–Gottlieb [**1**] transfer $B_+ \to E_+$ composed with the map induced by the

[2] This part of the argument was a suggestion of M. Weiss. It replaces a much more cumbersome argument involving calculations of transfer maps and the Kahn–Priddy theorem.

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projection $E_+ \to S^0$. In a recent paper [**4**] Dwyer, Weiss and Williams show this when X is a smooth manifold and E is a smooth bundle over B with section. Furthermore, they prove that in this case $\tilde{i} \circ f_E : B \to A(*)$ actually factors through QS^0.

PROPOSITION 4·4. *α factors through QS^0.*

Proof. Consider the universal $F_{g,1}$-bundle E over $B\,\mathrm{Diff}^+(F_{g,1};\partial) \simeq B\Gamma_{g,1}$. The natural map $B\Gamma_{g,1} \to B\Gamma_\infty^+$ composed with α is just f_E composed with the group completion map. But E is a smooth bundle. Hence by [**4**] $s \circ \tilde{i} \circ f_E$ is homotopic to tr_E and factors through QS^0. Let g go to infinity. Harer's stability theorem now implies the proposition. □

We note here, that the above considerations imply also that $s \circ \alpha$ could have been described directly using transfers of universal surface bundles. The proposition also implies that away from the prime two, the image of the map τ from $\mathbb{Z} \times B\Gamma_\infty^+$ to the K-theory of the integers $K(\mathbb{Z})$ is contained in its direct factor $\mathrm{Im}\, J_{(\frac{1}{2})}$.

4·5. *Factoring through symplectic Waldhausen K-theory.* The classical representation $\tau : \Gamma_{g,1} \to \mathrm{GL}_{2g}\mathbb{Z}$ factors through Siegel's modular group $\mathrm{SP}_{2g}\mathbb{Z}$ as the orientation preserving diffeomorphisms preserve the symplectic intersection pairing on the first homology $H_1(F_{g,1}) = H_1(F_g)$. Hence, also $\rho : \mathrm{Diff}^+(F_{g,1};\partial) \to H_{2g} \simeq \widehat{\mathrm{GL}}_{2g}QS^0$ factors through $\widehat{\mathrm{SP}}_{2g}QS^0$, defined by the pullback diagram

$$\begin{array}{ccc} \widehat{\mathrm{SP}}_k QS^0 & \longrightarrow & \mathrm{M}_{k\times k}QS^0 \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ \mathrm{SP}_k\mathbb{Z} & \longrightarrow & \mathrm{M}_{k\times k}\mathbb{Z}. \end{array}$$

Proposition 3·2 implies in particular that $\rho : \mathbb{Z} \times B\Gamma_\infty^+ \to A(*)$ is a map of infinite loop spaces. This can also be proved directly by adopting the argument from [**15**] where we proved that $\tau : B\Gamma_\infty^+ \to B\mathrm{SP}_\infty\mathbb{Z}^+$ is a map of infinite loop spaces. Indeed, this then also gives the stronger result that ρ induces a map of infinite loop spaces $B\Gamma_\infty^+ \to (B\widehat{\mathrm{SP}}_\infty QS^0)^+$. Unlike α this map is rationally non-trivial.

The above remarks lead to the following speculation. Let I be the homotopy fibre of $B\Gamma_\infty^+ \to (B\widehat{\mathrm{SP}}_\infty QS^0)^+$ and T be the pullback of the diagram $QS^0 \to A(*) \leftarrow (B\widehat{\mathrm{SP}}_\infty QS^0)^+$. By the above remarks, $\alpha : B\Gamma_\infty^+ \to A(*)$ factors through both these natural maps. So by the universal properties of the pullback, α factors through T. In particular, there is a map $B\Gamma_\infty^+ \to T$, and thus a map

$$I \to \Omega\,\mathrm{Wh}\,(*)$$

well defined up to homotopy, the latter being the induced map between homotopy fibres of the maps $B\Gamma_\infty^+ \to (B\widehat{\mathrm{SP}}_\infty QS^0)^+$ and $T \to (B\widehat{\mathrm{SP}}_\infty QS^0)^+$.

It is well-known that the rational cohomology of $\Omega\,\mathrm{Wh}\,(*)$ has exactly one generator b_k in dimensions $4k$ for $k \geqslant 0$. They are related to the Borel classes in $K(\mathbb{Z})$. On the other hand, the rational cohomology of $B\Gamma_\infty^+$ contains a Miller–Morita–Mumford class κ_i in dimension $2i$ for each $i \geqslant 0$. For odd i, κ_i is detected by the pullback of the odd rational Chern classes c_i under the map $B\Gamma_\infty^+ \to B\mathrm{SP}_\infty\mathbb{R} \simeq BU$ induced by the representation τ. It seems natural to expect that in the rational cohomology of I the pullback of the b_ks under the above constructed map detect the pullback of the κ_{2i}s.

https://doi.org/10.1017/S0305004199003485 Published online by Cambridge University Press

Acknowledgements. It was a suggestion of Marcel Bökstedt to use Waldhausen K-theory to achieve the splitting for which I am very grateful. I am also much indebted to Michael Weiss. A couple of suggestion by him much simplified my original proof (see footnotes). The initial work for this paper was done during a visit at the Universität Bonn. I would like to take the opportunity to thank the mathematical institute and in particular Carl-Friedrich Bödigheimer for their hospitality.

Recently, Fred Cohen announced a weaker version of Theorem C in which the splitting is just a splitting of spaces.

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