Math. Proc. Camb. Phil. Soc. (1989), **105**, 299
Printed in Great Britain

Invariants of tangles

BY TIM D. COCHRAN
Department of Mathematics, University of California, Berkeley, CA 94720, *U.S.A.*
AND DANIEL RUBERMAN
Department of Mathematics, Brandeis University, Waltham, MA 02254, *U.S.A.*

(*Received* 4 *August* 1987; *revised* 13 *April* 1988)

1. *Introduction*

A *tangle* is a pair of strings (t_0, t_1) properly embedded in a 3-ball. Tangles have been used in several approaches to the classification of knots (see [**1**, **4**, **15**]). In these investigations, one keeps track of the endpoints of the arcs, so that the sum of two tangles along their boundaries is well defined. In particular, the sum of a given tangle with a trivial tangle, and any invariants of the resulting link, are invariants of the tangle under the restricted relation of isotopy keeping the endpoints fixed.

Our purpose in this note is to define invariants of tangles which depend on the isotopy class of the triple (B^3, t_0, t_1), *without* the restriction that the endpoints stay fixed. Our approach is to close up the tangle by adding on a trivial tangle to obtain a link and then to apply standard link invariants. For this to work, we need to find invariants whose value does not depend on the manner in which the link is closed. The most naive link invariant, the linking number between components, is clearly not well defined, as it may be altered by dragging one component around the other before adding the trivial tangle. Similarly, the Sato–Levine invariant [**19**] can be altered by twisting the components to add in a Whitehead link. However, we show that the 'higher linking numbers' defined in [**3**] can be used to define tangle invariants in this manner. We go on to show that our tangle invariants detect a symmetry between the components and use this to distinguish some knots from their mutants (see [**4**]).

2. *Definitions and statement of theorem*

We introduce a little formalism so as to be able to talk sensibly about closing up tangles to give links.

Definition 2·1. A *tangle* T is a triple (B^3, t_0, t_1), where the t_i are properly embedded arcs in the 3-ball. Two tangles T and T' are equivalent if there is an isotopy of B^3 taking t_0 to t_0' and t_1 to t_1'.

Note that we distinguish components of T, but not their endpoints, so the tangles are unoriented. Suppose that we are given a fixed representative of T.

Definition 2·2. A *closing* of T is a self-homeomorphism ϕ of ∂B^3 taking ∂t_i to the points $i_\pm$ indicated in Fig. 1. The closure of T via ϕ, denoted $\mathscr{L}(T, \phi)$, is the link obtained by glueing T into the ball in Fig. 1 via ϕ.

https://doi.org/10.1017/S0305004100067785 Published online by Cambridge University Press

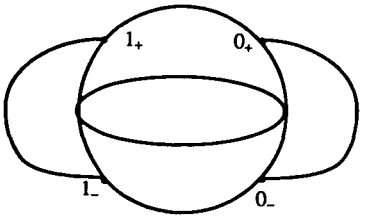


Fig. 1

There is no obvious canonical closure of a tangle. However the difference between two closures of T is expressed by a certain 2-bridge link. Suppose ϕ_0 and ϕ_1 are closings of the tangle T. Then the difference link $\mathscr{L}(\phi_0, \phi_1)$ is the (2-bridge) link

$$(B^3, u_0, u_1) \bigcup_{\phi_1 \phi_0^{-1}} (B^3, u_0, u_1).$$

Here (B^3, u_0, u_1) is the unlink in Fig. 1. The sense in which $\mathscr{L}(\phi_0, \phi_1)$ expresses the difference between the closures is the following lemma, whose straightforward proof we leave to the reader.

LEMMA 2·3. *The link $\mathscr{L}(T, \phi_1)$ can be expressed as a band connected sum*

$$\mathscr{L}(T, \phi_0) \#_b \mathscr{L}(\phi_0, \phi_1).$$

For an ordered, oriented link of two components, there are two sequences of higher-order linking numbers $\beta_x^i(L), \beta_y^i(L)$ $(i = 0, 1, \ldots)$ defined in [3]. (The x and y refer to the components of $\mathscr{L}$.) $\beta_x^0 = \beta_y^0$ is just the linking number between the components, which we will denote by λ; moreover the full collection $\{\beta_x^i, \beta_y^i\}$ is defined whenever λ vanishes. The first of the higher invariants to be defined was the Sato–Levine invariant $\beta^1(L) = \beta_x^1(L) = \beta_y^1(L)$, which is given as the self-linking number of the intersection of the Seifert surfaces for the two components (see [19]). Similarly, the higher invariants are defined by intersecting carefully chosen surfaces and computing the self-linkings of specific curves. These self-linking invariants do not depend on a choice of orientation for the link, but for $i > 2$ they do depend on an ordering of the components. Another important remark (see [3]) is that all of the β^i-invariants are additive under (oriented) band-connected sum.

LEMMA 2·4. *For any tangle T, there is a closure $\mathscr{L}(\phi)$ of T for which $\lambda(\mathscr{L}) = 0$ and $\beta^1(\mathscr{L}) = 0$.*

Proof. Choose any closing of T, and compute its linking number λ, with respect to any orientation. Then alter the homeomorphism by $\pm\lambda$ full twists between the endpoints of the two components to make the linking number zero. Now note that the Whitehead link can be viewed as a 2-bridge link, and hence copies of it (with either sense of clasp) may be introduced by varying the homeomorphism of S^2. As in Lemma 2·3, the resulting closures differ by addition of the Whitehead link; hence (by additivity) β^1 may be changed by an arbitrary amount. ∎

It follows from this lemma that there is no way to make a tangle invariant by considering the linking number or Sato–Levine invariant of a closure of the tangle, as these can be arbitrary. In contrast, we will show that each of the higher β^i's leads to a tangle invariant. We present the result first for $i = 2$.

https://doi.org/10.1017/S0305004100067785 Published online by Cambridge University Press

THEOREM 2·5. *Let ϕ be a closing of the tangle T with link $(\mathscr{L}) = \beta^1(\mathscr{L}) = 0$. Then $i(T) = \beta_x^2(\mathscr{L}) + \beta_y^2(\mathscr{L})$ is a* mod 2 *invariant of T, i.e. is independent of ϕ.*

Using the constructions of [3] it is easy to construct a tangle with $i(T)$ non-trivial. Note that any link of two components yields a tangle whose closure is the original link. In this way, example 4·6 or 4·5 (a) of [3], with $n = 1$, will give tangles with non-trivial $i(T)$.

Theorem 2·5 is a special case of a stronger theorem which shows that there are invariants made from each of the higher order β's. Theorem 4·1, proved in §4, states that $I^n = \beta_x^n - \beta_y^n$ of a closure of a tangle is independent of the particular choice of closure. We present the proof of Theorem 2·5 first, because the argument depends on a property of the Alexander polynomials of 2-bridge links that may have some independent interest. Theorem 4·1 will be proved in §4.

3. *Invariants of 2-bridge links*

Because of Lemma 2·3, in order to prove our theorem, we must establish some sort of vanishing result for 2-bridge links. First let us describe our conventions regarding 2-bridge links. Any such knot or link is described by a sequence of integers $[r_1, \ldots, r_n]$. If n is even we have a knot, while if n is odd we have the link drawn in Fig. 2.

The r_i give the number of full twists in Fig. 2. Note that the linking number, with the orientations as indicated, is given by $-\Sigma r_{2j+1}$.

For K a knot or link, let $\nabla_K(z)$ denote the Conway potential function (see [4]); it differs from the usual (1-variable) Alexander polynomial by a substitution. The symmetry property of the Alexander polynomial implies that the Conway polynomial of a *knot* has even degree (and constant term equal to 1). If K is a link of two components, ∇_K may be written as

$$\nabla(z) = z(a_0 + a_2 z^2 + a_4 z^4 + \ldots).$$

The vanishing result that leads to Theorem 2·5 is the following

PROPOSITION 3·1. *If $\mathscr{L}$ is a 2-bridge link, oriented as in Fig. 2, with $\lambda \equiv 0 (\mathrm{mod}\, 2)$, then $a_4 \equiv 0 (\mathrm{mod}\, 2)$.*

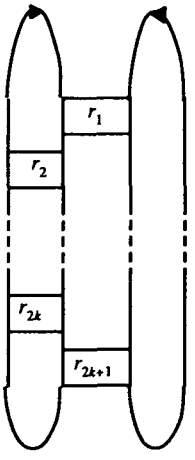


Fig. 2. Conventions for 2-bridge links ($n = 2k+1$).

https://doi.org/10.1017/S0305004100067785 Published online by Cambridge University Press

Proof. We use the calculation of the ∇-polynomial of a 2-bridge link given in [11]. The result is that, with the orientation of Fig. 2, the Conway polynomial is the $(1,1)$ entry in the matrix

$$\begin{bmatrix} r_1 z & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} r_2 z & 1 \\ 1 & 0 \end{bmatrix}\cdots\begin{bmatrix} r_{2k+1} z & 1 \\ 1 & 0 \end{bmatrix}.$$

We need only work with the polynomial modulo 2, and it is evident that $\nabla(\mathrm{mod}\,2)$ only depends on the $\{r_i\}\,(\mathrm{mod}\,2)$. If one of the r_i is 0, then the link has a picture with at most $2k-1$ sets of twists in it. Thus, inductively, we may assume that each of the r_i is 1.

Since we have assumed that the linking number is even, k must be odd, say $k = 2s+1$. Thus, to prove the proposition, it suffices to show that the coefficient of z^5 in the $(1,1)$ entry of M^{4s+3} is even, where M is the matrix

$$\begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}.$$

In the calculation, we may disregard multiples of z^n for $n \geqslant 6$, i.e. work modulo the ideal generated by z^6. Modulo this ideal, we find that

$$M^3 = \begin{bmatrix} z^3 & z^2+1 \\ z^2+1 & z \end{bmatrix}, \quad M^7 = \begin{bmatrix} 0 & z^4+1 \\ z^4+1 & z^5+z \end{bmatrix}, \quad \text{and} \quad M^8 = (z^4+1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since any M^{4s+3} is a product of these matrices, we see that there is never a z^5 in the $(1,1)$ position. ∎

Remark 3·2. A more elaborate argument shows that if $\lambda \equiv 0(\mathrm{mod}\,2)$, then all of the coefficients a_{4k} must in fact be even. We omit the proof, as it is not clear whether this fact can be used to manufacture further tangle invariants.

Proof of Theorem 2·5 *from Proposition* 3·1. The first ingredient is the relationship, due to the first author [2], between certain coefficients of ∇ and the β^i-invariants. In general, the first non-zero coefficient a_{2k} of ∇ is given in [2] as a linear function of Milnor's $\bar{\mu}$-invariants (see [14]), to which the $\{\beta^i\}$ are closely related. In particular, if $\lambda = 0$ and $\beta^1 = 0$, then $\nabla(z)$ may be written as $z(a_4 z^4 + \ldots)$. Moreover $a_4 \equiv \beta_x^2 + \beta_y^2$ modulo 2.

Now suppose that ϕ_0 and ϕ_1 are closings of the tangle T with $\lambda = \beta^1 = 0$. By Lemma 2·3 we have $\mathscr{L}(T,\phi_0) = \mathscr{L}(T,\phi_1)\,\#_b\,\mathscr{L}$, where $\mathscr{L}$ is a 2-bridge link. Orient the links so that the band connected sum is oriented coherently; the vanishing of β^1 and λ for the $\mathscr{L}(T,\phi_i)$ implies the same for $\mathscr{L}$. Also

$$(\beta_x^2+\beta_y^2)(\mathscr{L}(T,\phi_0)) = (\beta_x^2+\beta_y^2)(\mathscr{L}(T,\phi_1)) + (\beta_x^2+\beta_y^2)(\mathscr{L}),$$

so it suffices to show that $(\beta_x^2+\beta_y^2)(\mathscr{L}) \equiv 0\,(\mathrm{mod}\,2)$. Note that this statement is independent of orientation, so we may assume that $\mathscr{L}$ is a 2-bridge link, oriented as in Fig. 2. From Proposition 3·1, $a_4 \equiv 0\,(\mathrm{mod}\,2)$, so that $(\beta_x^2+\beta_y^2)$ is even as required. ∎

https://doi.org/10.1017/S0305004100067785 Published online by Cambridge University Press

4. *Further invariants and applications*

The proof of Theorem 2·5 suggests a way of defining many more tangle invariants. What one needs is an invariant of links which is additive (in an appropriate sense), and vanishes on 2-bridge links with linking number zero. In this section we show how this approach yields tangle invariants for each of the higher invariants of [3]. We will show that the resulting invariants detect a certain symmetry in tangles, and use this observation to distinguish certain knots and links from their 'mutants'.

Let $\mathscr{L}$ be any 2-component link with linking number zero, so that the higher β^i are defined. Recall that β^i_x, β^i_y are independent of link orientation, and additive under band connected sum. Recall also from [3] that if $r(\mathscr{L})$ denotes $\mathscr{L}$ with the opposite ordering of the components, then $\beta^i_x(\mathscr{L}) = \beta^i_y(r(\mathscr{L}))$. For $i > 1$, β^i strongly depends on the ordering of the components (cf. [3], example 4·5).

Let $T = (B^3, t_0, t_1)$ be any tangle and $\mathscr{L}(T)$ be any closure with linking number 0.

THEOREM 4·1. *For any positive integer n, the number $I^n(T) = \beta^n_x(\mathscr{L}) - \beta^n_y(\mathscr{L})$ is independent of the choice of closure $\mathscr{L}$ and hence is an integral invariant of T.*

Proof. As in Theorem 2·5, the main point is to establish a vanishing result for 2-bridge links. Following the proof of that theorem, it suffices to show that $\beta^n_x(\mathscr{L}) = \beta^n_y(\mathscr{L}) = \beta^n_x(r(\mathscr{L}))$ for $\mathscr{L}$ a 2-bridge link with linking number 0. Because $\beta^n_x(\mathscr{L})$ is a link invariant, any link for which there is an isotopy of S^3 interchanging the components will have this property. But 2-bridge links do have such a symmetry. This is presumably known to the experts, but we indicate a proof.

The main point is that one can transfer twists from the right-hand pair of strands in Fig. 2 to the left-hand pair of strands by 'flyping' [4]. Doing this for all of the twists labelled r_{2j}, and then rotating 180° around a vertical axis describes an isotopy from $\mathscr{L}$ to $r(\mathscr{L})$. ∎

Remark 4·2. (i) The invariant $I^2(T)$ reduces modulo 2 to the invariant $i(T)$ defined in §2. As for $i(T)$, all possible values of I^n are realized, using the procedure of [3].

(ii) We can also define integral invariants of unordered tangles by $T \mapsto (I^n(T))^2$, since I^n changes sign under reversing the roles of x and y.

The reader may wonder if there are other tangle invariants which can be derived using the same basic approach. The following observations arose in conversations with P. Kirk. By theorem 7·1 of [3], the sequence of invariants β^i_x for a 2-bridge link is eventually 0. Moreover, using the theses of Jin [8] and Kirk [10], one can show that any sequence of numbers which is eventually 0 can be realized as the sequence β^i_x for some 2-bridge link. Thus the property of having a finite number of non-zero β^i_x is preserved by addition of a 2-bridge link of linking number zero, and therefore defines a tangle invariant in the sense of this paper. Since this finiteness property and the symmetry property used above are the only restrictions on the pair of sequences β^i_x, β^i_y of 2-bridge links, it follows that this and our I^n-invariants are essentially the only tangle invariants which one can derive (in the manner done in this paper) from the β^i-invariants. Example 4·6 of [3] can be used to construct a tangle which is 'infinite'.

The following corollary follows easily from the invariance of I^n and shows its geometric content.

https://doi.org/10.1017/S0305004100067785 Published online by Cambridge University Press

COROLLARY 4·3. *If the tangle T is symmetric in the sense that there is an isotopy interchanging t_0 and t_1, then $I^n(T) = 0$ for every n.*

Because I^n can detect symmetry, it has a nice application in knot theory. Let τ denote any of the involutions of (S^4, 4 points) which interchanges the points in pairs.

Definition 4·4. Suppose K is a knot or link in S^3, and that S is a 2-sphere hitting K transversally in four points (S is called a Conway sphere). If τ is any of the above involutions, then the *mutation* of K via τ, denoted K^τ, is the knot or link resulting from cutting out the tangle on one side of S, and glueing it back in via τ.

Note that mutation takes knots to knots and links to links. In general, distinguishing a knot from its mutant is a difficult problem, as many standard invariants do not change under this operation. For instance, many of the 'abelian' knot invariants are unchanged [**5**] as are the generalized Jones polynomial [**6**], and geometric invariants such as Gromov's norm [**18**] and the η-invariant [**13**]. Ways of distinguishing K from K^τ include the Alexander module of a knot [**9**], computations [**16**] with non-abelian representations, and a geometric computation of the genus [**7**]. We will show that our I^n-invariants may be used to distinguish some knots from their mutants.

We will stick to the case of knots, although there is a completely parallel discussion which applies to links. For a given Conway sphere, there are two distinct types of mutations, referred to in [**18**] as *linked* and *unlinked*, respectively. For any mutation, there are two S^0's which are invariant under τ. These are either linked or not on the knot K, which gives rise to the terminology. We will be concerned with linked mutations, for the following reason. Suppose that $T = (B^3, t_0, t_1)$ is the tangle on one side of a Conway sphere, and that the involution τ interchanges ∂t_0 and ∂t_1. If T is symmetric in the sense mentioned above, then the mutation does not affect the knot. There is a rough converse to this, due to Bonahon, which states that in some circumstances $K = K^\tau$ implies that the tangle on one side or other will be symmetric. In particular, if the mutation is linked, then there will be an isotopy interchanging the components of one of the tangles. A more precise statement is as follows:

THEOREM 4·5. *Let S be a Conway sphere for a knot K splitting it into tangles T and T'. Suppose that there are no other Conway spheres and that T and T' are inequivalent tangles. If τ gives rise to a linked mutation of K, and $K = K^\tau$, then for each n, we have $I^n(T) = 0$ or $I^n(T') = 0$.*

Proof. If there is only one Conway sphere for K, then there is only one for K^τ. Hence, after an isotopy, we can assume that a diffeomorphism f taking K to K^τ will take the one Conway sphere to the other. Since the tangles on either side are inequivalent, f must take T to itself and T' to itself. If f preserves the strings on one side of S, then it will interchange the strings on the other side. By Corollary 4·3, the I^n-invariant of the side which is interchanged must vanish. ∎

We illustrate the usefulness of this approach with an example.

Example 4·6. Let T_m be the tangle illustrated in Fig. 3 below, where m refers to the number of whole twists between the indicated strands. The tangle T_m is essentially derived from fig. 6·9 of [**3**], and it may easily be computed that $I^2(T_m) = m$.

https://doi.org/10.1017/S0305004100067785 Published online by Cambridge University Press

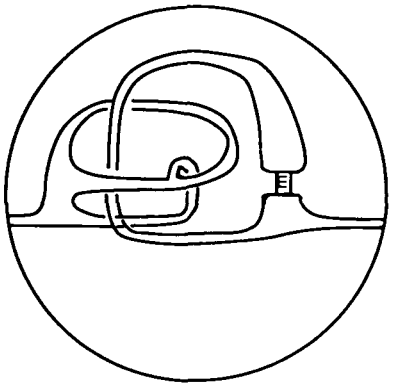


Fig. 3

Let $K_{m,n}$ be any knot obtained by adding T_m and T_n along their boundaries. One of the mutations of $K_{m,n}$ (depending on which endpoints are glued to which) will be unlinked; call that mutation τ. We claim that if $|m| \neq |n|$ and both are sufficiently large, then $K_{m,n}$ will not be equivalent to $K^{\tau}_{m,n}$. This of course uses Theorem 4·5: the tangles are inequivalent (even up to orientations of S^3) since their I^2-invariants are different, and Corollary 4·3 shows that neither is symmetric. Note that the individual strings in T_m are unknotted, so there is no 'first-order' way of seeing that the strings cannot be interchanged. We need to verify the technical condition in Theorem 4·5 that the obvious Conway sphere is the only one. We must show that for $|m|$ sufficiently large, any Conway sphere in B^3 (hitting T_m in four points) is isotopic to the boundary sphere, and that there are no essential discs hitting T_m in two points.

To do this, we observe that an essential Conway sphere yields an incompressible torus in M_m, the 2-fold cover of B^3 branched along T_m, and a disc yields an essential annulus. Conversely, using least-area surfaces in a now-standard way (see [**12**]), one need only show that the double cover has no essential tori or annuli. In particular, it would suffice to show that M_m is hyperbolic of finite volume. A moderately complicated calculation using the Kirby–Rolfsen calculus (see [**17**]) shows that M_m is obtained by $((-1/2m)+8)$-surgery on the solid component of the link in Fig. 1. (Details are available from the authors, upon request.)

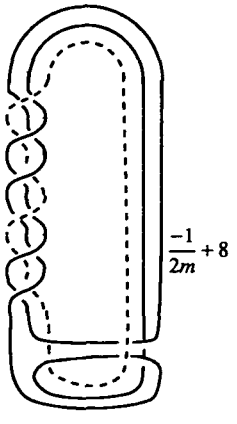


Fig. 4

https://doi.org/10.1017/S0305004100067785 Published online by Cambridge University Press

But L is a hyperbolic link, as can be shown using the fact that it is a 3-bridge link which is not a satellite link. (Alternatively, J. Weeks' computer programs 'carrot/snappea' found a hyperbolic structure in terms of an ideal triangulation of the complement of L.) Thurston's Dehn surgery theorem in §4 of [**20**] says that for all but finitely many $r \in \mathbb{Q}$, r-surgery on one component of a hyperbolic link will be hyperbolic of finite volume. Therefore, for m and n sufficiently large and not equal, the knot $K_{m,n}$ will have only the obvious Conway sphere and will not be equal to its mutant.

The authors thank the referee for his careful reading of their manuscript. The proof given of Proposition 3·1 is the referee's, and replaces a more involved version. They also thank Colin Adams for doing the computer calculation referred to in the last example. The first author is partially supported by an NSF Postdoctoral Fellowship and MSRI, and the second author is partially supported by NSF grant DMS 8502317 and MSRI.

REFERENCES

[1] F. BONAHON and L. SIEBENMANN. *Geometric Splittings of Knots and 3-Manifolds*. London Math. Soc. Lecture Note Ser. (To appear.)
[2] T. D. COCHRAN. Concordance invariance of coefficients of Conway's link polynomial. *Invent. Math.* **82** (1985), 527–541.
[3] T. D. COCHRAN. Geometric invariants of link cobordism. *Comment. Math. Helv.* **60** (1985), 291–311.
[4] J. H. CONWAY. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra* (Pergamon Press, 1970), pp. 329–358.
[5] D. COOPER. Signatures of surfaces in 3-manifolds and applications to knot and link cobordism. Ph.D. thesis, Warwick University (1982).
[6] P. FREYD, J. HOSTE, W. B. R. LICKORISH, K. C. MILLETT, A. OCNEANU and D. YETTER. A new polynomial invariant of knots and links. *Bull. Amer. Math. Soc.* **12** (1985), 239–246.
[7] D. GABAI. Foliations and genera of links. *Topology* **23** (1984), 381–394.
[8] G. T. JIN. Invariants of 2-component links. Ph.D. thesis, Brandeis University (1988).
[9] C. KEARTON. Mutation of knots. (Preprint, 1987.)
[10] P. KIRK. Link homotopy with one codimension two component. Ph.D. thesis, Brandeis University (1988).
[11] W. B. R. LICKORISH and K. C. MILLETT. A polynomial invariant of oriented links. *Topology* **26** (1987), 107–141.
[12] W. H. MEEKS and S. T. YAU. The classical Plateau problem and the topology of three-dimensional manifolds. *Topology* **21** (1982), 409–440.
[13] R. MEYERHOFF and D. RUBERMAN. Cutting and pasting and the η-invariant. *J. Differential Geom.* (To appear.)
[14] J. MILNOR. Isotopy of links. In *Algebraic Geometry and Topology: Symposium in Honor of S. Lefschetz* (Princeton University Press, 1957), 280–306.
[15] J. M. MONTESINOS. Revêtements ramifiés de noeuds, espaces de Seifert et scindements de Heegaard, 1980. (Preprint, 1980.)
[16] R. RILEY. Homomorphisms of knot groups on finite groups. *Math. Comp.* **25** (1971), 603–619.
[17] D. ROLFSEN. *Knots and Links* (Publish or Perish, 1976).
[18] D. RUBERMAN. Mutations and volumes of knots in S^3. *Invent. Math.* **90** (1987), 189–215.
[19] N. SATO. Cobordisms of semi-boundary links. *Top. Appl.* **18** (1984), 225–234.
[20] W. P. THURSTON. *The Geometry and Topology of 3-Manifolds*. Lecture notes, Princeton University (1978).

https://doi.org/10.1017/S0305004100067785 Published online by Cambridge University Press