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DISTANCE BETWEEN ARITHMETIC PROGRESSIONS AND PERFECT SQUARES

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Abstract. In this paper, we study how close the terms of a finite arithmetic progression can get to a perfect square. The answer depends on the initial term, the common difference and the number of terms in the arithmetic progression.

Many questions in number theory can be phrased as the study of the "distances" between two sequences of numbers. For instance, we have the famous conjecture that there are infinitely many primes of the form n^2+1. This can be interpreted as saying that the sequence of prime numbers and the sequence of perfect squares can get within unit distance from one another infinitely often. Another example is the conjecture of M. Hall, Jr. stating that

$$\text{if } x^2 \neq y^3 \quad \text{then } |x^2-y^3| \gg x^{1/2}.$$

This means that there are significant gaps between the sequence of perfect squares and the sequence of perfect cubes (apart from the sequence of sixth powers). In this paper, we are going to consider the distance between two simplest arithmetic sequences, namely arithmetic progressions and perfect squares.

More specifically, given integers $a \geq 0$, $d \geq 1$ and $N \geq 1$, we consider the arithmetic progression

$$\mathcal{A} = \mathcal{A}_{a,d,N} := \{a, a+d, a+2d, \dots, a+Nd\}.$$

We are interested in how close the terms of the above arithmetic progression can get to a perfect square $0^2, 1^2, 2^2, 3^2, \dots$. More precisely, we let

$$\delta = \delta_{a,d,N} := \min_{\substack{0\leq n\leq N\\ m\in\mathbb{Z}}} |a+nd-m^2|.$$

To avoid triviality, we require the interval $[a, a+Nd]$ to contain at least one perfect square. So we need

$$M := \sqrt{a+Nd}-\sqrt{a} \geq 1 \Leftrightarrow \sqrt{a+Nd} \geq 1+\sqrt{a} \Leftrightarrow a \leq \left(\frac{Nd-1}{2}\right)^2$$

which we will assume throughout the paper. Hence $a \ll N^2d^2$ in order for the question to be interesting. Clearly, we have the bound

$$\delta \leq \frac{d}{2}.$$

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Let us first present a heuristic investigation. With the above notation, there are about M perfect squares in the interval $[a, a+Nd]$. Suppose that these squares are "uniformly distributed" (mod d). Then we expect that they are spaced about d/M from one another (mod d) which implies that one can find a perfect square within a distance

$$\frac{d}{M} = \frac{d}{\sqrt{a+Nd}-\sqrt{a}} = \frac{\sqrt{a+Nd}+\sqrt{a}}{N} \approx \begin{cases} \sqrt{a}/N & \text{if } a \geq Nd, \\ \sqrt{d}/\sqrt{N} & \text{if } a \leq Nd \end{cases}$$

from some $a+nd$ with $0 \leq n \leq N$. There is a change of behavior depending on how a compares with Nd. The above heuristic makes sense only when $d/M \geq 1$ as the spacing between integers is at least one,

$$M \leq d \Leftrightarrow \sqrt{a+Nd}-\sqrt{a} \leq d \Leftrightarrow N \leq d+2\sqrt{a} \tag{1}$$

after some simple algebra. Thus we are led to the following.

CONJECTURE 1. *For any $\epsilon > 0$ and integers $a \geq 0$, $d \geq 1$, $N \geq 1$ such that $a \leq ((Nd-1)/2)^2$ and $N \leq d+2\sqrt{a}$,*

$$\delta \ll_\epsilon \frac{\sqrt{a+Nd}}{N} d^\epsilon$$

or, equivalently,

$$\delta \ll_\epsilon \begin{cases} \dfrac{\sqrt{a}}{N} d^\epsilon & \text{if } Nd \leq a \leq \left(\dfrac{Nd-1}{2}\right)^2, \\ \dfrac{\sqrt{d}}{\sqrt{N}} d^\epsilon & \text{if } a \leq Nd. \end{cases}$$

It may be worth mentioning that if $N \leq d+2\sqrt{a}$ and $Nd \leq a$, then $N^2 \leq Nd+2N\sqrt{a} \leq a+2N\sqrt{a}$ which implies $2N^2 \leq (\sqrt{a}+N)^2$. Hence $N \ll \sqrt{a}$ and the ratio $\sqrt{a}/N \gg 1$ in the first half of the conjecture. Meanwhile if $N \leq d+2\sqrt{a}$ and $a \leq Nd$, then $N \leq d+2\sqrt{Nd}$ which implies $2N \leq (\sqrt{d}+\sqrt{N})^2$. Hence $\sqrt{N} \ll \sqrt{d}$ and the ratio $\sqrt{d}/\sqrt{N} \gg 1$ in the second half of the conjecture.

What happens when $N > d+2\sqrt{a}$? Retracing the steps in (1), this means that $M > d$. So there are more perfect squares in the interval $[a, a+Nd]$ than the common difference d of the arithmetic progression. However, $m^2 \equiv (m+d)^2 \pmod{d}$. So having more than d consecutive perfect squares does not make them any closer to the arithmetic progression $\mathcal{A}$ than just having $M = d$ perfect squares. Hence, by putting in $N = d+2\sqrt{a}$ in Conjecture 1, we have the following conjecture.

CONJECTURE 2. *For any $\epsilon > 0$ and integers $a \geq 0$, $d \geq 1$, $N \geq 1$ such that $a \leq ((Nd-1)/2)^2$ and $N > d+2\sqrt{a}$,*

$$\delta \ll_\epsilon d^\epsilon.$$

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Towards Conjectures 1 and 2, we have the following theorems.

THEOREM 1. *For* $a \le N^2 d/1800$,

$$\delta \ll \begin{cases} \dfrac{a^{1/4}d^{1/2}}{N^{1/2}} & \text{if } \max(Nd, N^2) \le a \le \dfrac{N^2 d}{1800}, \\ \dfrac{d^{3/4}}{N^{1/4}} & \text{if } N^2 \le a \le Nd, \\ d^{1/2} & \text{if } a \le N^2. \end{cases}$$

THEOREM 2. *For* $a \le N^{4/3}d^{4/3}/200$,

$$\delta \ll \begin{cases} \dfrac{a^{1/2}}{N^{1/2}} & \text{if } \max(Nd, N^{2/3}d^{4/3}) \le a \le \dfrac{N^{4/3}d^{4/3}}{200}, \\ \dfrac{d}{a^{1/4}} & \text{if } \max(Nd, d^{4/3}) \le a \le N^{2/3}d^{4/3}. \end{cases}$$

COROLLARY 1. *For* $a \le N^2 d/1800$, $\delta \ll d^{3/4}$.

Proof of Corollary 1. It follows immediately from Theorem 1. Clearly, the second and third bounds in Theorem 1 are $\ll d^{3/4}$ while the first bound

$$\frac{a^{1/4}d^{1/2}}{N^{1/2}} \ll \frac{N^{1/2}d^{1/4}d^{1/2}}{N^{1/2}} = d^{3/4}.$$

Also the three cases cover all the possible ranges for $a \le N^2 d/1800$. □

Theorems 1 and 2 are far from Conjectures 1 and 2. However, if we assume a certain conjectural bound on an average of twisted incomplete Salié sums, we can prove that Conjecture 1 is true for a certain range of a, namely we have the following theorem.

THEOREM 3. *Assume Conjecture 3 in §2. If d is odd, then, for all $\epsilon > 0$, there exist some constants $C_1, C_2 > 0$ such that*

$$\delta \le C_1 \frac{\sqrt{a}}{N} d^{\epsilon} \quad \text{if } N^2 d^{2/3} \le a \le C_2 N^2 d^{1-\epsilon}.$$

However, in view of the above theorems, much is still unknown when a is big, namely $\max(N^2 d, N^{4/3}d^{4/3}) \ll a \ll N^2 d^2$.

Some notation. Throughout the paper, the notation $f(x) = O(g(x))$, $f(x) \ll g(x)$ and $g(x) \gg f(x)$ are all equivalent to $|f(x)| \le Cg(x)$ for some constant $C > 0$. Also $f(x) = O_\lambda(g(x))$, $f(x) \ll_\lambda g(x)$ or $g(x) \gg_\lambda f(x)$ mean that the implicit constant C may depend on λ.

§1. *Proof of Theorems 1 and 2.* Our main tool is a result of Huxley [**3**] on integer points close to a curve.

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THEOREM 4. *Let $M \geq 12$ be a positive integer, and let I be a closed interval of length M with integer endpoints. Let $f(x)$ be a real function, twice continuously differentiable on I with*

$$\frac{144}{M^2} \leq \frac{\Delta}{C} \leq |f''(x)| \leq C\Delta \leq 1$$

where $C \geq 1$ and $\Delta \leq 1$ are suitable real parameters. Let ϵ be a real number with

$$\max\left(3\sqrt{C^3\Delta}, 6C^2\sqrt{\frac{2}{M}}\right) \leq \epsilon \leq \frac{1}{4}.$$

Then there are distinct integers $m_1, \ldots, m_R$ in I with

$$\|f(m_r)\| \leq \epsilon \quad \text{for } 1 \leq r \leq R, \text{ and } R \geq \min\left(\frac{\epsilon^4 M}{2^4 3^4 C^7 \Delta}, \frac{\epsilon M}{2^7 3^2 C^4}\right).$$

We also keep using the following simple fact.

LEMMA 1. *For any $x \geq 0$, we can always find a perfect square that is within a distance $2\sqrt{x}$ from x.*

Proof. Observe that

$$0 \leq x - \lfloor\sqrt{x}\rfloor^2 = (\sqrt{x} - \lfloor\sqrt{x}\rfloor)(\sqrt{x} + \lfloor\sqrt{x}\rfloor) \leq 2\sqrt{x}. \qquad \square$$

Proof of Theorem 1. One can easily check that the theorem is true when $d = 1$. So we may assume $d \geq 2$ from now on. Consider $f(x) = (x^2 - a)/d$. Then $f''(x) = 2/d = \Delta$ and $C = 1$ in the notation of Theorem 4. Without loss of generality, we may assume that $\sqrt{a + Nd} - \sqrt{a} \geq 14$ for otherwise the theorem is true by picking a large enough implicit constant. Let I be the closed interval with endpoints $\lceil\sqrt{a}\rceil$ and $\lfloor\sqrt{a + Nd}\rfloor$. Then I has length $M = \lfloor\sqrt{a + Nd}\rfloor - \lceil\sqrt{a}\rceil \geq 12$. For what follows, we will use

$$\begin{aligned}\frac{Nd}{2(\sqrt{a + Nd} + \sqrt{a})} &= \frac{\sqrt{a + Nd} - \sqrt{a}}{2} \leq M \\ &\leq \sqrt{a + Nd} - \sqrt{a} = \frac{Nd}{\sqrt{a + Nd} + \sqrt{a}}.\end{aligned}$$

There are two cases.

Case 1: $a \geq Nd$. Then $Nd/5\sqrt{a} \leq M \leq Nd/2\sqrt{a}$. The condition $144/M^2 \leq \Delta/C$ is satisfied when $a \leq N^2 d/1800$. There are two subcases.

Subcase 1: $a \leq N^2$. Then $\sqrt{\Delta} \gg 1/\sqrt{M}$. So we can pick $\epsilon \ll \sqrt{\Delta} \ll 1/\sqrt{d}$ and $m_1 \in I$ with $\|f(m_1)\| \leq \epsilon$. This gives

$$\left|\frac{m_1^2 - a}{d} - n\right| \ll \frac{1}{\sqrt{d}} \Rightarrow |m_1^2 - a - nd| \ll d^{1/2}$$

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for some integer n. Since $m_1 \in I$, so $0 \le n \le N$. We have the third bound for Theorem 1 when $a \ge Nd$.

Subcase 2: $a \ge N^2$. Then $\sqrt{\Delta} \ll 1/\sqrt{M}$. So we can pick $\epsilon \ll 1/\sqrt{M} \ll a^{1/4}/N^{1/2}d^{1/2}$ and $m_1 \in I$ with $\|f(m_1)\| \le \epsilon$. This gives

$$\left|\frac{m_1^2 - a}{d} - n\right| \ll \frac{a^{1/4}}{N^{1/2}d^{1/2}} \Rightarrow |m_1^2 - a - nd| \ll \frac{a^{1/4}d^{1/2}}{N^{1/2}}$$

for some integer $0 \le n \le N$, and we have the first bound for Theorem 1.

Case 2: $a \le Nd$. Then $\sqrt{Nd}/5 \le M \le \sqrt{Nd}$. The condition $144/M^2 \le \Delta/C$ is satisfied when $N \ge 1800$. Again there are two subcases.

Subcase 1: $a \le N^2$. Then $\sqrt{\Delta} \gg 1/\sqrt{M}$. So we can pick $\epsilon \ll \sqrt{\Delta} \ll 1/\sqrt{d}$ and $m_1 \in I$ with $\|f(m_1)\| \le \epsilon$. This gives

$$\left|\frac{m_1^2 - a}{d} - n\right| \ll \frac{1}{\sqrt{d}} \Rightarrow |m_1^2 - a - nd| \ll d^{1/2}$$

for some integer $0 \le n \le N$. We have the third bound for Theorem 1 when $a \le Nd$. Note that if $N < 1800$, by Lemma 1, we can find a perfect square within a distance $2\sqrt{a} \le 2\sqrt{Nd} \ll d^{1/2}$ from a. So we still have the third bound.

Subcase 2: $a \ge N^2$. Then $\sqrt{\Delta} \ll 1/\sqrt{M}$. So we can pick $\epsilon \ll 1/\sqrt{M} \ll a^{1/4}/N^{1/2}d^{1/2}$ and $m_1 \in I$ with $\|f(m_1)\| \le \epsilon$. This gives

$$\left|\frac{m_1^2 - a}{d} - n\right| \ll \frac{1}{N^{1/4}d^{1/4}} \Rightarrow |m_1^2 - a - nd| \ll \frac{d^{3/4}}{N^{1/4}}$$

for some integer $0 \le n \le N$, and we have the second bound for Theorem 1. Note that if $N < 1800$, by Lemma 1, we can find a perfect square within a distance $2\sqrt{a} \le 2\sqrt{Nd} \ll d^{1/2} \ll d^{3/4}/N^{1/4}$ from a. So we still have the second bound. □

Proof of Theorem 2. We apply Theorem 4 to the inverse function $g(x) = f^{-1}(x) = \sqrt{a + xd}$ and I is the interval $[0, N]$ with length $M = N$. Then

$$g''(x) = -\frac{d^2}{4(a + xd)^{3/2}}.$$

We shall focus on the case $a \ge Nd$ only as the other case is covered by Theorem 1. We have

$$\frac{d^2}{16a^{3/2}} \le |g''(x)| \le \frac{d^2}{4a^{3/2}}.$$

So $\Delta = d^2/8a^{3/2}$ and $C = 2$ in the notation of Theorem 1. The condition $C\Delta \le 1$ is equivalent to $d^{4/3}/2^{4/3} \le a$. Meanwhile the condition $144/M^2 \le \Delta/C$ is satisfied when $a \le N^{4/3}d^{4/3}/200$. There are two subcases.

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Subcase 1: $a \le N^{2/3}d^{4/3}$. Then $\sqrt{\Delta} \gg 1/\sqrt{M}$. So we can pick $\epsilon \ll \sqrt{\Delta} \ll d/(a^{3/4})$ and $n \in I$ with $\|g(n)\| \le \epsilon$. This gives, for some integer m,

$$|\sqrt{a+nd} - m| \ll \frac{d}{a^{3/4}} \Rightarrow |a+nd-m^2| = |\sqrt{a+nd}-m||\sqrt{a+nd}+m| \ll \frac{d}{a^{1/4}}$$

as $a \gg d^{4/3}$ implies $d/a^{3/4} \ll a^{1/2}$. This gives the second bound for Theorem 2.

Subcase 2: $a \ge N^{2/3}d^{4/3}$. Then $\sqrt{\Delta} \ll 1/\sqrt{M}$. So we can pick $\epsilon \ll 1/\sqrt{M} = 1/\sqrt{N}$ and $n \in I$ with $\|g(n)\| \le \epsilon$. This gives, for some integer m,

$$|\sqrt{a+nd} - m| \ll \frac{1}{N^{1/2}} \Rightarrow |a+nd-m^2| = |\sqrt{a+nd}-m||\sqrt{a+nd}+m| \ll \frac{a^{1/2}}{N^{1/2}}.$$

This gives the first bound for Theorem 2. □

§2. *Twisted Salié sum and Theorem 3.* The same technique in [2] and [1] for the studies of $n^2\alpha \pmod 1$ and short intervals containing almost squares or sums of two squares can be used here. We recall the following conjectural bound on a certain average of twisted incomplete Salié sums.

CONJECTURE 3. *Let a, q be integers with $q \ge 2$ and $(a,q)=1$ where q is not a perfect square. Let H, $K \ge 1$ and λ, μ be any real numbers. Then, for any $\epsilon > 0$,*

$$\sum_{\substack{1\le h\le H\\(h,q)=1}} e(\lambda h) \sum_{0\le k<K} e(\mu k)\left(\frac{h}{q}\right) e\left(\frac{a\overline{h}k^2}{q}\right) \ll_\epsilon (H^{1/2}K^{1/2} + H^{3/4} + K + q^{-1/2}HK + q^{-1/2}K^2)q^\epsilon.$$

A consequence of the above conjecture is the following.

LEMMA 2. *Assume Conjecture 3. Let q be an odd number, H, $K \ge 1$ and λ, μ be any real numbers. We have*

$$\sum_{0\le k<K} e(\mu k) \sum_{\substack{1\le h\le H\\(h,q)=1}} e(\lambda h) G(h, \pm k; q) \ll_\epsilon (q^{1/2}H^{1/2}K^{1/2} + q^{1/2}H^{3/4} + q^{1/2}K + HK + K^2)q^\epsilon$$

where

$$G(a,b;q) := \sum_{n \pmod q} e\left(\frac{an^2+bn}{q}\right)$$

is the Gauss sum.

Proof. This is [1, Lemma 5.1]. □

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Proof of Theorem 3. The proof is very similar to the [1, Proof of Theorem 1.4]. Suppose that N and d are sufficiently large and $a \geq \max(Nd, N^2)$. Let

$$A := \left\lfloor \sqrt{a + \frac{N}{2}d} \right\rfloor.$$

We restrict our attention to the arithmetic progression $a + nd$ where $N/4 \leq n \leq 3N/4$. Suppose $a + nd = (A + m)^2$. Then

$$m = \sqrt{a + nd} - \sqrt{a + \frac{N}{2}d} + O(1) = \frac{(n - N/2)d}{\sqrt{a + nd} + \sqrt{a + Nd/2}} + O(1).$$

Let $M = \lfloor Nd/12\sqrt{a} \rfloor$. If $m \in [-M, M]$, then $n \in [N/4, 3N/4]$. Our goal is to find some $m \in [-M, M]$ such that

$$\left\| \frac{(A+m)^2 - a}{d} \right\| < \Delta \Rightarrow \left| \frac{(A+m)^2 - a}{d} - n \right| < \Delta \Rightarrow |(A+m)^2 - (a+nd)| < \Delta d$$

for some integer $n \in [N/4, 3N/4]$. For any $0 < \Delta < 1/2$, define

$$S := \#\left\{ m : |m| \leq M, \left\| \frac{(A+m)^2 - a}{d} \right\| < \Delta \right\}.$$

Let

$$f(x) := \begin{cases} 1 - |x| & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$t(x) := \begin{cases} 1 - |x/\Delta| & \text{if } -\Delta \leq x \leq \Delta, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(x) := \sum_{n=-\infty}^{\infty} t(x - n), \qquad g_\lambda(x) = g(x - \lambda).$$

Thus $f(x)$ can be used to detect numbers between -1 and 1, and $g_\lambda(x)$ can be used to detect numbers whose fractional part is between $\lambda - \Delta$ and $\lambda + \Delta$. In particular, with $\lambda = a/d$,

$$S \geq \sum_{m=-\infty}^{\infty} f\left(\frac{m}{M}\right) g_\lambda\left(\frac{(A+m)^2}{d}\right).$$

Now g_λ has the Fourier expansion

$$g_\lambda(x) = \sum_{h=-\infty}^{\infty} c(h) e(-\lambda h) e(hx)$$

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where

$$e(u) = \mathrm{e}^{2\pi i u} \quad \text{and} \quad c(h) = \Delta\left(\frac{\sin \pi \Delta h}{\pi \Delta h}\right)^2 \quad \text{when } h \neq 0,\ c(0) = \Delta.$$

Therefore,

$$S = \Delta M + \sum_{h \neq 0} c(h) e(-\lambda h) \sum_{m=-\infty}^{\infty} e\left(\frac{h(A+m)^2}{d}\right) f\left(\frac{m}{M}\right) =: \Delta M + R.$$

By Poisson summation, the inner sum over m in R is equal to

$$\begin{aligned}
&\sum_{t \,(\mathrm{mod}\, d)} e\left(\frac{h(A+t)^2}{d}\right) \sum_{m \equiv t \,(\mathrm{mod}\, d)} f\left(\frac{m}{M}\right) \\
&= \sum_{t \,(\mathrm{mod}\, d)} e\left(\frac{h(A+t)^2}{d}\right) \frac{M}{d} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{Mk}{d}\right) e\left(\frac{tk}{d}\right) \\
&= \frac{M}{d} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{Mk}{d}\right) \sum_{t \,(\mathrm{mod}\, d)} e\left(\frac{h(A+t)^2 + kt}{d}\right) \\
&= \frac{M}{d} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{Mk}{d}\right) e\left(-\frac{kA}{d}\right) \sum_{t' \,(\mathrm{mod}\, d)} e\left(\frac{ht'^2 + kt'}{d}\right) \\
&= \frac{M}{d} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{Mk}{d}\right) e\left(-\frac{kA}{d}\right) G(h, k; d)
\end{aligned}$$

where $\hat{f}(y) = ((\sin \pi y)/\pi y)^2$ is the Fourier transform of f and $G(h, k; d)$ is the Gauss sum. Thus

$$\begin{aligned}
R &= \frac{M}{d} \sum_{k=-\infty}^{\infty} \left(\frac{\sin(\pi k M/d)}{\pi k M/d}\right)^2 e\left(-\frac{kA}{d}\right) \sum_{h \neq 0} c(h) e(-\lambda h) G(h, k; d) \\
&= \frac{M}{d} \left(\sum_{|k| < d/M} + \sum_{d/M \leq |k| < L} + \sum_{|k| \geq L} \right)
\end{aligned}$$

where $L = (d/M)^\sigma$ for some parameter $\sigma > 1$ to be chosen later. At this point, the proof proceeds almost identically to that of [1, Lemma 6.1] by estimating $\sum_{|k| \geq L}$ trivially and the other two sums using Lemma 2. Apart from different choices of letters for the variables, the only difference in the argument is when applying partial summation, as we have

$$\begin{aligned}
\sum_{0 \leq k < d/M} &= \int_0^{d/M} \left(\frac{\sin(\pi u M/d)}{\pi u M/d}\right)^2 \\
&\quad \times d\left(\sum_{0 \leq k < u} e\left(-\frac{kA}{d}\right) \sum_{h \neq 0} c(h) e(-\lambda h) G(h, k; d) \right)
\end{aligned}$$

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for example (in contrast to [1, equation (13)]. Also we shall keep σ (i.e. L) in our bound instead of choosing $\sigma = 5$ in [1]. Consequently, we have

$$R \ll_\epsilon d^\epsilon \log \frac{d}{\Delta} \log L \left[M^{1/2}\Delta^{1/2} + \frac{M\Delta^{1/4}}{d^{1/2}} + \Delta d^{1/2} + 1 + \frac{d\Delta}{M} + \frac{d^2}{\Delta ML} \right].$$

Thus for $\Delta \geq 1/d$ and $a \leq N^2 d^2$, there exists some constant $C_\epsilon > 0$ such that

$$R \leq C_\epsilon \sigma d^{2\epsilon} \left[M^{1/2}\Delta^{1/2} + \frac{M\Delta^{1/4}}{d^{1/2}} + \Delta d^{1/2} + 1 + \frac{d\Delta}{M} + \frac{d^2}{\Delta ML} \right].$$

We are going to make each of the six pieces of the above bound for R less than $\Delta M/12$. Then it follows that $S > \Delta M/2 > 0$. The six pieces of requirements are achieved if (i) $\Delta \gg (\sqrt{a}/Nd)d^{4\epsilon}$, (ii) $\Delta \gg d^{3\epsilon}/d^{2/3}$, (iii) $a \ll N^2 d^{1-4\epsilon}$, (iv) $\Delta \gg (\sqrt{a}/Nd)d^{2\epsilon}$, (v) $a \ll N^2 d^{1-2\epsilon}$, and (vi) $\Delta \gg (N^{\sigma/2-1}/a^{\sigma/4-1/2})d^\epsilon$ respectively where the implicit constants may depend on ϵ and σ. Since ϵ is arbitrary, the above can be summarized as

$$\Delta \gg \frac{\sqrt{a}}{Nd} d^\epsilon, \qquad \Delta \gg \frac{1}{d^{2/3-\epsilon}}, \qquad a \ll N^2 d^{1-\epsilon}, \qquad \Delta \gg \frac{N^{\sigma/2-1}}{a^{\sigma/4-1/2}} d^\epsilon.$$

Now consider $N^2 d^{2/3} \leq a \ll_\epsilon N^2 d^{1-\epsilon}$. The first lower bound on Δ is the biggest by taking σ large enough. Therefore, there exist some $m \in [-M, M]$ and $n \in [N/4, 3N/4]$ such that

$$|(A+m)^2 - (a+nd)| < \Delta d \ll_\epsilon \frac{\sqrt{a}}{N} d^\epsilon$$

which proves Theorem 3. □

References

1. T. H. Chan, Finding almost squares. *Acta Arith.* **121**(3) (2006), 221–232.
2. J. B. Friedlander and H. Iwaniec, On the distribution of the sequence $n^2\theta$ (mod 1). *Canad. J. Math.* **39** (1987), 338–344.
3. M. N. Huxley, The integer points close to a curve. II. In *Analytic Number Theory,* Vol. 2 *(Allerton Park, IL, 1995)* (*Progress in Mathematics* **139**), Birkhäuser Boston (Boston, MA, 1996), 487–516.

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