COMPARING VARIANTS OF RAMSEY'S THEOREM USING UNIFORM REDUCIBILITIES

JUN LE GOH, ELLEN HAMMATT, HEER TERN KOH, AND KENG MENG NG

Abstract. We study the uniform computational content of Ramsey's theorem using both Weihrauch reducibility and a new variant of Weihrauch reducibility, where the functions in the reduction are required to be total. In the latter setting, we show that the strength of Ramsey's theorem varies significantly depending on how one represents its solutions, for example, using characteristic functions or using enumerations. Some of our results extend beyond variants of Ramsey's theorem. In particular, we show that RT^2_2 where solutions are represented using characteristic functions is not totally Weihrauch reducible to any computational problem whose solutions are represented using enumerations. Next, we study the computational problems RT^n_∞ which take as input a colouring of n-tuples which has some infinite homogeneous set and asks for any such set. The problem RT^1_∞ is fairly well-studied, as it is Weihrauch equivalent to the cluster point problem on $\mathbb{N}$. Our main result shows that RT^1_∞ is not Weihrauch reducible to $\mathsf{RT}^2_{\mathbb{N}}$, strengthening a result of Soldà and Valenti. We also show that the jump of RT^1_∞ is Weihrauch equivalent to SRT^2_∞.

§1. Introduction. Ramsey's theorem for cardinality n and k colours (RT^n_k) states that any k-colouring c of the size n subsets of natural numbers $[\mathbb{N}]^n$ is constant on all size n subsets of some infinite set $H \subseteq \mathbb{N}$. Such a set H is said to be homogeneous (for c). We view RT^n_k as a computational problem with instances c and corresponding solutions H. Specker constructed a computable 2-colouring of pairs with no infinite computable solution. In particular, Ramsey's theorem fails in a universe where all sets are computable. Jockusch [12] then established better bounds on the complexity of solutions, thereby initiating the endeavour of analysing Ramsey's theorem using various yardsticks of complexity, ranging from computability-theoretic (notions of reducibility) to proof-theoretic (notions of provability).

Much recent attention has been given to reductions which are (uniformly) computable, not merely pointwise computable. Among these, the most prominent is Weihrauch reducibility. The advantage of such frameworks is twofold: first, "natural" reductions in mathematics tend to be uniformly computable, if computable at all; second, uniformly computable reductions can explain differences between problems which are pointwise computable (a.k.a. computably true) or "computably equivalent," such as the infinite pigeonhole principles RT^1_k.

Different reductions to Ramsey's theorem utilise different aspects of the given homogeneous set. Some reductions only require the homogeneous colour, which

Received April 12, 2026 UTC.

2020 *Mathematics Subject Classification.* Primary 03D78, 05D10, Secondary 03D55, 03D80.

Key words and phrases. Ramsey's theorem, Weihrauch reducibility, total Weihrauch reducibility, stable Ramsey's theorem, choice principles, guessability.

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0022-4812/00/0000-0000
DOI:10.1017/jsl.2026.10220

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can be computed with access to finitely many elements of the solution and the colouring. On the other extreme, some reductions make use of a principal enumeration of the solution (as a fast-growing function). In a Weihrauch reduction to Ramsey's theorem, it does not matter whether the homogeneous set is given as a principal enumeration, an enumeration, or a characteristic function, because one can always compute a principal enumeration by unbounded search. However, such a procedure is not total: the search fails to terminate if its input is an enumeration or characteristic function of a finite set. This motivates us to consider a variant of Weihrauch reducibility where the reduction functionals are required to be total. We call this notion total Weihrauch reducibility. In this article we systematically compare variants of Ramsey's theorem under total Weihrauch reducibility.

The variants of Ramsey's theorem that we study are defined, roughly, as follows. An instance of each of the problems $\mathsf{RT}^n_\infty\chi$, $\mathsf{RT}^n_\infty\mathsf{c}$, $\mathsf{RT}^n_\infty\mathsf{e}$, and $\mathsf{RT}^n_\infty\mathsf{pe}$ is a colouring $c : [\mathbb{N}]^n \to \mathbb{N}$ which has some infinite homogeneous set $H \subseteq \mathbb{N}$. We do not require c to have bounded range. An $\mathsf{RT}^n_\infty\chi$-solution to c is the characteristic function of any such H. An $\mathsf{RT}^n_\infty\mathsf{c}$-solution to c is the colour of any such H. An $\mathsf{RT}^n_\infty\mathsf{e}$-solution to c is an enumeration of any such H, i.e., a function $f : \mathbb{N} \to \mathbb{N}$ whose range is H. An $\mathsf{RT}^n_\infty\mathsf{pe}$-solution to c is the principal enumeration of any such H, i.e., the increasing function $f : \mathbb{N} \to \mathbb{N}$ whose range is H.

For $k \in \mathbb{N} \cup \{\mathbb{N}\}$, $\mathsf{RT}^n_k\chi$ is the restriction of $\mathsf{RT}^n_\infty\chi$ to colourings with range bounded by k (for $k \in \mathbb{N}$) or to colourings with bounded range (for $k = \mathbb{N}$). $\mathsf{RT}^n_{<\infty}\chi$ is the variant of $\mathsf{RT}^n_{\mathbb{N}}\chi$ with instances being pairs (c, k) such that k bounds the range of c. For $k \in \mathbb{N} \cup \{\mathbb{N}\}$, $\mathsf{RT}^n_k\mathsf{c}$, $\mathsf{RT}^n_k\mathsf{e}$, and $\mathsf{RT}^n_k\mathsf{pe}$ are defined analogously.

We summarise our results as follows. The pigeonhole principles (i.e., those involving colourings of singletons) form a grid (Figure 1), where the vertical axis corresponds to the number of colours allowed and the horizontal axis corresponds to how solutions are represented, with $\mathsf{RT}^1_k\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{c} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{pe}$. All such reductions turn out to be strict except for the equivalences $\mathsf{RT}^1_{\mathbb{N}}\mathsf{e} \equiv^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}\mathsf{c}$ and $\mathsf{RT}^1_\infty\mathsf{e} \equiv^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_\infty\mathsf{c}$.

Apart from the aforementioned equivalences, our results suggest that the columns of the grid should be thought of as far apart. It is easy to see that $\mathsf{RT}^1_2\mathsf{c}$ is not reducible to $\mathsf{RT}^n_\infty\chi$ (Corollary 3.3). Using a notion of guessability (where guesses need to be total), we show that $\mathsf{RT}^1_2\mathsf{e}$ is not reducible to $\mathsf{RT}^n_{<\infty}\mathsf{c}$ (Corollary 3.7). The gap between enumerations and principal enumerations extends to problems beyond Ramsey's theorem: $\mathsf{RT}^1_2\mathsf{pe}$ is not reducible to any problem with solutions represented by enumerations (Proposition 3.9).

The picture for problems for colourings of pairs (and longer tuples) differs significantly from that for colourings of singletons. One expects $\mathsf{RT}^n_k\mathsf{c}$ to be weak, since there is no obvious way of recovering a homogeneous set for a colouring of n-tuples (for $n \geq 2$) from its colour. We confirm this expectation by showing that $\mathsf{RT}^1_3\chi$ (and therefore $\mathsf{RT}^2_2\mathsf{e}$ and $\mathsf{RT}^2_2\chi$) is not reducible to $\mathsf{RT}^n_2\mathsf{c}$ (Corollary 3.5). In fact, $\mathsf{RT}^2_2\chi$, $\mathsf{RT}^2_2\mathsf{c}$, and $\mathsf{RT}^2_2\mathsf{e}$ are incomparable, in contrast to $\mathsf{RT}^1_2\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_2\mathsf{c} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_2\mathsf{e}$. Among these separations, the most technical is perhaps our proof (using a priority argument) that $\mathsf{RT}^2_2\chi$ is not reducible to any problem with solutions represented by enumerations (Theorem 3.17).

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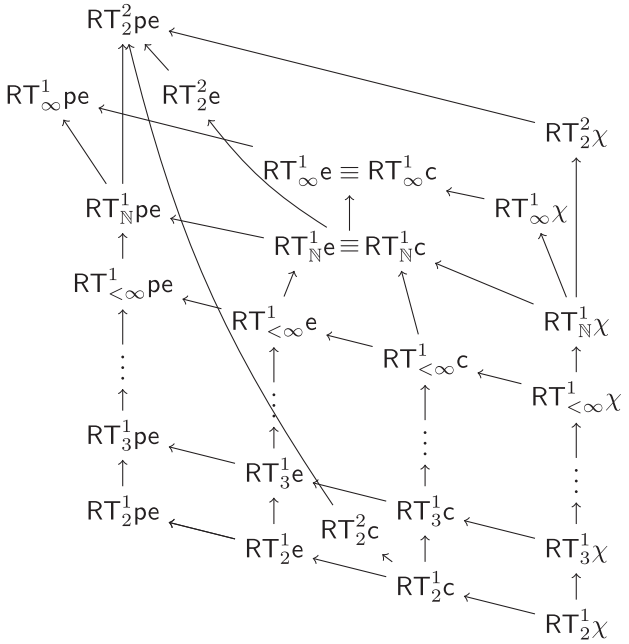


FIGURE 1. There are no additional $\leq^{\mathrm{t}}_{\mathrm{W}}$-reductions other than those implied by transitivity.

We turn to results concerning the RT^n_∞ problems. Here we view our primary contribution as the separation $\mathsf{RT}^1_\infty \not\leq_{\mathrm{W}} \mathsf{RT}^2_\mathbb{N}$ (Lemma 4.17), strengthening the $n = 1$ case of a result of Soldà and Valenti [15, Theorem 7.20]. We use a priority argument rather than algebraic properties of the Weihrauch degrees. The right-hand side of the aforementioned separation is in some sense optimal because RT^1_∞ is Weihrauch reducible to RT^3_2. This reduction may have been known but we did not find a valid proof in the literature. In any case, we refine the Weihrauch reduction to a total Weihrauch reduction $\mathsf{RT}^1_\infty\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^3_2\mathsf{pe}$ (Theorem 3.25).

We have further results delineating the difference in strength between $\mathsf{RT}^n_\infty\mathsf{e}$ and $\mathsf{RT}^n_\mathbb{N}\mathsf{e}$, at least for $n = 1, 2$. One such result states that $\mathsf{RT}^2_2\mathsf{c}$ (even LPO') is not reducible to $\mathsf{RT}^2_\mathbb{N}\mathsf{e}$ (even $\mathsf{RT}^n_\mathbb{N}\mathsf{e}$). Since $\mathsf{RT}^n_\infty\mathsf{c} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_\infty\mathsf{e}$ (Theorem 3.1(5)), this suggests that $\mathsf{RT}^2_\infty\mathsf{e}$ is significantly stronger than $\mathsf{RT}^2_\mathbb{N}\mathsf{e}$. We also show that $\mathsf{LPO}' \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_\infty\mathsf{e}$ (Proposition 3.15), which, when combined with the above separation, implies that $\mathsf{RT}^1_\infty\mathsf{e}$ (equivalently, $\mathsf{RT}^1_\infty\mathsf{c}$) is not reducible to $\mathsf{RT}^n_\mathbb{N}\mathsf{e}$.

Next, we discuss a result in the continuous Weihrauch degrees. It is well-known that RT^2_2 is not Weihrauch reducible to RT^1_∞, as the latter is computably true while the former is not. We strengthen this by showing that there is no continuous Weihrauch reduction (Corollary 3.23). Our proof (using a priority argument) shows that RT^2_2 is not continuously guessable in a certain sense.

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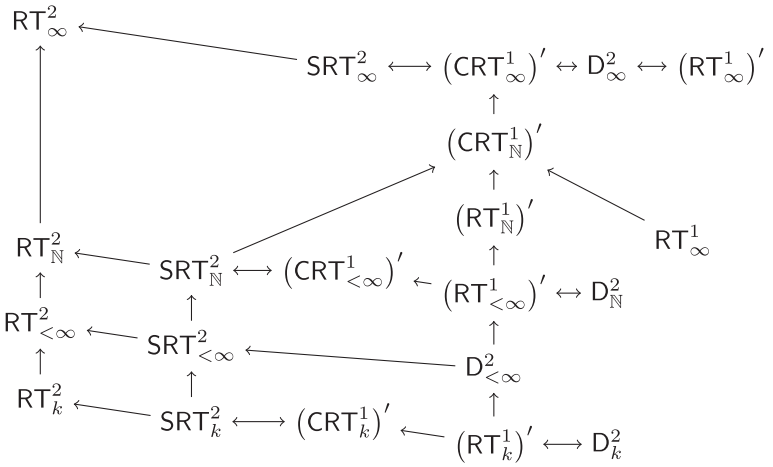


FIGURE 2. There are no additional $\leq_{\mathrm{W}}$-reductions other than those implied by transitivity.

Finally, we return to the setting of Weihrauch reducibility in Section 4. Our results completely determine the relationships between $\mathsf{RT}^2_\mathbb{N}$, $\mathsf{SRT}^2_\mathbb{N}$, and the jumps $\mathsf{C}'_\mathbb{N}$ ($\equiv_{\mathrm{W}} \mathsf{RT}^1_\infty$), $(\mathsf{RT}^1_{<\infty})'$, $(\mathsf{RT}^1_\mathbb{N})'$, and $(\mathsf{CRT}^1_\mathbb{N})'$, as depicted in Figure 2. One such result $\mathsf{RT}^1_\infty \not\leq_{\mathrm{W}} \mathsf{RT}^2_\mathbb{N}$ (Lemma 4.17) was mentioned above. Another notable result is the separation $(\mathsf{RT}^1_\mathbb{N})' \not\leq_{\mathrm{W}} \mathsf{RT}^2_\mathbb{N}$ (Lemma 4.1). This separation implies that $(\mathsf{CRT}^1_\mathbb{N})'$ cannot be Weihrauch equivalent to $\mathsf{SRT}^2_\mathbb{N}$, contrary to [7, Theorem 4.3] for $n = 1$, $k = \mathbb{N}$. Instead, we show that $\mathsf{SRT}^2_\mathbb{N}$ is Weihrauch equivalent to $(\mathsf{CRT}^1_{<\infty})'$ (Theorem 4.3). We also have results suggesting that $(\mathsf{RT}^1_\mathbb{N})'$ is much weaker than $(\mathsf{CRT}^1_\mathbb{N})'$: for all $k \in \mathbb{N}$, $(\mathsf{CRT}^1_\mathbb{N})'$ is not reducible to the iterated jump $(\mathsf{RT}^1_\mathbb{N})^{(k)}$ (Corollaries 4.8 and 4.16 both imply this). Our proof shows furthermore that there can be no omniscient continuous Weihrauch reduction.

In Section 4, we also study the problems SRT^2_∞ and D^2_∞. We show that they are Weihrauch equivalent to $(\mathsf{RT}^1_\infty)'$ and $(\mathsf{CRT}^1_\infty)'$ (Theorem 4.14).

§2. Preliminaries. We briefly recall notions from the theory of Weihrauch reducibility. We refer the reader to [7] for a detailed introduction.

To compare mathematical problems under Weihrauch reducibility, we begin by formalising them as partial multivalued functions $f :\subseteq X \rightrightarrows Y$, where X and Y are represented spaces (as defined below). We refer to such f as *problems*. We refer to each $x \in \mathrm{dom}(f)$ as an (f-)*instance* and each element of $f(x) \subseteq Y$ as an (f-)*solution (to x)*. If f is single-valued, we will abuse notation and refer to the unique solution to $x \in \mathrm{dom}(f)$ as $f(x)$.

A *represented space* (X, δ_X) is a set X together with a *representation* $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ that assigns *names* $p \in \mathbb{N}^\mathbb{N}$ to the element $\delta(p) \in X$. The partial function δ_X is required to be surjective, i.e., every element of X has some name (possibly many). If δ_X is clear from context, we typically omit it by writing X instead of (X, δ_X).

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In our proofs, we often abuse notation by referring to $p \in \mathbb{N}^{\mathbb{N}}$ as an element of X. For example, we may say that $p \in \mathbb{N}^{\mathbb{N}}$ is an f-instance whenever $\delta_X(p) \in \mathrm{dom}(f)$.

We say that $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a *realiser* for $f :\subseteq X \rightrightarrows Y$ if whenever p names some $x \in X$, then $F(p)$ names some element of $f(x)$. Notions such as computability or continuity of f can be defined via realisers, e.g., f is computable if it has a computable realiser F.

There are several benchmark problems, to which one compares problems of interest. Let $0^{\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ denote the sequence which is constantly 0. The following single-valued problems will appear in this article:

- $\mathsf{LPO} : \mathbb{N}^{\mathbb{N}} \to 2$, $\mathsf{LPO}(0^{\mathbb{N}}) = 0$ and $\mathsf{LPO}(p) = 1$ otherwise;
- $\lim :\subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, $\lim((p_n)_{n\in\mathbb{N}}) = \lim_{n\in\mathbb{N}} p_n$ if the limit exists.

Another important class of problems is the class of choice problems. While such problems can be formulated for a wide variety of spaces, we only need the following. For $k \in \mathbb{N}_{\geq 2} \cup \{\mathbb{N}\}$, the choice problem C_k is defined roughly as follows: Given an enumeration of a proper subset of k, produce any number which is not enumerated (see [5, Section 11.7] for a precise definition).

Definition 2.1. A problem f is *Weihrauch reducible* to a problem g, written $f \leq_{\mathrm{W}} g$, if there are computable functions $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that if p is a name for some $x \in \mathrm{dom}(f)$, then:

1. $\Phi(p)$ is a name for some $y \in \mathrm{dom}(g)$;
2. if q is a name for some g-solution of y, then $\Psi(q, p)$ is a name for some f-solution of x.

We say that f is *continuously Weihrauch reducible* to g, written $f \leq^*_{\mathrm{W}} g$, if there are continuous functions $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ satisfying (1) and (2).

We say that f is *strongly Weihrauch reducible* to g, written $f \leq_{\mathrm{sW}} g$, if there are computable functions $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ as above, satisfying (1) and

(3) if q is a name for some g-solution of y, then $\Psi(q)$ is a name for some f-solution of x.

In other words, Ψ does not have access to the initial f-instance p, only some g-solution q of y.

A problem f is *totally Weihrauch reducible* to a problem g, written $f \leq^{\mathrm{t}}_{\mathrm{W}} g$, if there are total computable functions $\Phi, \Psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that (1) and (2) hold.

In all the above reductions, we say that f is reducible to g via Φ and Ψ. We refer to Φ and Ψ as the pre-processing function and the post-processing function, respectively.

Observe that if $f \leq^{\mathrm{t}}_{\mathrm{W}} g$, then $f \leq_{\mathrm{W}} g$. To our knowledge, the reducibility $\leq^{\mathrm{t}}_{\mathrm{W}}$ is new and should not be confused with the reducibility $\leq_{\mathrm{tW}}$ defined by Brattka and Gherardi [3]. Roughly speaking, $f \leq_{\mathrm{tW}} g$ only requires that Φ and Ψ work on total realisers of g.

Remark 2.2. $\leq^{\mathrm{t}}_{\mathrm{W}}$ is *not* invariant under computably equivalent representations. To see this, recall [3, Examples 3.2 and 4.4]: For $p \in \mathbb{N}^{\mathbb{N}}$ such that $p(n) \geq 1$ for infinitely many n, let $p - 1 \in \mathbb{N}^{\mathbb{N}}$ denote the sequence which is obtained by concatenating $p(0) - 1, p(1) - 1, \ldots$, where -1 is treated as the empty string. Define

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the representation $\mathsf{id}^{\wp} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $\mathsf{id}^{\wp}(p) = p - 1$. It is easy to see that the representations $\mathsf{id}^{\wp}$ and id are computably equivalent.

Let $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a computable partial function with no total computable extension. We consider the problems $F :\subseteq (\mathbb{N}^{\mathbb{N}}, \mathsf{id}) \to (\mathbb{N}^{\mathbb{N}}, \mathsf{id})$ and $F_{\mathsf{id},\mathsf{id}^{\wp}} :\subseteq (\mathbb{N}^{\mathbb{N}}, \mathsf{id}) \to (\mathbb{N}^{\mathbb{N}}, \mathsf{id}^{\wp})$. The former has no total computable realiser, while the latter does (given $p \in \mathbb{N}^{\mathbb{N}}$, attempt to compute $F(p)(n) + 1$ for $n = 0, 1, \ldots$ in order; output 0 while waiting). Since the property of having a total computable realiser is easily seen to be preserved downwards under $\leq_{\mathrm{W}}^{\mathrm{t}}$, we deduce that $F \not\leq_{\mathrm{W}}^{\mathrm{t}} F_{\mathsf{id},\mathsf{id}^{\wp}}$.

Observe that $\leq_{\mathrm{W}}^{\mathrm{t}}$ is a pre-order. We denote the induced equivalence relation by $\equiv_{\mathrm{W}}^{\mathrm{t}}$. An equivalence class under $\equiv_{\mathrm{W}}^{\mathrm{t}}$ is called a $\equiv_{\mathrm{W}}^{\mathrm{t}}$-degree. As usual the pre-order $\leq_{\mathrm{W}}^{\mathrm{t}}$ induces a partial order on the $\equiv_{\mathrm{W}}^{\mathrm{t}}$-degrees, which we also denote by $\leq_{\mathrm{W}}^{\mathrm{t}}$.

Unlike $\leq_{\mathrm{W}}^{\mathrm{t}}$, the reducibilities $\leq_{\mathrm{W}}$, $\leq_{\mathrm{W}}^{*}$, and $\leq_{\mathrm{sW}}$ are well-studied. Our interest in $\leq_{\mathrm{W}}^{*}$ and $\leq_{\mathrm{sW}}$ is purely technical, insofar as they help us understand $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{W}}^{\mathrm{t}}$. We next list some well-known constructions and results.

For each $k \in \mathbb{N}_{\geq 1} \cup \{\mathbb{N}\}$, we fix computable pairing functions $\langle\cdot\rangle : (\mathbb{N}^{\mathbb{N}})^k \to \mathbb{N}^{\mathbb{N}}$. We denote the concatenation of finite strings or of finite strings and elements of $\mathbb{N}^{\mathbb{N}}$ using $^\frown$.

DEFINITION 2.3 (Constructions on representations). Suppose (X, δ_X) and (Y, δ_Y) are represented spaces.

- We represent elements of $X \times Y$ by $\langle p, q\rangle \mapsto (\delta_X(p), \delta_Y(q))$.
- For $k \in \mathbb{N}_{\geq 1} \cup \{\mathbb{N}\}$, we represent elements of X^k by $\langle p_i\rangle_{i\in k} \mapsto (\delta_X(p_i))_{i\in k}$.
- Let $X^* = \bigcup_{i\in\mathbb{N}}(\{k\} \times X^k)$. We represent elements of X^* by $k^\frown\langle p_i\rangle_{i\in k} \mapsto (k, p_0, \ldots, p_{k-1})$.
- The *jump* X' of the represented space (X, δ) is the same underlying set X with the representation $\delta' = \delta \circ \lim$.
- Recall the definition of $p - 1$ from Remark 2.2. The *completion* of (X, δ_X) is the space $\overline{X} = X \cup \{\bot\}$ (assuming $\bot \notin X$) equipped with the representation $\delta_{\overline{X}} : \mathbb{N}^{\mathbb{N}} \to \overline{X}$, defined by

$$\delta_{\overline{X}}(p) = \begin{cases} \delta_X(p-1) & p - 1 \text{ defined and } p - 1 \in \mathrm{dom}(\delta_X) \\ \bot & \text{otherwise.} \end{cases}$$

DEFINITION 2.4 (Constructions on problems). Suppose we have problems $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$.

- The *parallel product* $f \times g$ is defined by $f \times g :\subseteq X \times Z \rightrightarrows Y \times W$, $(f \times g)(x, z) = f(x) \times g(z)$, and $\mathrm{dom}(f \times g) = \mathrm{dom}(f) \times \mathrm{dom}(g)$.
- The *(infinite) parallelisation* $\widehat{f}$ of f is defined by $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$, $\widehat{f}((x_n)_n) = \prod_{n\in\mathbb{N}} f(x_n)$, and $\mathrm{dom}(\widehat{f}) = \mathrm{dom}(f)^{\mathbb{N}}$.
- The *jump* f' of f is defined to be the same problem as f, except that the domain is X' instead of X. In other words, a sequence $(p_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ names an instance w of f' if $\lim_n p_n$ exists and names some f-instance x. The f'-solutions for w are exactly the f-solutions for x. Note that each p_n may not name an f-instance, though for some problems one can assume this without loss of generality.
- Iterated jumps are denoted by $f^{(k)}$, e.g., $f^{(3)} = f'''$.

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FACT 2.5. *Let f and g be problems.*

1. *If $f \leq_{\mathrm{sW}} g$, then $f' \leq_{\mathrm{sW}} g'$. If $f \not\leq^*_{\mathrm{W}} g$, then $f' \not\leq^*_{\mathrm{W}} g'$.*
2. *If $f \leq_{\mathrm{W}} g$, then $\widehat{f} \leq_{\mathrm{W}} \widehat{g}$.*
3. *$\widehat{f \times g} \equiv_{\mathrm{sW}} \widehat{f} \times \widehat{g}$.*

PROOF. The first part of (1) was proved in [4, Proposition 5.6]. The second part of (1) follows from [6, Theorem 11] and the fact that every continuous function is computable relative to some $p \in \mathbb{N}^{\mathbb{N}}$. (2) and (3) were proved in [2, Propositions 4.2 and 4.5]. ⊣

We now introduce the variants of Ramsey's theorem studied in this article. Fix a computable family of bijections $\theta_n : \mathbb{N} \to [\mathbb{N}]^n$. Let $\mathcal{C}_n$ denote the space of all functions from $[\mathbb{N}]^n$ to $\mathbb{N}$, thought of as colourings, represented by $\delta_n : p \mapsto (\theta_n(i) \mapsto p(i))$. These spaces contain the instances for our variants of Ramsey's theorem. As for the solutions, for each $c \in \mathcal{C}_n$, let $\mathcal{H}_c$ be the set of infinite c-homogeneous subsets of $\mathbb{N}$. We represent $\mathcal{H}_c$ in the following ways:

- $\delta_\chi :\subseteq 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$, where $\delta_\chi(p)$ is the set whose characteristic function is p;
- $\delta_e : \mathbb{N}^{\mathbb{N}} \to \{\text{non-empty subsets of } \mathbb{N}\}$, where $\delta_e(p)$ is the range of p;
- the restriction $\delta_{pe} :\subseteq \mathbb{N}^{\mathbb{N}} \to \{\text{infinite subsets of } \mathbb{N}\}$ of δ_e to strictly increasing functions p.

For each non-empty $A \subseteq \mathbb{N}$, the elements of $\delta_e^{-1}(A)$ are *enumerations* of A. If A is infinite, the unique strictly increasing element of $\delta_e^{-1}(A)$ is called the *principal enumeration* of A.

DEFINITION 2.6. For all $n \geq 1$, we define:

- $\mathsf{RT}^n_\infty\chi :\subseteq (\mathcal{C}_n, \delta_n) \rightrightarrows (\mathcal{P}(\mathbb{N}), \delta_\chi)$, $\mathsf{RT}^n_\infty\chi(c) = \mathcal{H}_c$;
- $\mathsf{RT}^n_\infty\mathsf{c} :\subseteq (\mathcal{C}_n, \delta_n) \rightrightarrows \mathbb{N}$, $\mathsf{RT}^n_\infty\mathsf{c}(c)$ consists of all colours j such that c has an infinite homogeneous set of colour j;
- $\mathsf{RT}^n_\infty\mathsf{e} :\subseteq (\mathcal{C}_n, \delta_n) \rightrightarrows (\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}, \delta_e)$, $\mathsf{RT}^n_\infty\mathsf{e}(c) = \mathcal{H}_c$;
- $\mathsf{RT}^n_\infty\mathsf{pe} :\subseteq (\mathcal{C}_n, \delta_n) \rightrightarrows (\{\text{infinite subsets of } \mathbb{N}\}, \delta_{pe})$, $\mathsf{RT}^n_\infty\mathsf{pe}(c) = \mathcal{H}_c$;
- for $k \geq 2$ ($k = \mathbb{N}$ resp.), $\mathsf{RT}^n_k\chi$, $\mathsf{RT}^n_k\mathsf{c}$, $\mathsf{RT}^n_k\mathsf{e}$, and $\mathsf{RT}^n_k\mathsf{pe}$ are defined by restricting the respective problem above to the colourings with codomain k (finite range resp.);
- $\mathsf{RT}^n_{<\infty}\chi :\subseteq (\mathcal{C}_n, \delta_n) \times \mathbb{N} \rightrightarrows (\mathcal{P}(\mathbb{N}), \delta_\chi)$, where for all $c : [\mathbb{N}]^n \to k$, $\mathsf{RT}^n_{<\infty}\chi(c, k) = \mathcal{H}_c$. $\mathsf{RT}^n_{<\infty}\mathsf{c}$, $\mathsf{RT}^n_{<\infty}\mathsf{e}$, and $\mathsf{RT}^n_{<\infty}\mathsf{pe}$ are defined analogously.

Many of the above problems are Weihrauch equivalent to well-studied problems. For $k \in \mathbb{N}_{\geq 2} \cup \{<\infty, \mathbb{N}\}$, $\mathsf{RT}^n_k\chi$, $\mathsf{RT}^n_k\mathsf{e}$, and $\mathsf{RT}^n_k\mathsf{pe}$ are easily seen to be (strongly) Weihrauch equivalent to RT^n_k. On the other hand, $\mathsf{RT}^n_k\mathsf{c}$ is related to the following choice problems. For $k \in \mathbb{N}_{\geq 2}$, $\mathsf{RT}^1_k\mathsf{c}$ equals the cluster point problem CL_k, which is (strongly) Weihrauch equivalent to the jump C'_k of C_k [4, Theorem 9.4]. Similarly, $\mathsf{RT}^1_\infty\mathsf{c}$ equals $\mathsf{CL}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}'_{\mathbb{N}}$. For all $k \in \mathbb{N}_{\geq 2} \cup \{<\infty, \mathbb{N}\}$, observe that $\mathsf{RT}^1_k\mathsf{c}$ is Weihrauch equivalent to RT^1_k (and so to $\mathsf{RT}^1_k\chi$, $\mathsf{RT}^1_k\mathsf{e}$, and $\mathsf{RT}^1_k\mathsf{pe}$).

In Section 4, we will study some of the following problems using Weihrauch reducibility. A colouring $c : [\mathbb{N}]^n \to \mathbb{N}$ is *stable* if, for every $A \in [\mathbb{N}]^{n-1}$, the limit $\lim_{s > \max A} c(A \cup \{s\})$ exists. We call this limit the *stable (c-)colour* of A. A set $H \subseteq \mathbb{N}$ is *(c-)limit-homogeneous* if every $A \in [H]^{n-1}$ has the same stable colour.

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DEFINITION 2.7. For all $n \geq 1$, we define:

- RT^n_∞ is simply $\mathsf{RT}^n_\infty\chi$;
- $\mathsf{CRT}^n_\infty :\subseteq (\mathcal{C}_n, \delta_n) \rightrightarrows \mathbb{N} \times \mathcal{P}(\mathbb{N})$, $\mathsf{CRT}^n_\infty(c) = \{(j, X) : c \upharpoonright [X]^n = \{j\}\}$;
- SRT^n_∞ and CSRT^n_∞ are the restrictions of RT^n_∞ and CRT^n_∞ to stable colourings, respectively;
- $\mathsf{D}^n_\infty :\subseteq (\mathcal{C}_n, \delta_n) \rightrightarrows \mathcal{P}(\mathbb{N})$, $\mathsf{D}^n_\infty(c)$ is the set of infinite c-limit-homogeneous sets, where $c : [\mathbb{N}]^n \to \mathbb{N}$ is stable;
- for $k \geq 2$ ($k = \mathbb{N}$ resp.), RT^n_k, CRT^n_k, SRT^n_k, CSRT^n_k, and D^n_k are defined by restricting the respective problem above to the colourings with codomain k (finite range resp.);
- $\mathsf{RT}^n_{<\infty} :\subseteq (\mathcal{C}_n, \delta_n) \times \mathbb{N} \rightrightarrows \mathcal{P}(\mathbb{N})$, where $\mathsf{RT}^n_{<\infty}(c, k) = \mathsf{RT}^n_\infty(c)$ for all $c \in \mathrm{dom}(\mathsf{RT}^n_\infty)$ with codomain k. The problems $\mathsf{CRT}^n_{<\infty}$, $\mathsf{SRT}^n_{<\infty}$, CSRT^n_∞, and $\mathsf{D}^n_{<\infty}$ are defined analogously.

Observe that for $n \geq 1$ and $k \in \mathbb{N}_{\geq 2} \cup \{<\infty, \mathbb{N}\}$, we have $\mathsf{RT}^n_k \leq_{\mathrm{sW}} \mathsf{CRT}^n_k$, $\mathsf{SRT}^n_k \leq_{\mathrm{sW}} \mathsf{CSRT}^n_k$, and $\mathsf{SRT}^n_k \equiv_{\mathrm{W}} \mathsf{CSRT}^n_k \leq_{\mathrm{W}} \mathsf{CRT}^n_k \equiv_{\mathrm{W}} \mathsf{RT}^n_k$.

We end this preliminary section by stating a basic structural property of the $\equiv^{\mathrm{t}}_{\mathrm{W}}$-degrees. Since we do not use this property later, we refer the reader to [5, Definition 11.1.2] for relevant background.

PROPOSITION 2.8. *The $\equiv^{\mathrm{t}}_{\mathrm{W}}$-degrees form a lattice with the same meet and join operations as that of the Weihrauch lattice.*

PROOF. Let f_0 and f_1 be problems. To show that $f_0 \sqcap f_1 \leq^{\mathrm{t}}_{\mathrm{W}} f_i$, pre-process with $(p_0, p_1) \mapsto p_i$, and post-process with $((j, q), (p_0, p_1)) \mapsto q$. Suppose g is a problem such that $g \leq^{\mathrm{t}}_{\mathrm{W}} f_0, f_1$. For each i, fix Φ_i and Ψ_i which witness $g \leq^{\mathrm{t}}_{\mathrm{W}} f_i$. Then $g \leq^{\mathrm{t}}_{\mathrm{W}} f_0 \sqcap f_1$: Pre-process with $p \mapsto (\Phi_0(p), \Phi_1(p))$ and post-process with

$$((i, q), p) \mapsto \begin{cases} \Psi_i(q, p) & \text{if } i = 0, 1 \\ 0^{\mathbb{N}} & \text{otherwise.} \end{cases}$$

To show that $f_i \leq^{\mathrm{t}}_{\mathrm{W}} f_0 \sqcup f_1$, pre-process with $p \mapsto (i, p)$ and post-process with $((j, q), p) \mapsto q$. Suppose g is a problem such that $f_0, f_1 \leq^{\mathrm{t}}_{\mathrm{W}} g$. For each i, fix Φ_i and Ψ_i which witness $f_i \leq^{\mathrm{t}}_{\mathrm{W}} g$. Then $f_0 \sqcup f_1 \leq^{\mathrm{t}}_{\mathrm{W}} g$: Pre-process with

$$(i, p) \mapsto \begin{cases} \Phi_i(p) & \text{if } i = 0, 1 \\ 0^{\mathbb{N}} & \text{otherwise} \end{cases}$$

and post-process with

$$(q, (i, p)) \mapsto \begin{cases} \Psi_i(q, p) & \text{if } i = 0, 1 \\ 0^{\mathbb{N}} & \text{otherwise.} \end{cases}$$

⊣

§3. Variants of Ramsey's theorem under $\leq^{\mathrm{t}}_{\mathrm{W}}$.

THEOREM 3.1. *We have the following reductions*:

1. *For $k \in \mathbb{N}_{\geq 2} \cup \{<\infty, \mathbb{N}, \infty\}$, $\mathsf{RT}^1_k\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{c} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{pe}$.*
2. *For all n and all $k \geq 2$, $\mathsf{RT}^n_k\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{k+1}\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{<\infty}\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{\mathbb{N}}\chi$. Analogous statements hold for $\mathsf{RT}^n_k\mathsf{c}$, $\mathsf{RT}^n_k\mathsf{e}$, and $\mathsf{RT}^n_k\mathsf{pe}$.*

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3. $\mathsf{RT}^1_{\mathbb{N}}\mathsf{c} \equiv^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}\mathsf{e}$ *and* $\mathsf{RT}^1_{\infty}\mathsf{c} \equiv^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_{\infty}\mathsf{e}$.
4. *For all n and all* $k \in \mathbb{N}_{\geq 2} \cup \{<\infty, \mathbb{N}, \infty\}$, $\mathsf{RT}^n_k\chi$, $\mathsf{RT}^n_k\mathsf{c}$, *and* $\mathsf{RT}^n_k\mathsf{e}$ *are all* $\leq^{\mathrm{t}}_{\mathrm{W}}$*-reducible to* $\mathsf{RT}^n_k\mathsf{pe}$.
5. *For all n*, $\mathsf{RT}^n_{\infty}\mathsf{c} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_{\infty}\mathsf{e}$.
6. *For all n*, $\mathsf{RT}^n_{\mathbb{N}}\mathsf{pe} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^{n+1}_2\mathsf{pe}$, $\mathsf{RT}^n_{\mathbb{N}}\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^{n+1}_2\mathsf{e}$, *and* $\mathsf{RT}^n_{\mathbb{N}}\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^{n+1}_2\chi$.
7. *For all n and all* $k \in \mathbb{N}_{\geq 2} \cup \{<\infty, \mathbb{N}, \infty\}$, $\mathsf{RT}^n_k\mathsf{c} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^{n+1}_k\mathsf{c}$.

PROOF. (1): In all these desired reductions, we will pre-process with the identity. Fix $k \in \mathbb{N}_{\geq 2} \cup \{<\infty, \mathbb{N}, \infty\}$. To show $\mathsf{RT}^1_k\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{pe}$, post-process with the projection. For $\mathsf{RT}^1_k\mathsf{c} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{e}$, the post-processing takes the given colouring c and an enumeration f and returns $c(f(0))$. If f enumerates an $\mathsf{RT}^1_k\mathsf{e}$-solution to c, then $c(f(0))$ is an $\mathsf{RT}^1_k\mathsf{c}$-solution to c. Finally, to see that $\mathsf{RT}^1_k\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{c}$, the post-processing takes the given colouring c and a number j and returns the characteristic function χ defined by $\chi(x) = 1$ if $c(x) = j$, otherwise $\chi(x) = 0$. If j is an $\mathsf{RT}^1_k\mathsf{c}$-solution to c, then χ is an $\mathsf{RT}^1_k\chi$-solution to c.

(2): In all these desired reductions, we will post-process with the projection. To show $\mathsf{RT}^n_k\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{k+1}\chi$, pre-process with the identity. To show $\mathsf{RT}^n_{k+1}\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{<\infty}\chi$, the pre-processing function produces the given colouring and the bound $k+1$. To show $\mathsf{RT}^n_{<\infty}\chi \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{\mathbb{N}}\chi$, pre-process by forgetting the given bound.

(3): First, we show that $\mathsf{RT}^1_{\mathbb{N}}\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}\mathsf{c}$ and $\mathsf{RT}^1_{\infty}\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_{\infty}\mathsf{c}$. Given $c : \mathbb{N} \to \mathbb{N}$, define $d : \mathbb{N} \to \mathbb{N}$ by taking $d(x)$ to be the least number $(\leq x)$ which has the same c-colour as x. If c has bounded range, so does d. Also, if c is an $\mathsf{RT}^1_{\infty}\mathsf{e}$-instance (i.e., some colour appears infinitely often in c), then d is an $\mathsf{RT}^1_{\infty}\mathsf{c}$-instance. The post-processing takes a colouring c and a colour s_0 and produces the following enumeration f:

$$f(x) = \begin{cases} \max\{s \leq x : c(s) = c(s_0)\} & \text{if } x \geq s_0 \\ s_0 & \text{otherwise.} \end{cases}$$

If s_0 appears infinitely often in d, then infinitely many s_i will be enumerated.

(4): First notice that for any given infinite c-homogeneous set H, the principal enumeration of H can compute the characteristic function of H, an enumeration of H (via the identity function) and the colour of H (by computing the first n elements of H). Therefore, by using such a scheme for the post-processing, and the identity map for the pre-processing, one may obtain that $\mathsf{RT}^n_k\mathsf{c}, \mathsf{RT}^n_k\chi, \mathsf{RT}^n_k\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_k\mathsf{pe}$.

(5): To show that $\mathsf{RT}^n_{\infty}\mathsf{c} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_{\infty}\mathsf{e}$, let $\{p_i\}_{i\in\mathbb{N}}$ denote the primes in increasing order and $\lceil\cdot\rceil : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$ be the function such that $\lceil A\rceil = \prod_{i\in A} p_i$. This is evidently a primitive recursive embedding of $[\mathbb{N}]^{<\mathbb{N}}$ into $\mathbb{N}\setminus\{0\}$. Given $c : [\mathbb{N}]^n \to \mathbb{N}$, define $d(x,y) = 0$ if all of the following hold: (1) $x = \lceil A\rceil$, $y = \lceil B\rceil$ for some A, B finite subsets of $\mathbb{N}$; (2) $A \subseteq B$ or $B \subseteq A$; (3) A and B both have size at least n; and (4) A and B are c-homogeneous. Otherwise, define $d(x,y) = \lceil\{x,y\}\rceil$.

Assuming that c is an $\mathsf{RT}^n_{\infty}\mathsf{c}$-instance, we first show that there exists a d-homogeneous set. Let J be an infinite c-homogeneous set. Then for all sufficiently large m, $\{\lceil J_\ell\rceil : \ell \geq m\}$, where $J_\ell = \{x \in J : x \leq \ell\}$ is infinite d-homogeneous (with colour 0).

For the post-processing, given an enumeration f and a colouring c, first compute $f(0)$. If $f(0) = \lceil A\rceil$ for some A of size at least n, then apply c to the first n elements

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of A and return the resulting colour. Otherwise, return 0. If f enumerates an infinite d-homogeneous set H, then H must have d-colour 0 and $\bigcup_{\lceil A \rceil \in H} A$ is an infinite c-homogeneous set J. The above procedure then produces the c-colour of J.

(6): Given $c : [\mathbb{N}]^n \to \mathbb{N}$, the pre-processing function computes $d : [\mathbb{N}]^{n+1} \to 2$ by $d(A) = 0$ if $c \upharpoonright [A]^n$ is constant, otherwise $d(A) = 1$. If c has bounded range, then all infinite d-homogeneous sets have colour 0 because of Ramsey's theorem. The post-processing can be the projection because every d-homogeneous set of colour 0 is clearly also c-homogeneous.

(7): The pre-processing function takes a colouring $c : [\mathbb{N}]^n \to \mathbb{N}$ and computes the colouring $d : [\mathbb{N}]^{n+1} \to \mathbb{N}$ defined by $d(A) = c(A \setminus \{\max A\})$. If the pre-processing function is given some bound b, it outputs said bound as well. Notice that if b bounds the range of c, then b bounds the range of d as well (as they are equal). Also, it is easy to see that an infinite subset of $\mathbb{N}$ is c-homogeneous of colour j if and only if it is d-homogeneous of colour j. Therefore we can post-process with the projection. ⊣

Note that all pre-processing and post-processing functions defined in the proof of Theorem 3.1 are primitive recursive, not merely total computable.

The remaining results in this section will imply that there are no other reductions between the problems depicted in Figure 1:

- $\mathsf{RT}^1_2\mathsf{c} \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_\infty\chi, \mathsf{RT}^2_2\chi$ (Corollary 3.3);
- $\mathsf{RT}^1_{k+1}\chi \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{pe}$ and $\mathsf{RT}^1_{\mathbb{N}}\chi \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_{<\infty}\mathsf{pe}$ (essentially known, see Proposition 3.11);
- $\mathsf{RT}^1_3\chi \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_2\mathsf{c}$ (Corollary 3.5);
- $\mathsf{RT}^1_2\mathsf{e} \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_{<\infty}\mathsf{c}, \mathsf{RT}^2_2\mathsf{c}$ (Corollary 3.7);
- $\mathsf{RT}^1_2\mathsf{pe} \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_2\mathsf{e}, \mathsf{RT}^1_\infty\mathsf{e}$ (Corollary 3.10);
- $\mathsf{RT}^2_2\mathsf{c} \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_2\mathsf{e}, \mathsf{RT}^1_\infty\mathsf{pe}$ (Corollary 3.14 and Proposition 3.16);
- $\mathsf{RT}^2_2\chi \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_2\mathsf{e}, \mathsf{RT}^1_\infty\mathsf{pe}$ (Theorem 3.17 and Corollary 3.23);
- $\mathsf{RT}^2_2\mathsf{e} \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_\infty\mathsf{pe}$ (Corollary 3.23);
- $\mathsf{RT}^1_\infty\chi \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_2\mathsf{pe}$ (essentially known, see Proposition 3.24).

PROPOSITION 3.2. $\mathsf{C}_2 \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_\infty\chi$.

PROOF. Suppose for a contradiction that Φ and Ψ witness $\mathsf{C}_2 \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_\infty\chi$. Let g denote the empty enumeration, i.e., the C_2-instance with solution set $\{0, 1\}$. Since Ψ is total computable, we may fix some number s such that $\Psi(0^s, g \upharpoonright s) \downarrow = j$. We extend $g \upharpoonright s$ to a C_2-instance f by enumerating 0 (if $j \neq 1$) or 1 (if $j = 1$). Then j is not a C_2-solution for f, yet 0^s may be extended to an $\mathsf{RT}^n_\infty\chi$-solution to $\Phi(f)$, yielding a contradiction. ⊣

In the above proof, $\mathsf{RT}^n_\infty\chi$ can be replaced with any problem h whose codomain is represented by δ_χ with the following property.

> There is some $p \in 2^{\mathbb{N}}$ such that for every h-instance x, p lies in the closure of (the set of δ_χ-names for elements of) $h(x)$.

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COROLLARY 3.3. $\mathsf{RT}^1_2\mathsf{c} \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_\infty\chi$ *for all* n.

PROOF. By the previous proposition, it suffices to show that $\mathsf{C}_k \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_k\mathsf{c}$. The pre-processing function sends a given enumeration f to the following colouring c: $c(x)$ is the least number which has not been enumerated by f by stage x. If f is an instance of C_k, then c is an instance of $\mathsf{RT}^1_k\mathsf{c}$. We post-process by taking the projection. ⊣

PROPOSITION 3.4. *Let* $g :\subseteq X \rightrightarrows Y$ *be a problem. Suppose* $B :\subseteq X \to \overline{Y}^k$ *is a continuous function such that for all* $x \in \mathrm{dom}(g)$, *the finite sequence* $B(x)$ *contains some* g*-solution for* x. *Then* $\mathsf{RT}^1_{k+1} \not\leq^*_{\mathrm{W}} g$. *If, on the other hand, there is some such* $B :\subseteq X \to \overline{Y}^*$, *then* $\mathsf{RT}^1_{\mathbb{N}} \not\leq^*_{\mathrm{W}} g$.

PROOF. Suppose towards a contradiction that Φ and Ψ witness $\mathsf{RT}^1_{k+1} \leq^*_{\mathrm{W}} g$. We shall construct an initial segment $c_k :\subseteq \mathbb{N} \to k+1$ of an RT^1_{k+1}-instance by iterating through $j < k$, as follows. Let c_0 be the empty colouring. Having constructed c_j, we extend it to c_{j+1} as follows: If there is some finite extension c_{j+1} of c_j such that $\Psi(B(\Phi(c_{j+1}))_j, c_{j+1})(x) \downarrow= 1$ for some x, then extend c_j to the least such c_{j+1}. Otherwise, define $c_{j+1} = c_j$.

For each $j < k$, let x_j be the least number such that $\Psi(B(\Phi(c_k))_j, c_k)(x_j) \downarrow= 1$, if one exists. The set $\{c_k(x_j) : j < k\}$ is a proper subset of $k+1$, so we may fix some $i < k+1$ which differs from every $c_k(x_j)$. Define $c : \mathbb{N} \to k+1$ by extending c_k with $i^{\mathbb{N}}$. Then $\Phi(c) \in \mathrm{dom}(g)$. Fix some $j < k$ such that $B(\Phi(c))_j \in Y$ is a g-solution for $\Phi(c)$. Either x_j is undefined and for all x, it is not the case that $\Psi(B(\Phi(c))_j, c)(x) \downarrow= 1$, or x_j is defined and $\Psi(B(\Phi(c))_j, c)(x_j) = 1$. In both cases, $\Psi(B(\Phi(c))_j, c)$ is not an RT^1_{k+1}-solution for c.

To prove the second statement in the proposition, consider $B(\Phi(0^{\mathbb{N}})) \in \overline{Y}^*$. Fix some $s \in \mathbb{N}$ such that for all $\mathsf{RT}^1_{\mathbb{N}}$-instances p extending 0^s, $B(\Phi(p))$ has the same length k as $B(\Phi(0^{\mathbb{N}}))$. Let X_0 be the set of all $x \in \mathrm{dom}(g)$ such that $B(x)$ has length k. Since $\mathsf{RT}^1_{\mathbb{N}}$ is a fractal, we have $\mathsf{RT}^1_{\mathbb{N}} \leq^*_{\mathrm{W}} g \upharpoonright X_0$. However, $\mathsf{RT}^1_{k+1} \not\leq^*_{\mathrm{W}} g \upharpoonright X_0$ by applying the first statement in the proposition. This yields the desired contradiction. ⊣

For the following application, we only need the weaker version of Proposition 3.4, where $\overline{Y}^k$ and $\overline{Y}^*$ are replaced by Y^k and Y^*, respectively.

COROLLARY 3.5. *For all* n *and all* $k \geq 2$, $\mathsf{RT}^1_{k+1} \not\leq_{\mathrm{W}} \mathsf{RT}^n_k\mathsf{c}$ *and* $\mathsf{RT}^1_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{RT}^n_{<\infty}\mathsf{c}$.

PROOF. For $\mathsf{RT}^n_k\mathsf{c}$, define B to be the constant function $01\ldots(k-1)$. For $\mathsf{RT}^n_{<\infty}\mathsf{c}$, if B is given an $\mathsf{RT}^n_{<\infty}\mathsf{c}$-instance with bound b on the colours used, it returns the sequence $01\ldots(b-1)$ of length b. ⊣

PROPOSITION 3.6. *Suppose* $g :\subseteq X \rightrightarrows Y$ *is a problem and* $B :\subseteq X \to Y^*$ *is a continuous function such that for all* $x \in \mathrm{dom}(g)$, *the finite sequence* $B(x)$ *contains some* g*-solution for* x. *Then* $\mathsf{RT}^1_2\mathsf{e} \not\leq^{\mathrm{t}}_{\mathrm{W}} g$.

PROOF. Suppose towards a contradiction that Φ and Ψ witness $\mathsf{RT}^1_2\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} g$. Consider the finite sequence $B(\Phi(0^{\mathbb{N}}))$. Since Φ is computable, we may fix some s_0 such that for all c extending 0^{s_0}, the sequence $B(\Phi(c))$ has the same length k.

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Since Ψ is total, for each $j < k$, $\Psi(B(\Phi(0^{\mathbb{N}}))_j, 0^{\mathbb{N}})$ enumerates some number. Since Ψ is computable, we may fix some $s > s_0$ such that for each $j < k$, $\Psi(B(\Phi(0^s))_j, 0^s)$ enumerates some number below s. Then $0^s 1^{\mathbb{N}}$ is an $\mathsf{RT}^1_2\mathsf{e}$-instance, $\Phi(0^s 1^{\mathbb{N}})$ is a g-instance, and for every $j < k$, $\Psi(B(\Phi(0^s 1^{\mathbb{N}}))_j, 0^s 1^{\mathbb{N}})$ enumerates a number of colour 0 and is therefore not an $\mathsf{RT}^1_2\mathsf{e}$-solution for $0^s 1^{\mathbb{N}}$. This contradicts the fact that some $B(\Phi(0^s 1^{\mathbb{N}}))_j$ is a g-solution for $\Phi(0^s 1^{\mathbb{N}})$. ⊣

Corollary 3.7. $\mathsf{RT}^1_2\mathsf{e} \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{<\infty}\mathsf{c}$ *for all* n.

Remark 3.8. Propositions 3.4 and 3.6 can be reformulated using a variant of guessability (introduced by Pauly [13]). For the latter it is important that the codomain of B is Y^* instead of the completion $\overline{Y}^*$; indeed, $\mathsf{RT}^1_k\mathsf{e}$ *is* k-guessable in the sense of [13, Definition 4].

Proposition 3.9. *The following problems are not* $\leq^{\mathrm{t}}_{\mathrm{W}}$*-reducible to any problem with codomain* $(\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}, \delta_e)$: (1) $\mathsf{RT}^1_2\mathsf{pe}$ *and* (2) $\widehat{\mathsf{C}}_2$.

Proof. (1): Given a total computable function Ψ, we shall construct $c : \mathbb{N} \to 2$ such that for every $n \in \mathbb{N}$, there is some s such that $\Psi(n^s, c)$ does not extend to an $\mathsf{RT}^1_2\mathsf{pe}$-solution for c. Begin by fixing t_0 such that $\Psi(0^{t_0}, 0^{t_0})$ enumerates some number (less than t_0). Now $\Psi(0^{\mathbb{N}}, 0^{t_0}1^{\mathbb{N}})$ either enumerates some number greater than t_0, or enumerates some numbers out of order. Fix some s_0 such that $\Psi(0^{s_0}, 0^{t_0}1^{s_0})$ witnesses either of these. Define $\sigma_1 = 0^{t_0}1^{s_0}$.

In general, suppose that at stage n, we have constructed some σ_n. Fix t_n such that $\Psi(n^{t_n}, \sigma_n 0^{t_n})$ enumerates some number x (less than $|\sigma_n 0^{t_n}|$). Let $i < 2$ denote the colour $(\sigma_n 0^{t_n})(x)$. Then fix some s_n such that $\Psi(n^{s_n}, \sigma_n 0^{t_n}(1-i)^{s_n})$ enumerates some number greater than $|\sigma_n 0^{t_n}|$, or enumerates some numbers out of order. Define $\sigma_{n+1} = \sigma_n 0^{t_n}(1-i)^{s_n}$.

Finally, define $c = \bigcup_n \sigma_n$. For any non-empty set $X \subseteq \mathbb{N}$, there is some $n \in X$ and some enumeration g of X which extends n^{s_n}. Since $\Psi(n^{s_n}, c)$ cannot extend to an $\mathsf{RT}^1_2\mathsf{pe}$-solution for c, $\Psi(g, c)$ is not an $\mathsf{RT}^1_2\mathsf{pe}$-solution for c.

(2): Given a total computable function Ψ, we shall construct an instance $f = (f_i)_i$ of $\widehat{\mathsf{C}}_2$ such that for every $n \in \mathbb{N}$, there is some s such that $\Psi(n^s, f)$ does not extend to a $\widehat{\mathsf{C}}_2$-solution for f.

At the beginning of stage n, we have fully defined f_i for $i < n$. There may also be finitely many f_i which are partially defined as a finite initial segment of the empty enumeration. At stage n, let $g = (g_i)_i$ denote the $\widehat{\mathsf{C}}_2$-instance defined by $g_i = f_i$ if f_i is fully defined, otherwise g_i is the empty enumeration.

Since Ψ is total, $\Psi(n^{\mathbb{N}}, g)(n)$ is defined. We define f_n by enumerating 0 (if $\Psi(n^{\mathbb{N}}, g)(n) \neq 1$) or 1 (if $\Psi(n^{\mathbb{N}}, g)(n) = 1$). We also extend finitely many f_i (without enumerating any number) to ensure that $\Psi(n^{\mathbb{N}}, f)(n) = \Psi(n^{\mathbb{N}}, g)(n)$.

The remainder of the verification proceeds as above. ⊣

Corollary 3.10. *For all* n, $\mathsf{RT}^1_2\mathsf{pe} \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_\infty\mathsf{e}$ *and so* $\mathsf{RT}^1_2\mathsf{pe} \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_\infty\mathsf{c}$.

Proof. The second statement follows from the first, by applying Theorem 3.1(5). ⊣

Proposition 3.11. *For all* n *and all* $k \geq 2$, $\mathsf{RT}^n_{k+1}\chi \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_k\mathsf{pe}$ *and* $\mathsf{RT}^n_{\mathbb{N}}\chi \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{<\infty}\mathsf{pe}$.

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PROOF. It is known that $\mathsf{RT}^n_{k+1} \not\leq_{\mathrm{W}} \mathsf{RT}^n_k$ and $\mathsf{RT}^n_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{RT}^n_{<\infty}$ ([7, Theorem 4.22 and Corollary 4.23] and [10, Theorem 3.3]). The desired results follow from the facts that $\mathsf{RT}^n_k\chi \equiv_{\mathrm{W}} \mathsf{RT}^n_k$, $\mathsf{RT}^n_{<\infty}\mathsf{pe} \equiv_{\mathrm{W}} \mathsf{RT}^n_{<\infty}$, and $\mathsf{RT}^n_{\mathbb{N}}\chi \equiv_{\mathrm{W}} \mathsf{RT}^n_{\mathbb{N}}$. ⊣

We have now justified the relationships in Figure 1 between $\mathsf{RT}^1_k\mathsf{pe}$, $\mathsf{RT}^1_k\mathsf{e}$, $\mathsf{RT}^1_k\mathsf{c}$, and $\mathsf{RT}^1_k\chi$ for $k \in \mathbb{N} \cup \{<\infty, \mathbb{N}\}$. We turn our attention to $\mathsf{RT}^2_2\mathsf{pe}$, $\mathsf{RT}^2_2\mathsf{e}$, $\mathsf{RT}^2_2\mathsf{c}$, and $\mathsf{RT}^2_2\chi$.

PROPOSITION 3.12. *For all n and all $k \geq 2$, $(\mathsf{RT}^n_k\chi)' \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^{n+1}_k\chi$, $(\mathsf{RT}^n_k\mathsf{c})' \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^{n+1}_k\mathsf{c}$, $(\mathsf{RT}^n_k\mathsf{e})' \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^{n+1}_k\mathsf{e}$, and $(\mathsf{RT}^n_k\mathsf{pe})' \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^{n+1}_k\mathsf{pe}$.*

PROOF. Given a sequence $(c_i)_i$ (not necessarily convergent), we begin by replacing each c_i with $d_i : [\mathbb{N}]^n \to k$, defined by $d_i(A) = c_i(A)$ if $c_i(A) < k$, otherwise $d_i(A) = 0$. The result of pre-processing is the colouring $d : [\mathbb{N}]^{n+1} \to k$, defined by $d(A) = d_{\max A}(A \setminus \{\max A\})$.

We claim that if $(c_i)_i$ converges to a colouring $c : [\mathbb{N}]^n \to k$, then every infinite d-homogeneous set H is also c-homogeneous, with the same colour. This would mean that if we post-process by taking the projection, we obtain the claimed reductions. Suppose $A \in [H]^n$. Then for all $i > \max A$ in H, $d(A \cup \{i\}) = d_i(A)$ has the same colour, say $j < k$. Choose such i which is sufficiently large so that $c_i(A) = c(A) < k$. Then $d_i(A) = c_i(A) = c(A)$, so $c(A) = j$. ⊣

PROPOSITION 3.13. $\mathsf{LPO}' \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{\mathbb{N}}\mathsf{e}$ *for all n.*

PROOF. Suppose for a contradiction that Φ and Ψ witness $\mathsf{LPO}' \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^n_{\mathbb{N}}\mathsf{e}$. We shall construct an LPO'-instance $(f_s)_s$ in stages. Begin by defining $f_0 = 1^{\mathbb{N}}$. At stage s, suppose we have determined f_t for $t \leq s$. Below, we will abuse notation and use $(f_t)_{t\leq s}$ to denote an oracle for $(f_t)_{t\in\mathbb{N}}$ with the partial information $(f_t)_{t\leq s}$. That is, if a computation queries any f_t for $t > s$, then it diverges.

Case 1. If there is some $m \in \mathbb{N}$ such that $\Psi(m^s, (f_t)_{t\leq s}) \downarrow = 1$ and it is not the case that $\Psi(m^{s-1}, (f_t)_{t\leq s-1}) \downarrow = 1$, then we define f_{s+1} as follows: $f_{s+1}(x) = 0$ if $x \leq s$ and $f_{s+1}(x) = 1$ otherwise.

Case 2. If no such m exists, we define $f_{s+1} = f_s$.

This completes the construction of $(f_s)_s$. We claim that if Case 1 occurs infinitely often, then $f = \lim_s f_s$ equals $0^{\mathbb{N}}$; otherwise, f contains some 1. Suppose first that Case 1 occurs infinitely often. Given $x \in \mathbb{N}$, fix some stage $s > x$ at which Case 1 occurs. We define $f_{s+1}(x) = 0$. For all $t > s + 1$, $f_t(x) = 0$ as well, so $f(x) = 0$.

On the other hand, suppose Case 1 occurs only finitely often. For all $x \in \mathbb{N}$ which is greater than all stages at which Case 1 occurs, we have $f_s(x) = 1$ for all $s \in \mathbb{N}$.

It remains to argue that the failure of Φ and Ψ is witnessed in either situation. If Case 1 occurs infinitely often, there must be infinitely many m for which $\Psi(m^{\mathbb{N}}, (f_s)_s) = 1$. By Ramsey's theorem applied to the restriction of $\Phi(f)$ to the set of such m, some such m must be contained in some infinite $\Phi(f)$-homogeneous set. Fix such an m and fix some t such that $\Psi(m^t, (f_s)_s) \downarrow = 1$. Then, m^t may be extended to an $\mathsf{RT}^n_{\mathbb{N}}\mathsf{e}$-solution for $\Phi(f)$, but $\mathsf{LPO}(f) = 0$.

If Case 1 only occurs finitely often, then let us fix an arbitrary infinite $\Phi(f)$-homogeneous set H. Since Ψ is total, $\Psi(m^{\mathbb{N}}, (f_s)_s)$ is defined for each $m \in H$. Fix some $m \in H$ such that $\Psi(m^{\mathbb{N}}, (f_s)_s) \neq 1$. Fix some t such that $\Psi(m^t, (f_s)_s) \downarrow = \Psi(m^{\mathbb{N}}, (f_s)_s)$. Then, m^t may be extended to an $\mathsf{RT}^n_{\mathbb{N}}\mathsf{e}$-solution for $\Phi(f)$, but $\mathsf{LPO}(f) = 1$. ⊣

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It is easy to see that $\mathsf{LPO}' \leq_{\mathrm{W}}^{\mathrm{t}} (\mathsf{RT}^1_2\mathsf{c})'$, so by Propositions 3.12 and 3.13 we have the following corollary.

COROLLARY 3.14. $(\mathsf{RT}^1_2\mathsf{c})' \not\leq_{\mathrm{W}}^{\mathrm{t}} \mathsf{RT}^n_{\mathbb{N}}\mathsf{e}$ *and* $\mathsf{RT}^2_2\mathsf{c} \not\leq_{\mathrm{W}}^{\mathrm{t}} \mathsf{RT}^n_{\mathbb{N}}\mathsf{e}$ *for all* n.

The occurrence of $\mathsf{RT}^n_{\mathbb{N}}\mathsf{e}$ in Proposition 3.13 cannot be replaced with $\mathsf{RT}^n_{\infty}\mathsf{e}$. This follows from the proposition below, which refines the known reduction $\mathsf{LPO}' \leq_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}}$ [15, Propositions 7.26 and 7.29].

PROPOSITION 3.15. $\mathsf{LPO}' \leq_{\mathrm{W}}^{\mathrm{t}} \mathsf{RT}^1_{\infty}\mathsf{c}$.

PROOF. Given $(f_s)_s$, the pre-processing function defines a colouring $c : \mathbb{N} \to \mathbb{N}$ as follows. Given $x \in \mathbb{N}$, if $\min\{y < x : f_x(y) = 1\}$ exists and equals $\min\{y < x - 1 : f_{x-1}(y) = 1\}$, then define $c(x) = y + 1$. Otherwise, define $c(x) = 0$.

Suppose $(f_s)_s$ is an LPO'-instance. Let $f = \lim_s f_s$. If $\mathsf{LPO}(f) = 1$, let y be the least number such that $f(y) = 1$. Then $c(s) = y + 1$ for all sufficiently large s. On the other hand, suppose $f = 0^{\mathbb{N}}$. For each $y \in \mathbb{N}$, $c(s) \neq y + 1$ for all sufficiently large s. Furthermore, there are infinitely many s such that $c(s) = 0$: either $\{\min\{y < x : f_x(y) = 1\} : x \in \mathbb{N}\}$ is infinite, in which case $c(x) = 0$ whenever the minimum changes value, or $\{\min\{y < x : f_x(y) = 1\} : x \in \mathbb{N}\}$ is finite, in which case $c(s) = 0$ for all sufficiently large s. Therefore c is an $\mathsf{RT}^1_{\infty}\mathsf{e}$-instance.

The post-processing function works as follows. Given a colour j, we return 0 if $j = 0$ and return 1 otherwise. Our analysis in the previous paragraph implies that this works correctly. ⊣

We do not know if the right-hand side of Corollary 3.14 can be replaced by $\mathsf{RT}^n_{\infty}\mathsf{e}$. For $n = 1$, it follows from the proposition below that $(\mathsf{RT}^1_2\mathsf{c})' \not\leq_{\mathrm{W}}^{\mathrm{t}} \mathsf{RT}^1_{\infty}\mathsf{e}$ (even $\mathsf{RT}^1_{\infty}\mathsf{pe}$).

PROPOSITION 3.16. $(\mathsf{RT}^1_2\mathsf{c})' \not\leq_{\mathrm{W}} \mathsf{RT}^1_{\infty}$.

PROOF. It is known that $\mathsf{RT}^1_2\mathsf{c}$ is not continuously Weihrauch reducible to $\mathsf{C}_{\mathbb{N}}$. Since we did not find a source (specifically for the topological separation), we give a proof. Given continuous functions Φ and Ψ which witness the reduction, we shall construct a colouring $c : \mathbb{N} \to 2$ as follows.

Case 1. For all n and all finite partial colourings $\sigma :\subseteq \mathbb{N} \to 2$, there is some $\tau \supseteq \sigma$ such that $\Phi(\tau)$ enumerates n. Then we construct c in stages, as follows. At stage n, suppose we have constructed a finite partial colouring σ_n. Define σ_{n+1} to be the least extension of σ_n such that $\Phi(\sigma_{n+1})$ enumerates n. Define $c = \bigcup_n \sigma_n$. Then $\Phi(c)$ is not a $\mathsf{C}_{\mathbb{N}}$-instance because it enumerates every $n \in \mathbb{N}$.

Case 2. Otherwise, fix some n and some $\sigma :\subseteq \mathbb{N} \to 2$ such that for all $\tau \supseteq \sigma$, $\Phi(\tau)$ does not enumerate n. In particular, n is a $\mathsf{C}_{\mathbb{N}}$-solution for $\Phi(\sigma^\frown 0^{\mathbb{N}})$, so $\Psi(n, \sigma^\frown 0^{\mathbb{N}})$ produces the unique $\mathsf{RT}^1_2\mathsf{c}$-solution 0 for $\sigma^\frown 0^{\mathbb{N}}$. Fix some sufficiently large s such that $\Psi(n, \sigma^\frown 0^s)$ produces 0. Then $\Psi(n, \sigma^\frown 0^s 1^{\mathbb{N}})$ produces 0 but 0 is not an $\mathsf{RT}^1_2\mathsf{c}$-solution for $\sigma^\frown 0^s 1^{\mathbb{N}}$, yielding a contradiction.

Since $\mathsf{RT}^1_2\mathsf{c}$ is not continuously Weihrauch reducible to $\mathsf{C}_{\mathbb{N}}$, we conclude by [6, Theorem 11] that $(\mathsf{RT}^1_2\mathsf{c})' \not\leq_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{RT}^1_{\infty}$. ⊣

THEOREM 3.17. $(\mathsf{RT}^1_2\chi)'$ *is not* $\leq_{\mathrm{W}}^{\mathrm{t}}$*-reducible to any problem with codomain* $(\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}, \delta_e)$.

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PROOF. Let g be a problem with codomain $(\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}, \delta_e)$. Suppose that Φ and Ψ witness $(\mathsf{RT}^1_2\chi)' \leq^{\mathrm{t}}_{\mathrm{W}} g$. We employ a finite injury priority argument, constructing for each s, $f_s : \mathbb{N} \to \{0,1\}$ such that $f(x) = \lim_s f_s(x)$ exists for all x. In the informal description, we shall think of f as a Δ^0_2 function instead of specifying the definition of f_s for each s. Fix some ordering of all non-empty finite sets, $D_0, D_1, \ldots$, such that if $D_i \subseteq D_j$, then $i \leq j$.

Intuitively, the 'advantage' that an enumeration has over the characteristic function is that an enumeration may delay revealing (small) elements for arbitrarily long. Since Ψ is total, it must continue to reveal more bits of the characteristic function even with constant enumerations as its oracle. However, if the constant enumeration does not name a valid g-solution, Ψ is not obliged to produce a valid solution to $f = \lim_s f_s$. The tension here would be to make the enumeration 'sufficiently slow' whilst producing a potential solution to g in the limit.

The idea is that we will diagonalise against any possible g-solution X (a finite or infinite set) as follows. Initially let f be the all zero colouring. Let the sequence of finite sets $D_{i_1} \subsetneq D_{i_2} \subsetneq \cdots \subseteq X$ be such that $X = \bigcup_j D_{i_j}$, $|D_{i_j}| = j$ and $\max D_{i_{j+1}} > \max D_{i_j}$ for each j. We shall denote $\max D_k = d_k$ for each $k \in \mathbb{N}$. The goal now is to construct some enumeration δ of X so that $\Psi(\delta, (f_s)_s)$ is not the characteristic function of an f-homogeneous set. First consider the function $\Psi(d^{\mathbb{N}}_{i_1}, (f_s)_s)$. If there is some $t \in \mathbb{N}$ and $x, y \in \mathbb{N}$ such that $x < y$ and $\Psi(d^t_{i_1}, (f_s)_s)(x) = \Psi(d^t_{i_1}, (f_s)_s)(y) = 1$, then we define $f(y) = 1 - f(x)$, ensuring that $\{x, y\}$ cannot be extended to a homogeneous set. Therefore, we have successfully diagonalised against X.

However, as observed earlier, $\Psi(d^{\mathbb{N}}_{i_1}, (f_s)_s)$ is not at all obliged to produce any elements as D_{i_1} may not be a g-solution. So suppose $\Psi(d^{\mathbb{N}}_{i_1}, (f_s)_s)(x) = 1$ for at most one $x \in \mathbb{N}$. While we must give 'new information' in the enumeration (to ensure it is a valid solution), we can exploit the fact that we can force Ψ to commit on arbitrarily long initial segments of $\mathbb{N}$.

Then once we have seen that $|\{x < z \mid \Psi(d^{\mathbb{N}}_{i_1}, (f_s)_s)(x) = 1\}| < 2$ for some 'sufficiently large' z (chosen by the construction), we wish to begin adding d_{i_2} to the enumeration. That is, we begin to consider D_{i_2}. Now we carefully choose the oracle that we will consider. The idea is for us to repeat the first bit, d_{i_1}, long enough to force Ψ to commit on the string that we have seen so far before adding the next bit. That is, we will monitor $\Psi(d^t_{i_1} {}^\frown d^{\mathbb{N}}_{i_2}, (f_s)_s)$, where t is such that $\Psi(d^t_{i_1}, (f_s)_s)(x) \downarrow$ for sufficiently many x. If the characteristic function now produces at least two elements, say $x < y$, we may then define $f(y) = 1 - f(x)$ as before, ensuring that Ψ never produces an f-homogeneous set on enumerations beginning with $d^t_{i_1} {}^\frown d^{t'}_{i_2}$ for some sufficiently large t'. If the second bit does not produce such an x, y, then we will repeat this strategy to add the next bit (of the potential solution to g).

To ensure no injury to higher priority requirements, each finite set D_i of size n is given some 'restraint' $r(i)$ that it must respect. Additionally this restraint defines the length of the string we wait for Ψ to commit to on the enumeration of the first $n-1$ elements. That is, we are looking for Ψ to produce two elements x, y such that $x < y$ and $y \geq r(i)$. Then the element y cannot have been 'used' by any higher priority requirement, and we may safely define $f(y) = 1 - f(x)$ without injuring any higher priority requirements. That is, the colour of $x < r(i)$ can only be changed by $D_{i'}$

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with $i' < i$. The restraints will be maintained stage-wise and have the property that for all $i, s \in \mathbb{N}$, $r(i)[s] \leq r(i)[s+1]$ and $r(i)[s] < r(i+1)[s]$. When we act for D_i, each $r(i')$ will be kicked to some 'large' value for each $i' > i$ as is standard in a movable marker construction.

In the construction, we ensure that any solution $X \subseteq \mathbb{N}$, possibly finite, has an enumeration δ such that $\Psi(\delta, f)$ does not output a valid solution by doing the strategy as described for each finite set. The idea is that to deal with D_i, we consider the (unique) sequence of sets that build up the set one by one in order of its principal enumeration. We then use the strategy as described for this sequence of sets. We now detail the formal construction.

Construction: Fix some ordering of all finite non-empty sets, $D_0, D_1, \ldots$, such that if $D_i \subseteq D_j$, then $i \leq j$. For each D_i we will consider the following unique sequence of sets: $D_{i_1} \subsetneq D_{i_2} \subsetneq \cdots \subsetneq D_{i_n} = D_i$, where $n = |D_i|$ and $d_{i_1} < d_{i_2} < \cdots < d_{i_n}$ (recall that $d_{i_j} = \max D_{i_j}$). During the construction, each D_i will be either *active*, *inactive*, or *waiting*. These states correspond to when the strategy for D_i is currently in progress, or has already acted, or not yet begun, respectively. For D_i which are currently active, we will also use $\delta_i[s] \in \mathbb{N}^{\mathbb{N}}$ to denote the current enumeration of D_i being used as an oracle in the computation D_i is monitoring for its strategy.

At stage 0: Declare all D_i to be in the waiting state and leave each $\delta_i[0]$ undefined. Define $r(j)[0] = j$ for all $j \in \mathbb{N}$. Let f be the all zero colouring.

At stage $s+1$: Let $i \leq s+1$ be the least such that one of the following holds:

(i) D_i is currently active and there exists x, y such that $x > y \geq r(i-1)[s]$, and $\Psi(\delta_i[s], (f_t)_{t \leq s})(x) = \Psi(\delta_i[s], (f_t)_{t \leq s})(y) = 1$. For such an i, we act for D_i, declaring $f(y) = 1 - f(x)$. Also define for each e, $r(e)[s+1]$ to be $r(e)[s]$ if $e < i$, and the least number $> \max\{r(e)[s], r(e-1)[s+1]\}$ otherwise. Declare D_i to now be inactive, and for each $j > i$, declare D_j to be in the waiting state and $\delta_j[s+1]$ to be undefined. Finally, for $e \leq i$, define $\delta_e[s+1] = \delta_e[s]$ and proceed to the next stage.

(ii) D_i is in the waiting state and for each $j < n$, where $n = |D_i|$, D_{i_j} is currently active. Then do the following for the least $i < s$ such that one of the following applies (do nothing and go to the next stage if no such i exists):

- If $|D_i| > 1$, then let l be the least such that for each $x < r(i)[s]$, $\Psi(\delta_{i_{n-1}}[s] \upharpoonright l, (f_t)_{t \leq s})(x) \downarrow$. Such an l must exist, since Ψ is assumed to be total. Then define $\delta_i[s+1] = (\delta_{i_{n-1}}[s] \upharpoonright l)^\frown d_i^{\mathbb{N}}$. Finally, define for each e, $r(e)[s+1] = r(e)[s]$, and if $e \neq i$, $\delta_e[s+1] = \delta_e[s]$, whenever defined, and proceed to the next stage.
- If $|D_i| = 1$, then declare D_i active.

Verification: Now we verify that the construction works; each D_i acts only finitely many times; $f : \mathbb{N} \to \{0,1\}$ is Δ^0_2; and for any set X that is a solution to the instance $\Phi((f_s)_s)$, there is some enumeration δ of X such that $\Psi(\delta, (f_s)_s)$ is not a solution to f.

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From the construction, once D_i acts, it is declared inactive and never again acts unless it gets initialised and placed in the waiting state by some higher priority requirement. Evidently, this ensures that each D_i acts at most finitely often and thus, for each i, $\lim_s r(i)[s] < \infty$. Additionally, after stage s^*, where $r(i)[s^*] = \lim_s r(i)[s]$, $f(x)$ is never changed again for all $x < r(i)[s^*]$. Therefore, f is Δ^0_2 and $(f_s)_s$ is a valid instance to $(\mathsf{RT}^1_2\chi)'$.

Now suppose that $X \subseteq \mathbb{N}$ is a solution to the g-instance $\Phi((f_s)_s)$. Note that X can be infinite or finite. Let $D_{i_1} \subsetneq D_{i_2} \subsetneq \cdots \subseteq X$ be a sequence of finite sets such that for each n, $|D_{i_n}| = n$, $d_{i_n} < d_{i_{n+1}}$, and $X = \bigcup_j D_{i_j}$. For each j, let t_j be the least stage such that $r(i_j)[t_j] = \lim_s r(i_j)[s]$.

From the construction, for stages $s > t_j$, no D_k for any $k \leq i_j$ acts at stage s. That is to say, if D_{i_j} was declared inactive at stage t_j, then it must have acted during stage t_j. Since D_{i_j} acted, there exists x, y such that $x > y \geq r(i_j - 1)[t_j - 1]$ and $\Psi(\delta_{i_j}[t_j - 1], f)(x) = \Psi(\delta_{i_j}[t_j - 1], f)(y) = 1$ and $f(y)[t_j] = 1 - f(x)$. Then for all $e \geq i_j$ and $s \geq t_j$, $r(e)[s] > x, y$. Therefore no D_e with $e > i_j$ can further change $f(x)$ or $f(y)$. Similarly, by the definition of t_j, none of D_k for any $k < i_j$ acts after stage t_j. This implies that $f(y) = 1 - f(x)$ and therefore $\Psi(\delta_{i_j}[t_j], (f_s)_s)$ does not output a valid solution. In particular, for any set $X \supseteq D_{i_j}$, $\Psi((\delta_{i_j}[t_j] \upharpoonright l)^\frown \delta^X, (f_s)_s)$, where δ^X is the principal enumeration of X and l is chosen such that $\Psi(\delta_{i_j}[t_j] \upharpoonright l, (f_s)_s)(z) \downarrow$ for each $z < r(i_j)[t_j]$ cannot be a valid solution to f.

On the other hand, if D_{i_j} does not act during stage t_j, its state never again changes for all stages $s \geq t_j$. That is, either D_{i_j} remains forever in the waiting state, implying that some D_{i_k}, where $k < j$ is inactive (since D_{i_1} cannot be in the waiting state for cofinitely many stages), or D_{i_j} itself remains in the active state for cofinitely many stages. In the former case, D_{i_j} is necessarily a superset of some D_{i_k} which remains inactive and thus we are done. Thus, we may assume that for all j, D_{i_j} remains in the active state for all stages $s \geq t_j$. For each j, since we never act for any higher priority set after stage t_j, or for D_{i_j} itself, then for all j, $|\{x \mid \Psi(\delta_{i_j}[t_j], (f_s)_s)(x) = 1\} \cap \{y \mid y \geq \lim_s r(i_j - 1)[s]\}| = 0$. Therefore $\Psi(\lim_j \delta_{i_j}[t_j], (f_s)_s)$ produces only a finite set which cannot be a valid solution to $(f_s)_s$ but $\lim_j \delta_{i_j}[t_j]$ is an enumeration of X. ⊣

It follows from Proposition 3.12 that:

COROLLARY 3.18. *$\mathsf{RT}^2_2\chi$ is not $\leq^{\mathrm{t}}_{\mathrm{W}}$-reducible to any problem with codomain $(\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}, \delta_e)$.*

Next, observe that $\mathsf{RT}^2_2 \not\leq_{\mathrm{W}} \mathsf{RT}^n_\infty \mathsf{c}, \mathsf{RT}^1_\infty$ because the former problem is not computably true, unlike the latter problems. We show below that there are not even *continuous* Weihrauch reductions (Corollary 3.23).

DEFINITION 3.19. Let $f :\subseteq X \rightrightarrows Y$ be a problem. We define $\breve{f}_\omega :\subseteq X \rightrightarrows \overline{Y}^{\mathbb{N}}$ by $(p_n)_n \in \breve{f}_\omega(x)$ if and only if $p_n \in f(x)$ for some $n \in \mathbb{N}$. We say that f is *continuously countably guessable* if $\breve{f}_\omega$ is continuous.

REMARK 3.20. Following the treatment of completions by Pauly and Soldà [14], we note that the Weihrauch degree of $\breve{f}_\omega$ does not depend on the choice of completion $\overline{Y}$.

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REMARK 3.21. For each problem f, the problem $\check{f}_\omega$ is Weihrauch reducible to the stashing Σf [1, Definition 1.6], by pre-processing with $x \mapsto (x)_n$.

It is easy to see that

- being continuously countably guessable is closed downwards under continuous Weihrauch reduction, and
- if a problem is computably true, i.e., every instance x has an x-computable solution, then it is continuously countably guessable.

LEMMA 3.22. *$(\mathsf{RT}^1_2)'$ is not continuously countably guessable.*

PROOF. Suppose Ψ is a continuous map such that if $(c_s)_s$ is an $(\mathsf{RT}^1_2)'$-instance, then $\Psi((c_s)_s) = (p_n)_n$, where each $p_n \in \overline{\mathbb{N}^{\mathbb{N}}}$. We shall construct an $(\mathsf{RT}^1_2)'$-instance $(c_s)_s$ with limit colouring c such that all requirements

$$R_n : p_n \text{ is not infinite } c\text{-homogeneous}$$

are satisfied.

Construction: Arrange these requirements in order of priority. At stage s of our construction, we shall define $c_s : \mathbb{N} \to 2$. During the construction, we will refer to p_n, with the understanding that we only have partial information about p_n given by applying Ψ to the colourings c_s defined thus far. For each n, the strategy for R_n may act to *protect* certain numbers which have been enumerated into p_n. We will use $P_n[s]$ to denote the set of numbers protected by R_n at stage s.

Begin by defining all $P_n[0]$ to be empty.

Stage s: Define c_s as follows: $c_s(x) = 0$ unless there are some $y < x$ and $n < s$ such that $x, y \in P_n[s]$ and $x, y \notin P_m[s]$ for any $m < n$. If such y (and n) exist, then pick the least such y and define $c_s(x) = 1 - c_s(y)$.
For $n \leq s$, we define $P_n[s+1]$ as follows. If x is enumerated into p_0 and $|P_0[s]| \leq 1$ then define $P_0[s+1] = P_0 \cup \{x\}$. For each $0 < n \leq s$, if some x is enumerated into p_n then take the least such x and do as follows:

- If there are some $y < z$ such that $y, z \in P_n[s]$ and $y, z \notin P_m[s]$ for any $m < n$, then define $P_n[s+1] = P_n[s]$.
- Otherwise, define $P_n[s+1] = P_n[s] \cup \{x\}$.

This completes stage s of the construction.

Verification: We first show that for each requirement R_n, the limit $P_n = \lim_s P_n[s]$ exists and is finite. It is clear from construction that P_0 exists and $|P_0| \leq 2$. Suppose for all $m < n$, $P_m = \lim_s P_m[s]$ exists and is finite. We claim that $P_n = \lim_s P_n[s]$ exists and $\left|P_n \cup \bigcup_{m<n} P_m\right| \leq \left|\bigcup_{m<n} P_m\right| + 2$. Indeed, after every P_m for $m < n$ has stabilised, at most two numbers outside $\bigcup_{m<n} P_m$ can be added to P_n.

Next, we prove by induction on x that for each x, $c(x) = \lim_s c_s(x)$ exists. We have $c_s(0) = 0$ for all s. As for $x > 0$, if there are some $y < x$ and n such that $x, y \in P_n \setminus \bigcup_{m<n} P_m$, then for the least such y, we have $\lim_s c_s(x) = 1 - c(y)$.

Finally, we show that each R_n is satisfied. Suppose p_n is infinite, otherwise we are done. Then there must be exactly two elements in $P_n \setminus \bigcup_{m<n} P_m$, say $y < x$. As shown above, we must have $c(x) = 1 - c(y)$ and thus p_n is not c-homogeneous. ⊣

COROLLARY 3.23. *$(\mathsf{RT}^1_2)' \not\leq^*_{\mathrm{W}} \mathsf{RT}^n_\infty\mathsf{c}$ and $(\mathsf{RT}^1_2)' \not\leq^*_{\mathrm{W}} \mathsf{RT}^1_\infty$.*

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PROOF. $\mathsf{RT}^n_\infty\mathsf{c}$ and RT^1_∞ are both computably true, hence continuously countably guessable. ⊣

To complete the justification for the relationships in Figure 1, we apply a result of Soldà and Valenti [15] in the Weihrauch degrees. We will later strengthen their result. As a consequence, we will strengthen the following result as well (see Corollary 4.18).

PROPOSITION 3.24. $\mathsf{RT}^1_\infty\chi \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_2\mathsf{pe}$.

PROOF. By [15, Theorem 7.20], it is known that $\mathsf{C}'_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{RT}^2_2$. The desired result follows from the equivalences $\mathsf{RT}^1_\infty\chi \equiv_{\mathrm{W}} \mathsf{RT}^1_\infty \equiv_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}}$ and $\mathsf{RT}^2_2\mathsf{pe} \equiv_{\mathrm{W}} \mathsf{RT}^2_2$. ⊣

We have now justified all relationships in Figure 1. We end this section by giving an upper bound for $\mathsf{RT}^1_\infty\mathsf{e}$. This upper bound is optimal among the problems $\mathsf{RT}^n_k\mathsf{pe}$ for $n \geq 1$ and $k \in \mathbb{N} \cup \{<\infty, \mathbb{N}\}$, as a consequence of Lemma 4.17 (which is our aforementioned strengthening of [15, Theorem 7.20]).

THEOREM 3.25. $\mathsf{RT}^1_\infty\mathsf{e} \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^3_2\mathsf{pe}$.

PROOF. We first show that $\mathsf{RT}^1_\infty\mathsf{e}$ is equivalent to $\mathsf{RT}^1_\infty\mathsf{e}^*$, which we define as $\mathsf{RT}^1_\infty\mathsf{e}$ with the restriction that instances have a unique colour which occurs infinitely often. Evidently, $\mathsf{RT}^1_\infty\mathsf{e}^* \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^1_\infty\mathsf{e}$. Let $f : \mathbb{N} \to \mathbb{N}$, an instance of $\mathsf{RT}^1_\infty\mathsf{e}$ be given. We construct $f^* : \mathbb{N} \to \mathbb{N}$, an instance of $\mathsf{RT}^1_\infty\mathsf{e}^*$ as follows.

For each $i, s \in \mathbb{N}$, we maintain that at each stage s, $x_{i,s} < x_{i+1,s}$, and $x_{i,s} \leq x_{i,s+1}$ (if defined). The intention here is that there is exactly one $x_i = \lim_s x_{i,s}$ such that $x_i < \infty$, and x_i is part of an infinite f-homogeneous set. Initially set $x_{i,0} \uparrow$ for all $i \in \mathbb{N}$. At each stage s, let $x_{i,s}$ be the least x in $(\max_{j<i} x_{j,s}, s]$ such that $f(x) = i$ and for each $y \in [x, s]$, $f(y) \geq i$, if such x exists, and let $x_{i,s} \uparrow$ otherwise.

For each $y \in \mathbb{N}$, define $f^*(y) = x_{i,y}$ if $f(y) = i$ and $x_{i,y} \downarrow$. Otherwise, define $f^*(y) = y$. Now we verify that f^* is a valid instance of $\mathsf{RT}^1_\infty\mathsf{e}^*$. First, observe that if there are infinitely many y such that $f(y) = i$, then for each $j > i$, $\lim_s x_{j,s} = \infty$. Second, if there are only finitely many y for which $f(y) = j$, then either $x_{j,s} \uparrow$ for cofinitely many s, or $\lim_s x_{j,s}$ exists and is bounded by $\max\{y \mid f(y) = j\}$.

Let i be the least i such that there are infinitely many y for which $f(y) = i$. Clearly, $\lim_s x_{i,s} = y$ for the least y such that $f(y) = i$ and $f(z) \geq i$ for each $z \geq y$. It is thus evident that y is the unique f^*-colour repeated infinitely often.

Given an $\mathsf{RT}^1_\infty\mathsf{e}^*$-solution $S^* : \mathbb{N} \to \mathbb{N}$ for f^*, we define an $\mathsf{RT}^1_\infty\mathsf{e}$-solution $S : \mathbb{N} \to \mathbb{N}$ for f as follows. Let $S(0) = S^*(0)$. For each $x \in \mathbb{N}$, if $f(x) = f(S^*(0))$, then define $S(x) = x$, otherwise define $S(x) = S^*(0)$. Clearly, the range of S is an infinite f-homogeneous set of colour $f(S^*(0))$, and S is computable uniformly in S^* and f.

Now we show that $\mathsf{RT}^1_\infty\mathsf{e}^* \leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^3_2\mathsf{pe}$. Given $f : \mathbb{N} \to \mathbb{N}$, we compute the following colouring $g : [\mathbb{N}]^3 \to 2$:

$$g(x, y, z) = \begin{cases} 0 & \text{if } |\{f(x') \mid x' \leq x\} \cap \{f(y') \mid y \leq y' \leq z\}| = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Assuming f is an $\mathsf{RT}^1_\infty\mathsf{e}^*$-instance, we shall prove that all infinite g-homogeneous sets have colour 0. Suppose $\{x_0 < x_1 < x_2 < \dots\}$ is g-homogeneous. Pick the least i such that there is some $x \leq x_i$ for which $f(x)$ is the unique f-colour, n, repeated

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infinitely often. Also pick $j > i$ such that for all $x \geq x_j$, either $f(x) = n$, or $f(x) \neq f(x')$ for any $x' \leq x_i$. Finally, pick $k > j$ such that there is some x, where $x_j \leq x \leq x_k$ and $f(x) = n$. Since n is the unique f-colour that repeats infinitely often, such i, j, and k must exist. By definition of g, we have $g(x_i, x_j, x_k) = 0$ as desired.

For the post-processing, suppose we are given some $S : \mathbb{N} \to \mathbb{N}$. We shall compute some $H : \mathbb{N} \to \mathbb{N}$ as follows. First compute and compare the sets $\{f(x') \mid x' \leq S(0)\}$ and $\{f(y') \mid S(1) \leq y' \leq S(2)\}$. If they do not have exactly one colour in common, then S cannot be an RT^3_2pe-solution for g, so we simply return $H = \mathrm{id}$. Otherwise, we may identify the unique colour n in common and the least $x \leq S(0)$ such that $f(x) = n$. Define $H(0) = x$. For each $i > 0$, define $H(i) = i$ if $f(i) = n$, and x otherwise. In this case, it is clear that the range of H is f-homogeneous of colour n.

Suppose S is an RT^3_2pe-solution for g. We shall show that the range of H is infinite, in other words, that n appears infinitely often in f. It suffices to verify that for any $0 < i < j$,

$$\{f(x') \mid x' \leq S(0)\} \cap \{f(y') \mid S(i) \leq y' \leq S(j)\} = \{n\}.$$

Suppose for a contradiction that there exists i, j such that the above fails. In particular, $\{f(x') \mid x' \leq S(0)\} \cap \{f(y') \mid S(i) \leq y' \leq S(j)\}$ contains some $n' \neq n$. However, this implies that $\{f(x') \mid x' \leq S(0)\} \cap \{f(y') \mid S(1) \leq y' \leq S(j)\}$ contains both n and n', contrary to our assumption on S. ⊣

§4. Weihrauch jump inversions for colourings of pairs. Brattka and Rakotoniaina [7, Proposition 4.1] proved that $(\mathsf{CRT}^n_k)' \leq_{\mathrm{sW}} \mathsf{CSRT}^{n+1}_k$ for $k \geq 1$. They also claimed that $(\mathsf{CRT}^n_{\mathbb{N}})' \leq_{\mathrm{sW}} \mathsf{CSRT}^{n+1}_{\mathbb{N}}$. However, their proof for finite k does not seem to generalise to $k = \mathbb{N}$. We show below that their claim is false, at least for $n = 1$. This affects the following results in [7]: Proposition 4.1, Theorem 4.3, and Corollary 4.28 for $n = 1$, $k = \mathbb{N}$, Corollary 4.2 for $m = n = 1$, $k = \mathbb{N}$, as well as Theorem 7.9 for $n = 2$ (see also our Lemma 4.17, which refutes Theorem 7.9 for $n = 2$).

LEMMA 4.1. $(\mathsf{RT}^1_{\mathbb{N}})' \not\leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$. *In particular*, $(\mathsf{CRT}^1_{\mathbb{N}})' \not\leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{CRT}^2_{\mathbb{N}}$.

PROOF. Suppose that Φ and Ψ witness $\left(\mathsf{RT}^1_{\mathbb{N}}\right)' \leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$. In the discussion that follows, we directly construct a Δ^0_2 instance f of $\mathsf{RT}^1_{\mathbb{N}}$, which may be turned into a computable instance $(f_s)_{s\in\mathbb{N}}$ of $\left(\mathsf{RT}^1_{\mathbb{N}}\right)'$ in the obvious way. Let $g = \Phi((f_s)_{s\in\mathbb{N}})$. Throughout this discussion, we will also think of Ψ as a principal enumeration of an f-homogeneous set given as oracle a g-homogeneous set. To facilitate the discussion, we make the following definitions. Fix a finite set $D = \{x_0 < x_1 < \cdots < x_{n-1}\}$ and a string $\sigma \in \mathbb{N}^{<\mathbb{N}}$ of length n.

- Define $H(D, \sigma) = \{y > \max D \mid g(x_i, y) = \sigma(i) \text{ for each } i\}$.
- The pair (D, σ) is *consistent* if for each i, j, where $i < j < n$, $g(x_i, x_j) = \sigma(i)$.
- Finally, for each $k \in \mathbb{N}$, define $P_k(D, \sigma) = \{x_i \in D \mid \sigma(i) = k\}$.

It is easy to see that if (D, σ) is consistent, then $P_k(D, \sigma)$ is a g-homogeneous set (possibly empty) of colour k for each $k \in \mathbb{N}$.

We first restrict ourselves to the case, where g uses only two colours, say $0, 1$. Following from the definitions, if $H(D, \sigma)$ is infinite, then for any $y \in H(D, \sigma)$, at least one of $H(D \cup \{y\}, \sigma^\frown 0)$ or $H(D \cup \{y\}, \sigma^\frown 1)$ is infinite. Moreover, if (D, σ)

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is also consistent, then one of $P_0(D,\sigma)$ or $P_1(D,\sigma)$ must be extendable to an infinite g-homogeneous set. In particular, by ensuring that f colours only finitely many bits with the colours of $\Psi(P_0(D,\sigma),(f_s)_{s\in\mathbb{N}})(0)$ or $\Psi(P_1(D,\sigma),(f_s)_{s\in\mathbb{N}})(0)$ if they have converged, ensures that Φ and Ψ fail to witness $(\mathsf{RT}^1_{\mathbb{N}})' \leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$. Evidently, it may be that neither $\Psi(P_0(D,\sigma),(f_s)_{s\in\mathbb{N}})(0)$ nor $\Psi(P_1(D,\sigma),(f_s)_{s\in\mathbb{N}})(0)$ has converged as we have yet to provide 'sufficient information' in the oracle. We 'add' information by defining a new finite set $D' = D \cup \{\mu y \in H(D,\sigma)\}$. Let $\sigma' = \sigma^\frown i$ such that $H(D',\sigma^\frown i)$ is infinite. Note that provided that $H(D,\sigma)$ is infinite, at least one of $H(D',\sigma^\frown 0)$ or $H(D',\sigma^\frown 1)$ is infinite. Observe also that if (D,σ) is consistent, then so are $(D',\sigma^\frown i)$ for both $i = 0,1$. In this way, given a consistent (D,σ) so that $H(D,\sigma)$ is infinite, we can obtain a sequence of consistent (D_m,σ_m) for which $H(D_m,\sigma_m)$ is infinite for each $m \in \mathbb{N}$. Then at least one of $\bigcup_m P_0(D_m,\sigma_m)$ or $\bigcup_m P_1(D_m,\sigma_m)$ must be an infinite g-homogeneous set. That is, if $\Psi(P_0(D_m,\sigma_m),((f_s)_{s\in\mathbb{N}})(0),\Psi(P_1(D_m,\sigma_m),(f_s)_{s\in\mathbb{N}})(0)$ never converges, Φ and Ψ must also fail to witness the reduction $(\mathsf{RT}^1_{\mathbb{N}})' \leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$.

We extend the ideas above to the general case as follows. During stage s of the construction, we say that a pair (D_i,σ) *fires* if the following conditions hold:

- (D_i,σ) is consistent.
- $|H(D_i,\sigma)\upharpoonright s| > |H(D_i,\sigma)\upharpoonright (s-1)|$.
- For any $k < |D_i|$, $|\{n \in rng(g[s]) \mid \Psi(P_n(D_i,\sigma),f)(0)\downarrow\}| > |\{n \in rng(g[s]) \mid \Psi(P_n(D_i\upharpoonright k,\sigma\upharpoonright k),f)(0)\downarrow\}|$.

Then we wish to define $f(s)$ to be the least colour which avoids the current colour of each of $\Psi(P_n(D_j,\tau),(f_t)_{t<s})(0)\downarrow$ for 'higher priority' (D_j,τ). If we are not careful, we may end up in a situation, where f defined by us has no infinite homogeneous set. In particular, g may continually use new colours n, and pick new elements via $\Psi(P_n(D,\sigma),(f_t)_{t\in\mathbb{N}})(0)$, causing us to restrain infinitely many colours from f. Whilst g is also not a valid instance of $\mathsf{RT}^2_{\mathbb{N}}$ in this case, Φ and Ψ do not necessarily fail since f as constructed by us is also not a valid instance of $(\mathsf{RT}^1_{\mathbb{N}})'$. To circumvent this, we will define $f(x) = 0$ for each $x < s$ at stage s whenever we witness that $|rng(g[s])| > |rng(g[s-1])|$. This ensures that if $|rng(g)| = \infty$, then f is the all zero colouring and thus a valid instance of $(\mathsf{RT}^1_{\mathbb{N}})'$. Finally, provided that $|rng(g)| < \infty$, there must be some 'highest priority' (D_i,σ) which fires infinitely often during the construction. For such a pair, we shall argue that for at least one $n \in rng(g)$, $P_n(D_i,\sigma)$ is extendable to an infinite g-homogeneous set and that f colours only finitely many bits with one of the colour of $\Psi(P_n(D_i,\sigma),(f_s)_{s\in\mathbb{N}})(0)$ whenever defined. We now detail the construction.

Construction: Let $\{D_i\}_{i\in\mathbb{N}}$ be a list of the finite sets with the property that if $D_i \subseteq D_j$, then $i \leq j$, and arrange the pairs (D_i,σ), where $\sigma \in \mathbb{N}^{|D_i|}$ in order of priority as follows. $(D_i,\sigma) < (D_j,\tau)$ if $i < j$, or if $i = j$ and $\sigma <_{lex} \tau$. (Note that if $i = j$, then $|\sigma| = |\tau| = |D_i|$.) Define $C(\emptyset,\varepsilon)[s] = 0$ for all s.

Stage s: For a consistent pair (D_i,σ), do the following that applies.

- If for each $(D_j,\tau) \leq (D_i,\sigma)$, where $|D_j| = |\tau| < s$, and each $n \in rng(g[s])$, $\Psi(P_n(D_j,\tau),(f_t)_{t<s})(0)\uparrow$. Define $C(D_i,\sigma)[s] = 0$.

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- Otherwise, define $C(D_i, \sigma)[s]$ to be the least y larger than the current colours of $\Psi(P_n(D_j, \tau), (f_t)_{t<s})(0)$ for any $(D_j, \tau) \leq (D_i, \sigma)$ and $n \in rng(g[s])$, whenever defined.

If $|rng(g[s])| > |rng(g[s-1])|$, then for each $x \leq s$, define $f_s(x) = 0$. Otherwise, there must be some (D_i, σ), where $0 < |D_i| = |\sigma| < s$ which fires. Let the pair (D_i, σ) be the highest priority pair which fires and also such that for any consistent pair (D_j, τ), where $|D_j| = |\tau| < s$, $D_j \supsetneq D_i$, and $\tau \supsetneq_{pref} \sigma$, (D_j, τ) does not fire at stage s. Then let y be the least such that y is larger than $C(D_j, \tau)[s]$ (wherever defined) for each D_j, τ such that $D_j \supseteq D_i$, $\tau \supseteq_{pref} \sigma$, and $|D_j| = |\tau| < s$. Define $f_s(s) = y$ and for each $x < s$, if $f_{s-1}(x) > y$, define $f_s(x) = y$. Otherwise, let $f_s(x) = f_{s-1}(x)$.

Verification: Now we verify that the construction works. It is easy to see from the construction that $f = \lim_s f_s$ exists, as we have that $f_s(x) \geq f_{s+1}(x)$ for any $x, s \in \mathbb{N}$. Then we have that $\lim_s C[s]$ also exists. Now, if $|rng(g)| = \infty$, then we obtain that f is the all zero colouring, and thus, Φ and Ψ must fail to witness $(\mathsf{RT}^1_{\mathbb{N}})' \leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$. We may thus assume that $|rng(g)| = N < \infty$. Now let (D_i, σ) be the highest priority pair satisfying the following:

- (D_i, σ) fires infinitely often.
- For each (D_j, τ) such that $D_j \supsetneq D_i$ and $\tau \supsetneq_{pref} \sigma$, (D_j, τ) fires only finitely often.

To see why such a pair (D_i, σ) must exist, suppose for a contradiction that for any pair (D_i, σ) which fires infinitely often, there is some (D_j, τ), where $D_j \supsetneq D_i$ and $\tau \supsetneq_{pref} \sigma$ such that (D_j, τ) also fires infinitely often. We then obtain infinite sequences $D_{i_0} \supsetneq D_{i_1} \supsetneq \ldots$ and $\sigma_0 \subsetneq_{pref} \sigma_1 \subsetneq_{pref} \ldots$, such that $|\{n \in rng(g) \mid \Psi(P_n(D_{i_k}, \sigma_k), f)(0)\downarrow\}|$ is strictly increasing in k, a contradiction to the assumption that $|rng(g)| < \infty$.

For the pair (D_i, σ) which satisfies the desired properties, it follows that for at least one $n \in rng(g)$, $P_n(D_i, \sigma)$ is extendable to an infinite g-homogeneous set of colour n. Now consider some stage large enough such that all of the following holds:

- For each $(D_j, \tau) \leq (D_i, \sigma)$, $C(D_j, \tau)$ never again changes.
- For each (D_k, γ) such that $D_k \supsetneq D_i$ and $\gamma \supsetneq_{pref} \sigma$, $C(D_k, \gamma)$ never again changes and (D_k, γ) never again fires.

By choice of (D_i, σ) and the definition of C, such a stage must exist. After such a stage, there must be some fixed y such that whenever (D_i, σ) fires, the new bit of f is defined to be y, and all earlier bits with values larger than y is now changed to y. By choice of (D_i, σ), we obtain that $f(x) \leq y$ for all $x \in \mathbb{N}$, thus, f as defined is a valid instance of $(\mathsf{RT}^1_{\mathbb{N}})'$.

It remains to argue that for each n, where $\Psi(P_n(D_i, \sigma), (f_s)_{s\in\mathbb{N}})(0)\downarrow = x_n$, there are only finitely many bits coloured $f(x_n)$ in f. From the construction, it is evident that after the stage where each of $\Psi(P_n(D_i, \sigma), (f_t)_{t\in\mathbb{N}})(0)\downarrow$, we must define $f(s) > f(x_n)$ for each n if we defined $f(s)$ due to a lower priority pair than (D_i, σ) firing. Now for higher priority pairs (D_j, τ), by choice of (D_i, σ), either (D_j, τ) fires only finitely often, or it must be that $D_j \subsetneq D_i$ and $\tau \subsetneq_{pref} \sigma$. The former only potentially colours finitely many more bits of f with one of $f(x_n)$, and may thus be ignored. In the latter case, it follows from the construction that whenever $f(s)$ is defined due to (D_j, τ) firing, it is defined to be larger than the value $C(D_i, \sigma)[s]$, and thus cannot

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cause infinitely many bits of f to be coloured with one of the $f(x_n)$. Recall that $C(D_i, \sigma)[s]$ is defined to be some number larger than $f_s(x_n)$ for each $n \in rng(g[s])$, where the computation x_n is defined. In particular, none of $\Psi(P_n(D_i, \sigma), f)$ are extendable to an enumeration of an infinite f-homogeneous set. Since at least one of $P_n(D_i, \sigma)$ must be extendable to an infinite g-homogeneous set of colour n, Φ, and Ψ must fail to witness $(\mathsf{RT}^1_{\mathbb{N}})' \leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$. ⊣

If we consider $\mathsf{CRT}^n_{<\infty}$ instead of $\mathsf{CRT}^n_{\mathbb{N}}$, we recover analogues of [7, Proposition 4.1 and Theorem 4.3], with analogous proofs. We include details for completeness.

PROPOSITION 4.2. *For all $n \geq 1$ and $k \geq 2$, we have*

$$(\mathsf{CRT}^n_{<\infty})' \leq_{\mathrm{sW}} \mathsf{CSRT}^{n+1}_{\mathbb{N}}, \quad (\mathsf{RT}^n_{<\infty})' \leq_{\mathrm{sW}} \mathsf{D}^{n+1}_{\mathbb{N}}, \quad \textit{and} \quad (\mathsf{RT}^n_k)' \leq_{\mathrm{sW}} \mathsf{D}^{n+1}_k.$$

PROOF. An instance of $(\mathsf{CRT}^n_{\infty})'$ is a sequence $(p_s)_{s\in\mathbb{N}}$ which converges to an instance of $\mathsf{CRT}^n_{<\infty}$. Without loss of generality, we may assume that each p_s names a pair (k_s, f_s) such that $k_s \in \mathbb{N}$, $f_s : [\mathbb{N}]^n \to \mathbb{N}$, and $f := \lim_s f_s$ uses only colours $< \lim_s k_s$. We pre-process by computing the colouring $g : [\mathbb{N}]^{n+1} \to \lim_s k_s$ by $g(A \cup \{s\}) = \min\{k_s - 1, f_s(A)\}$ for all $A \in [\mathbb{N}]^n$ and $s > \max(A)$. Since $\lim_s k_s$ exists, g is stable and has finite range (as $\max_s k_s$ exists). Next, we claim that for each $A \in [\mathbb{N}]^n$ and $s > \max(A)$, $\lim_s g(A \cup \{s\}) = f(A)$. To prove this, fix some $t \in \mathbb{N}$ such that $k_s = k_t$ for all $s > t$. Since $\lim_s f_s$ uses only colours $< \lim_s k_s$, we may fix $t^* \geq t$ such that $f_s(A) < \lim_s k_s$ for all $s > t^*$. Then for all $s > t^*$, we have $f_s(A) < k_s$ and so $g(A \cup \{s\}) = f_s(A)$. This proves our claim.

It follows that every g-limit-homogeneous set is also f-homogeneous. Therefore $(\mathsf{RT}^n_{<\infty})' \leq_{\mathrm{sW}} \mathsf{D}^{n+1}_{\mathbb{N}}$. To show that $(\mathsf{CRT}^n_{<\infty})' \leq_{\mathrm{sW}} \mathsf{CSRT}^{n+1}_{\mathbb{N}}$, we use the same pre-processing and post-processing as above. It suffices to observe that by the claim, if $H \subseteq \mathbb{N}$ is g-homogeneous of colour ℓ, then H is f-homogeneous of colour ℓ.

Finally, to show that $(\mathsf{RT}^n_k)' \leq_{\mathrm{sW}} \mathsf{D}^{n+1}_k$, suppose we are given a convergent sequence $(f_s)_{s\in\mathbb{N}}$ such that $f_s : [\mathbb{N}]^n \to \mathbb{N}$ and $f := \lim_s f_s$ uses only colours $< k$. We pre-process by first replacing each f_s with the colouring $g_s : [\mathbb{N}]^n \to k$, defined by $g_s(A) = f_s(A)$ if $f_s(A) < k$, otherwise $g_s(A) = 0$. Observe that $\lim_s g_s = f$. Next, we apply the pre-processing in the previous paragraph to the convergent sequence $(k, g_s)_{s\in\mathbb{N}}$. This yields some colouring $g : [\mathbb{N}]^{n+1} \to k$. As reasoned above, g is stable (and is hence a D^{n+1}_k-instance) and every g-limit-homogeneous set is homogeneous for $\lim_s g_s = f$. ⊣

THEOREM 4.3. *For all $n \geq 1$ and $k \geq 2$, we have*

$$(\mathsf{CRT}^n_{<\infty})' \equiv_{\mathrm{W}} \mathsf{CSRT}^{n+1}_{\mathbb{N}}, \quad (\mathsf{RT}^n_{<\infty})' \equiv_{\mathrm{sW}} \mathsf{D}^{n+1}_{\mathbb{N}}, \quad \textit{and} \quad (\mathsf{RT}^n_k)' \equiv_{\mathrm{sW}} \mathsf{D}^{n+1}_k.$$

PROOF. By the previous proposition, it suffices to show that $\mathsf{CSRT}^{n+1}_{\mathbb{N}} \leq_{\mathrm{W}} (\mathsf{CRT}^n_{<\infty})'$, $\mathsf{D}^{n+1}_{\mathbb{N}} \leq_{\mathrm{sW}} (\mathsf{RT}^n_{<\infty})'$, and $\mathsf{D}^{n+1}_k \leq_{\mathrm{sW}} (\mathsf{RT}^n_k)'$. To reduce $\mathsf{D}^{n+1}_{\mathbb{N}}$ to $(\mathsf{RT}^n_{<\infty})'$, suppose we are given a stable colouring $f : [\mathbb{N}]^{n+1} \to \mathbb{N}$ with bounded range. Compute a sequence $(k_s, g_s)_{s\in\mathbb{N}}$ by

$$k_s = \max\{f(B) + 1 : B \in [\mathbb{N}]^{n+1},\ \max(B) < s\}$$

$$g_s(A) = \begin{cases} f(A \cup \{s\}) & s > \max(A) \\ 0 & \text{otherwise.} \end{cases}$$

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Then $\lim_s k_s = \max\{f(B)+1 : B \in [\mathbb{N}]^{n+1}\}$ and $\lim_s g_s(A) = \lim_s f(A \cup \{s\})$ (which exists because f is stable). Furthermore, $g := \lim_s g_s$ uses only colours $< \lim_s k_s$. Observe that every g-homogeneous set is f-limit-homogeneous. This shows that $\mathsf{D}^{n+1}_{\mathbb{N}} \leq_{\mathrm{sW}} (\mathsf{RT}^n_{<\infty})'$.

The proof that $\mathsf{D}^{n+1}_k \leq_{\mathrm{sW}} (\mathsf{RT}^n_k)'$ proceeds similarly: Given a stable colouring $f : [\mathbb{N}]^{n+1} \to k$, compute $(g_s)_{s\in\mathbb{N}}$ as above.

To show that $\mathsf{CSRT}^{n+1}_{\mathbb{N}} \leq_{\mathrm{W}} (\mathsf{CRT}^n_{<\infty})'$, we use the same pre-processing as above. Given (k, H) such that H is an infinite g-homogeneous set of colour k, we may use f to compute an infinite f-homogeneous set M of colour k as follows. The set $M = \bigcup_i M_i$ will be defined inductively using sets $M_i \in [H]^{n+i}$. Begin by choosing M_0 to be the first n elements of H. Having defined $M_i \in [H]^{n+i}$, search for the least $s > \max(M_i)$ such that $f(A \cup \{s\}) = k$ for all $A \in [M_i]^n$. Such s exists because for each $A \in [M_i]^n$, we have $A \in [H]^n$ and so $k = g(A) = \lim_s f(A \cup \{s\})$. Then $M = \bigcup_i M_i$ is infinite and f-homogeneous of colour k. ⊣

For $n = 1$, we observe that the right side of the equivalences in Theorem 4.3 cannot be replaced with $\mathsf{SRT}^2_{<\infty}$ or $\mathsf{D}^2_{<\infty}$.

Lemma 4.4. $\mathsf{D}^2_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{RT}^2_{<\infty}$.

Proof. This follows easily from a result of Hirschfeldt and Jockusch [10]: If $\mathsf{D}^2_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}^2_{<\infty}$, there would be some open set U and some $k \geq 2$ such that $\mathsf{D}^2_{\mathbb{N}}|_U \leq_{\mathrm{W}} \mathsf{RT}^2_k$. However, $\mathsf{D}^2_{\mathbb{N}}$ is a fractal (this can be checked directly or by applying Theorem 4.3 and the fact that jumps are fractals [4, Proposition 5.8]) so that implies $\mathsf{D}^2_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{RT}^2_k$, contrary to [10, Theorem 3.4]. ⊣

Corollary 4.5. $\mathsf{SRT}^2_{<\infty} <_{\mathrm{W}} \mathsf{SRT}^2_{\mathbb{N}}$ *and* $\mathsf{D}^2_{<\infty} <_{\mathrm{W}} \mathsf{D}^2_{\mathbb{N}}$.

Our next result shows that [7, Corollary 4.10] remains valid. We give a brief proof (see [7] for relevant background).

Corollary 4.6 [7, Corollary 4.10]. $\mathsf{SRT}^{n+1}_k \leq_{\mathrm{W}} \mathsf{RT}^n_k * \mathsf{lim}$ *for all* $n \geq 1$, $k \in \mathbb{N} \cup \{\mathbb{N}\}$.

Proof. This was proved for $k \in \mathbb{N}$ in [7]. For $k = \mathbb{N}$, observe that $\mathsf{CRT}^n_{<\infty} \leq_{\mathrm{sW}} \mathsf{CRT}^n_{\mathbb{N}}$ and $\mathsf{CRT}^n_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{RT}^n_{\mathbb{N}}$, so

$$(\mathsf{CRT}^n_{<\infty})' \leq_{\mathrm{sW}} (\mathsf{CRT}^n_{\mathbb{N}})' \leq_{\mathrm{W}} \mathsf{CRT}^n_{\mathbb{N}} * \mathsf{lim} \equiv_{\mathrm{W}} \mathsf{RT}^n_{\mathbb{N}} * \mathsf{lim}.$$

The desired result then follows from Theorem 4.3. ⊣

Next, we show that $\mathsf{SRT}^2_{<\infty} \not\leq_{\mathrm{W}} \mathsf{D}^2_{\mathbb{N}}$. Our results establish the positions of $(\mathsf{RT}^1_{\mathbb{N}})'$ and $(\mathsf{CRT}^1_{\mathbb{N}})'$ relative to $\mathsf{SRT}^2_{\mathbb{N}}$ and $\mathsf{D}^2_{\mathbb{N}}$.

Lemma 4.7. SRT^2_2 *is not omnisciently continuously Weihrauch reducible to* $\mathsf{RT}^1_{\mathbb{N}}$, *i.e., for every continuous* Ψ, *there is an* SRT^2_2*-instance* f *such that for all* $\mathsf{RT}^1_{\mathbb{N}}$*-instances* g, *there is a solution* H *for* g *such that* $\Psi(H, f)$ *is not a solution for* f.

This lemma strengthens the known result that SRT^2_2 is not omnisciently Weihrauch reducible to $\mathsf{D}^2_{<\infty}$ ([8, Theorem 3.2], see also [9, before Definition 1.8]).

Proof. Suppose for a contradiction that Ψ satisfies the negation of the desired statement.

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Given some function f, we say that a finite set D is *0-ready* if $\Psi(D, f)$ satisfies $(*)$, some property (depending on f) to be specified. For $n \geq 1$, we say a finite set D is *n-ready* if all of the following hold:

- $\Psi(D, f)$ satisfies $(*)$;
- $D = \bigsqcup_{i<k} D_i$, where each D_i is $(n-1)$-ready;
- for each possible subset $E = \{x_0, x_1, \ldots, x_{k-1}\}$ of D, where $x_i \in D_i$, $\Psi(E, f)$ satisfies $(*)$.

We claim that if D is n-ready, then for any $(n+1)$-colouring of (singletons from) D, there is some homogeneous set $E \subseteq D$ such that $\Psi(E, f)$ satisfies $(*)$. This claim can be proved by induction on n, as follows. For $n = 0$, simply take $E = D$. For $n \geq 1$, consider an $(n+1)$-colouring g of an n-ready set D. Fix $(n-1)$-ready sets D_i which witness that D is n-ready. If g uses at most n colours on any D_i, then we conclude by applying the inductive hypothesis to $g \restriction D_i$. Otherwise, for each i, we choose some $x_i \in D_i$ of colour 0. Then the set of x_i's satisfies the desired property. This completes the proof by induction.

Once we have defined an SRT^2_2-instance f, by assumption on Ψ, there is an $\mathsf{RT}^1_{\mathbb{N}}$-instance g such that for all infinite g-homogeneous sets H, $\Psi(H, f)$ is a solution for f. If the range of g is bounded by $n+1$ and D is n-ready, then regardless of the value of $g(x)$ for each $x \in D$, we still successfully obtain a diagonalisation. Note that the definition of n-ready depends solely on Ψ and f. We also note that this n-ready technique will be used in the proof of Lemma 4.15.

Let $f : [\mathbb{N}]^2 \to \{0, 1\}$ be defined as follows. We will search for n-ready sets, where $\Psi(A, f)$ satisfies the property $(*)$ if $\Psi(A, f)$ produces at least two elements larger than s_{n-1}, that is, the stage at which an $(n-1)$-ready set was found in the construction. The idea is that, each element x initially looks like it has stable colour 0, until some n-ready set is found, after which all elements $< s_n$ are now defined to have stable colour 1. It follows from the property $(*)$ that for any n-ready set D_n that is found, $\Psi(E, f)\downarrow$ cannot be extended to a solution to f for any $E \subseteq D_n$.

Construction: Define $s_0 = 0$ and begin searching for a 1-ready set, while defining $f(x, y) = 0$ for $x, y \leq s$. Once a 1-ready set is found, say at stage s_1, call this set D_1 and define $f(x, y) = 1$ for $x < s_1$ and $y \geq s$. Start searching for a 2-ready set disjoint from D_1. Recursively suppose that D_n, an n-ready set has been found at stage s_n. Then we continue searching for an $(n+1)$-ready set D_{n+1}, disjoint from D_m for each $m \leq n$, with the property $(*)$. While searching define $f(x, y) = 0$ for $s_n \leq x < y \leq s$. Once D_{n+1} is found, let s_{n+1} be the current stage number and similarly define $f(x, y) = 1$ for each x, y, where $s_n \leq x < s_{n+1}$ and $y \geq s_{n+1}$.

Verification: It is clear that f is an instance of SRT^2_2, therefore, we may fix an $\mathsf{RT}^1_{\mathbb{N}}$-instance g as above. We claim that every infinite g-homogeneous set H contains an n-ready set. This claim can be proved by induction on n, as follows. The claim for $n = 0$ follows from our choice of g, in particular that $\Psi(H, f)$ is a solution for f. For $n \geq 1$, begin by applying the inductive hypothesis to generate a sequence $D_0, D_1, \ldots$ of $(n-1)$-ready subsets of H such that $\max(D_i) < \min(D_{i+1})$ for all i. (This can be done because every final segment of H is still infinite g-homogeneous.) Observe that $\bigsqcup_{i<k} D_i$ satisfies $(*)$ for all sufficiently large k, because $\bigsqcup_{i\in\mathbb{N}} D_i$ is infinite g-homogeneous and so $\Psi(\bigsqcup_{i\in\mathbb{N}} D_i, f)$ is infinite. Consider the finitely branching tree T consisting of all strings σ such that (1) for each $i < |\sigma|$, $\sigma(i) \in D_i$ and (2) $\mathrm{rng}(\sigma)$

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does not satisfy $(*)$. If T has an infinite path P, then P is infinite g-homogeneous, so $\Psi(P, f)$ is infinite. But this means that $\mathrm{rng}(\sigma)$ satisfies $(*)$ for some initial segment σ of P. This contradiction implies (by König's lemma) that T has finite height. It follows that for sufficiently large k, the set $\bigsqcup_{i<k} D_i$ is n-ready. This concludes the proof of our claim.

Thus in the construction, we must always succeed in finding n-ready sets for each $n \in \mathbb{N}$, say $D_0, D_1, \ldots$. Note that D_n may not be g-homogeneous.

Nonetheless, since g is an $\mathsf{RT}^1_{\mathbb{N}}$-instance, there is some k such that $g(x) < k$ for each $x \in \mathbb{N}$. In particular, for each $n \geq k$, D_n may only be coloured with colours $< k$. By our previous claim, there must be some subset E_n of D_n such that E_n is g-homogeneous. By the infinite pigeonhole principle, there is a sequence of disjoint sets $E_{n_0}, E_{n_1}, \ldots$, where $E_{n_i} \subseteq D_{n_i}$, all of which satisfy $(*)$ and have the same g-colour. Then $\bigsqcup_i E_{n_i}$ is an infinite g-homogeneous set but $\Psi(E_{n_0}, f)\downarrow$ is not extendable to a solution of f, contradicting our assumption on g. ⊣

Corollary 4.8. *For all* $k \in \mathbb{N}$, $\mathsf{SRT}^2_2 \not\leq_{\mathrm{W}} (\mathsf{RT}^1_{\mathbb{N}})^{(k)}$. *Therefore* $\mathsf{SRT}^2_{<\infty} \not\leq_{\mathrm{W}} (\mathsf{RT}^1_{<\infty})' \equiv_{\mathrm{W}} \mathsf{D}^2_{\mathbb{N}}$.

Proof. If Φ, and Ψ witness $\mathsf{SRT}^2_2 \not\leq_{\mathrm{W}} (\mathsf{RT}^1_{\mathbb{N}})^{(k)}$, then consider the SRT^2_2-instance f given by Lemma 4.7. There is a solution H for the $\mathsf{RT}^1_{\mathbb{N}}$-instance $\lim^{(k)}(\Phi(f))$ such that $\Psi(H, f)$ is not a solution for f. ⊣

Corollary 4.9. *Under Weihrauch reducibility,* $\mathsf{SRT}^2_{\mathbb{N}}$ *and* $(\mathsf{RT}^1_{\mathbb{N}})'$ *are incomparable problems (strictly) above* $\mathsf{D}^2_{\mathbb{N}}$ *and (strictly) below* $(\mathsf{CRT}^1_{\mathbb{N}})'$.

Proof. Since $\mathsf{RT}^1_{<\infty} \leq_{\mathrm{sW}} \mathsf{RT}^1_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{CRT}^1_{\mathbb{N}}$, we have $(\mathsf{RT}^1_{<\infty})' \leq_{\mathrm{sW}} (\mathsf{RT}^1_{\mathbb{N}})' \leq_{\mathrm{sW}} (\mathsf{CRT}^1_{\mathbb{N}})'$. We have $(\mathsf{RT}^1_{<\infty})' \equiv_{\mathrm{W}} \mathsf{D}^2_{\mathbb{N}}$ by Theorem 4.3. The incomparabilities follow from Lemma 4.1 and Corollary 4.8. ⊣

Finally, we analyse the problems RT^1_{∞}, $(\mathsf{RT}^1_{\infty})'$ and $(\mathsf{CRT}^1_{\infty})'$.

Proposition 4.10. $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{CRT}^1_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{RT}^1_{\infty}$, *so* $\mathsf{RT}^1_{\infty} \equiv_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}} \leq_{\mathrm{sW}} (\mathsf{CRT}^1_{\mathbb{N}})' \leq_{\mathrm{sW}} (\mathsf{RT}^1_{\infty})'$.

Proof. First we show that $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{CRT}^1_{\mathbb{N}}$. Let W be an instance of $\mathsf{C}_{\mathbb{N}}$, interpreted as an enumeration. We construct $f : \mathbb{N} \to \mathbb{N}$ as follows: $f(s)$ is the least element yet to be enumerated by W by stage s. Since W is an instance of $\mathsf{C}_{\mathbb{N}}$, it follows that f is eventually constant, and $\lim_s f(s)$ is the least element not enumerated by W. That is, the colour of the homogeneous set is the solution to W.

Now we show that $\mathsf{CRT}^1_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{RT}^1_{\infty}$. Given f, an instance of $\mathsf{CRT}^1_{\mathbb{N}}$, define

$$g(\langle x, y\rangle) = \begin{cases} \langle 0, y\rangle & \text{if } f(x) = y \\ \langle x+1, y+1\rangle & \text{otherwise.} \end{cases}$$

To see that g is an instance of RT^1_{∞}, note that it must have an infinite homogeneous set of colour $\langle 0, y\rangle$ for the y such that f has an infinite homogeneous set of colour y. Furthermore, any infinite homogeneous set of g necessarily has colour $\langle 0, y\rangle$ for some y. In fact, any infinite homogeneous set of g is of the form $\{\langle x, y\rangle \mid f(x) = y\}$. Given such a set, it is easy to obtain both the infinite homogeneous set of f and its colour. ⊣

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The following results imply that the reductions in Proposition 4.10 are strict. We thank Manlio Valenti for suggesting the following proof and allowing us to include it here. This proof uses the following well-known problems:

- König's lemma, denoted KL, has as instances infinite finitely branching subtrees $T \subseteq \mathbb{N}^{<\mathbb{N}}$ (represented via characteristic functions), such that $\mathsf{KL}(T)$ is the set of infinite paths through T.
- For each $A \subseteq \mathbb{N}^{\mathbb{N}}$, the problem id_A is the restriction of the identity function to A.

LEMMA 4.11. *RT^1_2 is not continuously Weihrauch reducible to $\mathsf{C}_{\mathbb{N}}$.*

PROOF. Suppose RT^1_2 is continuously Weihrauch reducible to $\mathsf{C}_{\mathbb{N}}$ via Φ and Ψ. Fix some $p \in \mathbb{N}^{\mathbb{N}}$ such that Φ and Ψ are p-computable. It follows that $\mathsf{RT}^1_2 \times \mathsf{id}_{\{p\}} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$: Given an RT^1_2-instance f and p, one can compute the $\mathsf{C}_{\mathbb{N}}$-instance $\Phi(f)$; given a $\mathsf{C}_{\mathbb{N}}$-solution q to $\Phi(f)$ and (f, p), one can compute $\Psi(f, q)$.

Parallelising both sides, we obtain (using [4, Fact 3.5 and Proposition 3.8] and [7, Fact 2.3, Proposition 3.4(2), and Theorem 5.13]) that

$$\mathsf{KL} \times \mathsf{id}_{\{p\}} \equiv_{\mathrm{W}} \widehat{\mathsf{RT}^1_2} \times \widehat{\mathsf{id}_{\{p\}}} \equiv_{\mathrm{W}} \widehat{\mathsf{RT}^1_2 \times \mathsf{id}_{\{p\}}} \leq_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}} \equiv_{\mathrm{W}} \lim.$$

However, the above cannot hold because there is an infinite p-computable finitely branching tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ with no p'-computable path (constructed by relativising a classical argument (see, e.g., [11, Corollary 5.2])). This contradiction shows that RT^1_2 is not continuously Weihrauch reducible to $\mathsf{C}_{\mathbb{N}}$. ⊣

LEMMA 4.12. *RT^1_∞ is not continuously Weihrauch reducible to $\mathsf{RT}^1_{\mathbb{N}}$.*

PROOF. Since $\mathsf{RT}^1_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}\mathsf{c}$, it suffices to show that $\mathsf{RT}^1_\infty \not\leq^*_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}\mathsf{c}$. Suppose for a contradiction that Φ and Ψ witness $\mathsf{RT}^1_\infty \leq^*_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}\mathsf{c}$. Fix X to be an oracle such that Φ and Ψ are both X-computable. We shall work relative to X in the remainder of the proof. We define f an (X-computable) instance of RT^1_∞ as follows. Let $g = \Phi(f)$, with the understanding that we only have partial information about g given by applying Φ to the initial segment of f defined thus far. Define $f(0) = 0$. For each $s > 0$, if $|rng(g[s])| > |rng(g[s-1])|$, then define $f(s) = 0$. Otherwise, define $f(s)$ to be the least y distinct from $\Psi(n, f \upharpoonright s)$ for each $n \in rng(g[s])$ such that the computation has converged.

Now we verify that the construction works. If $|rng(g)| = \infty$, then we evidently have that f has a homogeneous set of colour 0 and is thus a valid instance of RT^1_∞. Since $|rng(g)| = \infty$, then g fails to be an instance of $\mathsf{RT}^1_{\mathbb{N}}\mathsf{c}$ and thus Φ cannot be part of a reduction from RT^1_∞ to $\mathsf{RT}^1_{\mathbb{N}}\mathsf{c}$. Conversely, if $|rng(g)| < \infty$, then it must have some $\mathsf{RT}^1_{\mathbb{N}}\mathsf{c}$-solution n. For such an n, either $\Psi(n, f)$ converges, in which case f clearly stops colouring new elements with the colour $\Psi(n, f)$ eventually, or $\Psi(n, f)$ never converges. In either case, we obtain that Φ and Ψ fail to witness $\mathsf{RT}^1_\infty \leq^*_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}\mathsf{c}$. ⊣

COROLLARY 4.13. *$\mathsf{RT}^1_\infty <_{\mathrm{W}} (\mathsf{CRT}^1_{\mathbb{N}})' <_{\mathrm{W}} (\mathsf{RT}^1_\infty)'$ and $\mathsf{C}_{\mathbb{N}} <_{\mathrm{sW}} \mathsf{CRT}^1_{\mathbb{N}} <_{\mathrm{sW}} \mathsf{RT}^1_\infty$.*

PROOF. The above reductions were established in Proposition 4.10. The strictness of the latter two reductions follow from the previous two lemmas. The strictness of the first two reductions follow from applying [6, Theorem 11] to the previous two lemmas. ⊣

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THEOREM 4.14. *The principles* $\mathsf{SRT}^2_\infty, \left(\mathsf{CRT}^1_\infty\right)', \mathsf{D}^2_\infty, \left(\mathsf{RT}^1_\infty\right)'$ *are all Weihrauch equivalent*.

PROOF. We show the sequence of reductions $\mathsf{D}^2_\infty \leq_{\mathrm{W}} \left(\mathsf{RT}^1_\infty\right)' \leq_{\mathrm{W}} \left(\mathsf{CRT}^1_\infty\right)' \leq_{\mathrm{W}} \mathsf{SRT}^2_\infty \leq_{\mathrm{W}} \mathsf{D}^2_\infty$. The first three reductions are straightforward, as explained in the following paragraph.

First we note that the reduction $\left(\mathsf{RT}^1_\infty\right)' \leq_{\mathrm{W}} \left(\mathsf{CRT}^1_\infty\right)'$ is trivial. Now, we show that $\mathsf{D}^2_\infty \leq_{\mathrm{W}} \left(\mathsf{RT}^1_\infty\right)'$. Given $f : [\mathbb{N}]^2 \to \mathbb{N}$, for each $x, s \in \mathbb{N}$, define $g_s(x) = f(x, s)$. If f is stable and there is some infinite f-limit-homogeneous set, then it is evident that $(g_s)_{s\in\mathbb{N}}$ is a valid instance of $\left(\mathsf{RT}^1_\infty\right)'$. Let $g = \lim_s g_s$. It is easy to verify that any g-homogeneous set is also an f-limit-homogeneous set, thus, we have that $\mathsf{D}^2_\infty \leq_{\mathrm{W}} \left(\mathsf{RT}^1_\infty\right)'$. To obtain the reduction $\left(\mathsf{CRT}^1_\infty\right)' \leq_{\mathrm{W}} \mathsf{SRT}^2_\infty$, define the pre-processing to take $(f_s)_{s\in\mathbb{N}}$ to the function $g(x, s) = f_s(x)$ and define the post-processing to return the homogeneous set $\{x_0, x_1, \dots\}$ given as a solution to g together with $g(x_0, x_1)$ as the colour.

The remainder of the proof of this theorem is devoted to proving $\mathsf{SRT}^2_\infty \leq_{\mathrm{W}} \mathsf{D}^2_\infty$. Let $f : [\mathbb{N}]^2 \to \mathbb{N}$ be an instance of SRT^2_∞. We will define an instance of D^2_∞, $g : [\mathbb{N}]^2 \to \mathbb{N}$.

The main challenge lies in 'thinning' out the limit-homogeneous set to obtain a homogeneous set. Roughly speaking, given two elements x, y, if they currently look like they have the same stable colour (with respect to f), but different from the value of $f(x, y)$, then we would like to 'change' the stable colour of one of x or y (with respect to g, defined by us) to some unique value ensuring that they are never a part of the same limit-homogeneous set. Conversely, if the current stable colours of x, y agree with $f(x, y)$, then we ensure that they continue to 'agree' in the new colouring g. We provide the formal details below.

Stage s: For each $x < s$, define $g(x, s)$ as follows:
- If there is some $y < x$ such that $f(y, x) = f(x, s) = f(y, s)$, then pick the least such y and define $g(x, s) = g(y, s)$.
- Otherwise, define $g(x, s)$ to be the least $n \notin \{g(y, s) \mid y < x\}$.

Verification: We now verify that the construction works. First we show that g is a valid instance of D^2_∞ by proving that g is stable and has an infinite limit-homogeneous set. We conclude the verification by showing that any limit-homogeneous set with respect to g is a homogeneous set with respect to f.

Firstly, $\lim_t g(0, t)$ exists as $g(0, t) = 0$ for all $t \in \mathbb{N}$. Inductively suppose that there is some x so that for all $y < x$, $\lim_t g(y, t)$ exists. Now pick s large enough so that for each $z \leq x$, for each $y < x$, and for each $t \geq s$, $f(z, s) = f(z, t)$ and $g(y, s) = g(y, t)$. Such an s must exist by the inductive hypothesis and the assumption that f is stable. Since the definition of $g(x, t)$ only possibly depends on the values of $f(y, t), f(x, t), f(x, y)$, and $g(y, t)$, by the choice of s, it follows that $g(x, s) = g(x, t)$ for any $t \geq s$.

To show that there is some infinite limit-homogeneous set with respect to g, let x^* be the least such that x^* is contained in some infinite homogeneous set with

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respect to f. Also let $H = \{x^* < x_0 < x_1 < \dots\}$ be one such infinite homogeneous set containing x^*. We claim that for each $i \in \mathbb{N}$, $\lim_t g(x_i, t) = \lim_t g(x^*, t)$. By the assumption that H is homogeneous and that f is stable, $f(x^*, x_i) = \lim_t f(x^*, t) = \lim_t f(x_i, t)$. Then, there exists some s^* such that for all $t \geq s^*$, $f(x^*, s^*) = f(x_i, s^*) = f(x^*, t) = f(x_i, t)$, and for each $y < x^*$, $f(y, s^*) = f(y, t)$. In particular, for all such $t \geq s^*$, whenever $g(x_i, t)$ is being defined, there is always some $y < x_i$ such that $f(x_i, y) = f(y, t) = f(x_i, t)$. From the definition of g, $g(x_i, t)$ is defined to be $g(y, t)$ for the least y which satisfies the aforementioned property. Clearly, if x^* is the least y, then we are done. Suppose for a contradiction that the least such y is less than x^*. By the choice of s^* we have that $f(x_i, t) = \lim_s f(x_i, s)$ and $f(y, t) = \lim_s f(y, s)$. Since H is homogeneous, it also follows that for any $m, n \in \mathbb{N}$, $f(x_m, x_n) = f(x_i, t) = f(y, t)$. However this would imply that $\{y\} \cup \{x_n \mid x_n \geq s^*\}$ is an infinite homogeneous set with respect to f, contradicting the minimality of x^*. Therefore, H is an infinite limit-homogeneous set with respect to g.

Finally, we show that any limit-homogeneous set with respect to g is a homogeneous set with respect to f. That is, $\lim_t g(x, t) = \lim_t g(y, t)$, then $f(x, y) = \lim_t f(x, t) = \lim_t f(y, t)$. Let $x, y \in \mathbb{N}$ be given so that $\lim_t g(x, t) = \lim_t g(y, t)$. Without loss of generality, suppose that $y < x$. For any $t \in \mathbb{N}$, following the definition of $g(x, t)$, if $g(x, t) = g(y, t)$, then $f(x, y) = f(x, t) = f(y, t)$. Since $\lim_t g(x, t) = \lim_t g(y, t)$, then there must be an s so that for all $t \geq s$, $g(x, t) = g(y, t)$, and thus, $f(x, y) = f(x, t) = f(y, t)$. ⊣

Note that $\mathsf{RT}^2_2 \not\leq_{\mathrm{W}} \left(\mathsf{RT}^1_\infty\right)'$ because every instance of the latter has a solution computable in one Turing jump of the instance, while the former has a computable instance with no $\emptyset'$-computable solution.

We end by showing that $(\mathsf{CRT}^1_{\mathbb{N}})'$ is the best upper bound for RT^1_∞ in terms of Weihrauch reducibility, among the problems we have considered.

LEMMA 4.15. *RT^1_∞ is not omnisciently continuously Weihrauch reducible to $\mathsf{RT}^1_{\mathbb{N}}$, i.e., for every continuous Ψ, there is an RT^1_∞-instance f such that for all $\mathsf{RT}^1_{\mathbb{N}}$-instances g, there is a solution H for g such that $\Psi(H, f)$ is not a solution for f.*

PROOF. Suppose for a contradiction that Ψ satisfies the negation of the desired statement. We use the technique of searching for n-ready sets as in Lemma 4.7. We restate the definition of n-ready sets here for convenience. Note that we will use a different $(*)$ property from the one used in Lemma 4.7.

A finite set D is *0-ready* if $\Psi(D, f)$ satisfies $(*)$. For $n \geq 1$, a finite set D is *n-ready* if it satisfies all of the following:

- $D = \bigsqcup_{i<k} D_i$, where each of D_i is $(n-1)$-ready.
- For each possible subset $E = \{x_0, x_1, \dots, x_{k-1}\}$ of D where $x_i \in D_i$, $\Psi(E, f)$ satisfies $(*)$.
- $\Psi(D, f)$ also satisfies $(*)$.

In this case we define $(*)$ as the property that $\Psi(A, f)$ produces at least one element not of f-colour 0. One may show (as we did in the proof of Lemma 4.7) that if D is n-ready, then for any $(n+1)$-colouring of D, there is some homogeneous set $E \subseteq D$ such that $\Psi(E, f)$ satisfies $(*)$.

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Construction: We begin by searching for a 1-ready set D_1 while defining $f(s) = 1$ at stage s. Let the stage that D_1 is found be s_1. Define $f(s_1) = 0$ and begin searching for a 2-ready set disjoint from D_1. Recursively suppose that an n-ready set, D_n, has been found at stage s_n. While searching for an $(n+1)$-ready set D_{n+1} disjoint from D_m for $m < n+1$, define $f(x) = n+1$ for each $x > s_n$. Once D_{n+1} is found, say at stage s_{n+1}, define $f(s_{n+1}) = 0$, and $f(x) = n+2$ for subsequent $x > s_{n+1}$ while beginning to search for an $(n+2)$-ready set.

Verification: Observe that if there is some $n \geq 1$ for which no n-ready set is ever found, then $f(x) = n$ for cofinitely many x. On the other hand, if for each $n \geq 1$, some n-ready set is found, then $f(x) = 0$ for infinitely many x. Thus, f as defined is an RT^1_∞-instance. Therefore, by assumption on Ψ, we may fix an $\mathsf{RT}^1_{\mathbb{N}}$-instance $g : \mathbb{N} \to k$ such that for all infinite g-homogeneous sets H, $\Psi(H, f)$ is a solution for f.

Unlike Lemma 4.7, we have two cases to consider.

First suppose that some n-ready set D_n is found for every n. For all $n \geq k$, there is a g-homogeneous set $E_n \subseteq D_n$ such that $\Psi(E_n, f)$ satisfies $(*)$. By the infinite pigeonhole principle, there is a disjoint sequence $E_{n_0}, E_{n_1}, \ldots$, where $E_{n_i} \subseteq D_{n_i}$, such that each E_{n_i} satisfies $(*)$ and $\bigsqcup_i E_{n_i}$ is g-homogeneous. Then $\Psi(\bigsqcup_i E_{n_i}, f)$ produces an element not of f-colour 0, yet all infinite f-homogeneous sets have colour 0. This contradicts our choice of g.

Otherwise, let $n \geq 1$ be the least such that an n-ready set is never found. Recall that by construction, $f(x) = n$ for cofinitely many x. It then follows from our choice of g that for every infinite g-homogeneous set H, there is some initial segment D of H such that $\Psi(D, f)$ satisfies $(*)$. This fact allows us to prove (by induction on n, following the proof of Lemma 4.7) that every infinite g-homogeneous set contains an n-ready set. This contradicts the case assumption, however. ⊣

COROLLARY 4.16. *For each* $k \in \mathbb{N}$, $\mathsf{RT}^1_\infty \not\leq_{\mathrm{W}} (\mathsf{RT}^1_{\mathbb{N}})^{(k)}$.

LEMMA 4.17. $\mathsf{RT}^1_\infty \not\leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$.

PROOF. Suppose that Φ and Ψ witness $\mathsf{RT}^1_\infty \leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$. In the discussion that follows, we shall think of RT^1_∞ as the Weihrauch equivalent problem $\mathsf{RT}^1_\infty\mathsf{C}$. That is, Ψ simply produces a colour which it believes f will colour infinitely many elements with. The proof here will closely follow the techniques of the proof of Lemma 4.1. See the proof of Lemma 4.1 for a detailed discussion of these techniques. Here we recall the main definitions used: Let $g = \Phi(f)$. For a finite set $D_i = \{x_0 < x_1 < \cdots < x_{n-1}\}$ and a string $\sigma \in \mathbb{N}^{<\mathbb{N}}$ of length n,

- $H(D_i, \sigma) = \{y > \max D_i \mid g(x_k, y) = \sigma(k) \text{ for each } k\}$.
- (D_i, σ) is *consistent* if for each $k < k' < n$, $g(x_k, x_{k'}) = \sigma(k)$.
- For each $m \in \mathbb{N}$, $P_m(D_i, \sigma) = \{x_k \in D_i \mid \sigma(k) = m\}$.

During stage s of the construction, we say that a pair (D_i, σ) *fires* if the following conditions hold:

- (D_i, σ) is consistent.
- $|H(D_i, \sigma) \upharpoonright s| > |H(D_i, \sigma) \upharpoonright (s-1)|$.
- For any $k < |D_i|$, $|\{n \in rng(g[s]) \mid \Psi(P_n(D_i, \sigma), f)(0) \downarrow\}| > |\{n \in rng(g[s]) \mid \Psi(P_n(D_i \upharpoonright k, \sigma \upharpoonright k), f)(0) \downarrow\}|$.

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As in Lemma 4.1, the construction will define f based on the pairs (D_i, σ) which fires. The main idea is to ensure that for some pair (D_i, σ), at least one of $P_n(D_i, \sigma)$ is extendable to an infinite g-homogeneous set whilst colouring only finitely many bits in f with some colour $\Psi(P_n(D_i, \sigma), f)$. The main difference is that f now needs to be computable (rather than Δ^0_2), but the range of f may now possibly be infinite. We now describe the construction.

Construction: Let $\{D_i\}_{i\in\mathbb{N}}$ be an enumeration of all the finite sets such that if $D_i \subseteq D_j$, then $i \leq j$. During the construction, we shall monitor the pairs (D_i, σ) for which $|D_i| = |\sigma|$ and σ mentions only values currently present in $rng(g)$. Arrange the pairs in order of priority as follows: $(D_i, \sigma) < (D_j, \tau)$ if $i < j$, or if $i = j$ and $\sigma <_{lex} \tau$. Now define $C(\emptyset, \varepsilon)[s] = 0$ for all s.

Stage s: For consistent pairs (D_i, σ), do the following that applies.

- If for each $(D_j, \tau) \leq (D_i, \sigma)$, and each $n \in rng(g[s])$, $\Psi(P_n(D_j, \tau), f) \uparrow$. Define $C(D_i, \sigma)[s] = 0$.
- Otherwise, define $C(D_i, \sigma)[s]$ to be the least y larger than the current values of $\Psi(P_n(D_j, \tau), f)$ for any (D_j, τ) and $n \in rng(g[s])$ whenever the computation is defined.

Let (D_i, σ) be the highest priority pair satisfying the following:

- (D_i, σ) *fires* at stage s: $|H(D_i, \sigma)[s]| > |H(D_i, \sigma)[s-1]|$, (D_i, σ) is consistent, and for any $k < |D_i|$, $|\{n \in rng(g[s]) \mid \Psi(P_n(D_i, \sigma), f) \downarrow\}| > |\{n \in rng(g[s]) \mid \Psi(P_n(D_i \upharpoonright k, \sigma \upharpoonright k), f) \downarrow\}|$.
- For any (D_j, τ), where $D_j \supsetneq D_i$ and $\tau \supsetneq_{pref} \sigma$, (D_j, τ) does not fire at stage s.

If no such pair (D_i, σ) exists or if $|rng(g[s])| > |rng(g[s-1])|$, then define $f(s) = 0$. Otherwise, define $f(s)$ to be $C(D_j, \tau)[s]$ for the lowest priority pair (D_j, τ) such that $C(D_j, \tau)[s] \downarrow$, and $D_j \supseteq D_i$ and $\tau \supseteq_{pref} \sigma$.

Verification: Now we verify that the construction works. First, if $|rng(g)| = \infty$, then we evidently have that f has a homogeneous set of colour 0, and Φ and Ψ fail as a reduction from RT^1_∞ to $\mathsf{RT}^2_\mathbb{N}$. Now suppose that $|rng(g)| = N$. Then let (D_i, σ) be the highest priority pair which fires infinitely often and so that for any (D_j, τ) such that $D_j \supsetneq D_i$ and $\tau \supsetneq_{pref} \sigma$, (D_j, τ) fires only finitely often.

To see why such a (D_i, σ) exists, first observe that for any g-homogeneous set X of colour n, there must be some s so that $H(X \upharpoonright s, n^s)$ fires infinitely often, or $\Psi(H \upharpoonright s, f) \uparrow$ for all s, a contradiction. Now suppose that for any (D_i, σ) which fires infinitely often, there is some (D_j, τ) such that $D_j \supsetneq D_i$ and $\tau \supsetneq_{pref} \sigma$ which fires infinitely often. This immediately contradicts the definition of a pair firing as we assumed that $|rng(g)| = N < \infty$.

Finally, for the highest priority pair (D_i, σ) which satisfies our desired properties, we necessarily have that for any n such that $\Psi(P_n(D_i, \sigma), f) \downarrow$, f only colours finitely many bits with $\Psi(P_n(D_i, \sigma), f) \downarrow$. This follows from the fact that any higher priority (D_j, τ) which fires infinitely often is necessarily such that $D_j \subsetneq D_i$ and $\tau \subsetneq_{pref} \sigma$. That is, whenever $f(s)$ is defined due to some such (D_j, τ) firing, we have that $f(s) \geq C(D_i, \sigma)[s] > \Psi(P_n(D_i, \sigma), f)$ for any n where the computation converges. Since $\lim_s C[s]$ clearly exists, we obtain that f has no infinite homogeneous set of colour $\Psi(P_n(D_i, \sigma), f)$ for any $n \in rng(g)$ for which the computation converges.

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Thus, Φ and Ψ fail to witness $\mathsf{RT}^1_\infty \leq_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$ as one of $P_n(D_i, \sigma)$ must be extendable to an infinite g-homogeneous set. ⊣

Lemma 4.17 strengthens a result of Soldà and Valenti [15, Theorem 7.20] for $n = 1$ ($\mathsf{C}'_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{RT}^2_k$ for all $k \geq 2$). We can now strengthen Proposition 3.24.

COROLLARY 4.18. $\mathsf{RT}^1_\infty\chi \not\leq^{\mathrm{t}}_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}\mathsf{pe}$.

PROOF. The desired result follows from Lemma 4.17 and the equivalences $\mathsf{RT}^1_\infty\chi \equiv_{\mathrm{W}} \mathsf{RT}^1_\infty \equiv_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}}$ and $\mathsf{RT}^2_{\mathbb{N}}\mathsf{pe} \equiv_{\mathrm{W}} \mathsf{RT}^2_{\mathbb{N}}$. ⊣

Acknowledgments. The authors thank the referee for their helpful comments.

Funding. K.M.N. was supported by the Ministry of Education, Singapore, under its Academic Research Fund Tier 2 (MOE-T2EP20222-0018) and Academic Research Fund Tier 1 (RG104/24). E.H. was partially supported by the Austrian Science Fund (FWF) through grant 10.55776/STA139.

REFERENCES

[1] V. BRATTKA, *Stashing and parallelization pentagons*. ***Logical Methods in Computer Science***, vol. 17 (2021), no. 4, Article no. 20, 29 pp.

[2] V. BRATTKA and G. GHERARDI, *Weihrauch degrees, omniscience principles and weak computability*. ***Journal of Symbolic Logic***, vol. 76 (2011), no. 1, pp. 143–176.

[3] ———, *Weihrauch goes Brouwerian*. ***Journal of Symbolic Logic***, vol. 85 (2020), no. 4, pp. 1614–1653.

[4] V. BRATTKA, G. GHERARDI, and A. MARCONE, *The Bolzano-Weierstrass theorem is the jump of weak König's lemma*. ***Annals of Pure and Applied Logic***, vol. 163 (2012), no. 6, pp. 623–655.

[5] V. BRATTKA, G. GHERARDI, and A. PAULY, *Weihrauch complexity in computable analysis*, ***Handbook of Computability and Complexity in Analysis***, (V. Brattka and P. Hertling, editors), Theory and Applications of Computability, Springer, Cham, 2021, pp. 367–417.

[6] V. BRATTKA, R. HÖLZL, and R. KUYPER, *Monte Carlo computability*, ***34th Symposium on Theoretical Aspects of Computer Science***, (H. Vollmer and B. Vallée, editors), LIPIcs. Leibniz International Proceedings in Informatics, 66, Schloss Dagstuhl Leibniz-Zentrum für Informatik, Wadern, 2017, Article no. 17, 14 pp.

[7] V. BRATTKA and T. RAKOTONIAINA, *On the uniform computational content of Ramsey's theorem*. ***Journal of Symbolic Logic***, vol. 82 (2017), no. 4, pp. 1278–1316.

[8] D. D. DZHAFAROV, *Strong reductions between combinatorial principles*. ***Journal of Symbolic Logic***, vol. 81 (2016), no. 4, pp. 1405–1431.

[9] D. D. DZHAFAROV and L. PATEY, *COH, SRT^2_2, and multiple functionals*. ***Computability***, vol. 10 (2021), no. 2, pp. 111–121.

[10] D. R. HIRSCHFELDT and C. G. JOCKUSCH, JR., *On notions of computability-theoretic reduction between Π^1_2 principles*. ***Journal of Mathematical Logic***, vol. 16 (2016), no. 1, p. 1650002, 59 pp.

[11] C. G. JOCKUSCH, A. LEWIS, and J. B. REMMEL, *Π^0_1-classes and Rado's selection principle*. ***Journal of Symbolic Logic***, vol. 56 (1991), no. 2, pp. 684–693.

[12] C. G. JOCKUSCH, JR., *Ramsey's theorem and recursion theory*. ***The Journal of Symbolic Logic***, vol. 37 (1972), pp. 268–280.

[13] A. PAULY, *More on the indivisibility of $\mathbb{Q}$*, preprint, 2024, arXiv:2407.03722.

[14] A. PAULY and G. SOLDÀ, *Sequential discontinuity and first-order problems*, ***Twenty Years of Theoretical and Practical Synergies*** (L. L. Patey, E. Pimentel, L. Galeotti, and F. Manea, editors), Springer Nature, Cham, 2024, pp. 351–365.

[15] G. SOLDÀ and M. VALENTI, *Algebraic properties of the first-order part of a problem*. ***Annals of Pure and Applied Logic***, vol. 174 (2023), no. 7, Article no. 103270, 41 pp.

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DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
SINGAPORE
E-mail: gohjunle@nus.edu.sg

TECHNISCHE UNIVERSITÄT WIEN
AUSTRIA
E-mail: ellen.hammatt@tuwien.ac.at

SCHOOL OF COMPUTER SCIENCE and ENGINEERING
UNIVERSITY OF ELECTRONIC SCIENCE AND TECHNOLOGY OF CHINA
CHINA
E-mail: heertern001@e.ntu.edu.sg

SCHOOL OF PHYSICAL & MATHEMATICAL SCIENCES
NANYANG TECHNOLOGICAL UNIVERSITY
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