THE GENUS OF A DEVELOPABLE SURFACE

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Cayley's remark* that the formula by which the genus of a surface, according to Clebsch's definition, may presumably be computed leads to a negative number in the case of a cone, or a developable surface, or a ruled surface in general, has great importance in the history of the theory. But it would appear, from various indications, that, for a developable surface at least, it is more often quoted than read. I have thought therefore that the following simplifying remarks may have a use. Cayley uses formulae, due to Salmon and Cremona, without reference to the memoir† where these are given in detail. Of two of these, for the number of tangents of a curve which meet it again, and for the number of triple points of the nodal curve, proofs by the theory of correspondence are extant; for the present purpose it is only necessary to have the sum of these two numbers. I do not know whether it has been remarked that there exists a remarkable formula for this sum, very similar to, and including the ordinary formula for the number of triple points of a general ruled surface (and like this probably capable of a direct proof by the theory of correspondence). For the genus of the nodal curve, deduced by Cayley from the Salmon-Cremona formulae, a proof by the theory of correspondence (in the general case, sufficient for the purpose in hand, in which $i=\tau=\delta=\delta'=0$) is added here, which seems to have a certain interest.

1. We start with Cayley's formula for the genus of a developable surface‡ which he gives with a reference to his memoir on reciprocal surfaces§, namely, using $(n-1,3)$ for $(n-1)(n-2)(n-3)/6$,

$$(n-1,3)-(n-3)(b+c)+\tfrac{1}{2}(q+r)+2t+\tfrac{7}{2}\beta+\tfrac{5}{2}\gamma+i-\tfrac{1}{8}\theta,$$

wherein n is the order of the surface, supposed to be formed by the tangents of a curve of order c and rank r; the surface has a double curve of order b and rank q, having t triple points, each an intersection of three tangents of the curve c. The number β is that of the stationary points of the curve c, which is a cuspidal curve on the surface under consideration; γ is the number of tangents of the curve c which meet the curve again, and it can be shown that these are stationary points

* Cayley, *Math. Ann.* 3 (1871), 526–9; *Collected Papers*, 8, 394–7.

† Cayley, *Quarterly Journal*, 11 (1871), 294–317; *Collected Papers*, 8, 72–91.

‡ Cayley, *Collected Papers*, 8, 395.

§ Cayley, *Phil. Trans.* 159 (1869), 201–29 (411); *Collected Papers*, 6, 356.

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on the curve b*. The number i is that of the stationary planes of the curve c, and is the number of intersections of the curves c and b which are not stationary on either curve†. θ is a number which Cayley assumes to vanish.

If we substitute for r and q in terms of the genus p of the curve c, and the genus p' of the curve b, putting

$$r = 2c + 2p - 2 - \beta, \quad q = 2b + 2p' - 2 - \gamma,$$

and at the same time replace n by r, the order of the developable formed by the tangents of the curve c, and also *use* κ, *instead of* β, for the number of stationary points on the curve c, *and* κ', *instead of* i, for the number of stationary planes of the curve c, *and* n, ν *instead of* c, b, for the orders of the original curve c, and the nodal curve b, respectively, the formula becomes (omitting θ)

$$(r-1, 3) - (r-4)(n+\nu) + 2t + (p-1+p'-1) + 3\kappa + 2\gamma + \kappa'$$

and what is required is to show that, if we suppose to be zero each of the four characters of the curve denoted by Salmon and Cayley by H, G, υ, ω, which we denote here respectively by δ, δ', i, τ (namely the number of actual double points, of double osculating planes, of inflexions and of double tangents of the curve c), then this expression reduces to $-p$. But we remark that if we assume‡ that $3\kappa + 2\gamma + \kappa'$ is the number of intersections of the original curve (n) with the nodal curve (ν), and use Noether's formula for the genus of a composite curve§, and so write for the order of the composite double curve of the developable, $N = n + \nu$, and in terms of the genus P of this composite curve, put

$$(p-1+p'-1) + 3\kappa + 2\gamma + \kappa' = P - 1,$$

then the formula $$(r-1, 3) - (r-4)N + 2t + P - 1$$

is the usual one for the arithmetic genus of a surface of order r, with a double curve of order N and genus P with t triple points, but without isolated nodes.

To verify the formula in its original form we use the well-known formulae

$$n + \nu = \tfrac{1}{2}(r-1)(r-2) - (p + i + \tau),$$

$$\kappa + \kappa' = 2(r + 4p - 4) - 2i,$$

also the remarkable formula referred to above, namely

$$t + \gamma + 4\delta + \kappa = (r-4)\{\tfrac{1}{6}(r-2)(r-3) - (p + i + \tau)\},$$

which can be verified from the Salmon-Cremona formulae‖, and lastly the formula $$p - 1 + p' - 1 = (r-13)(p-1) + \tfrac{1}{2}(r^2 - 9r) - (\tau + \delta + \delta'),$$

* Cayley, *Collected Papers*, 8, 85.

† In Cayley's *Collected Papers*, 8, 395, it is stated that the (β) points are stationary on b; and the (γ) points stationary on c; and this misprint is copied in Salmon, *Geometry of three dimensions* (1882), 558, and reproduced in the new edition, Salmon-Rogers, 2 (1915), 268.

‡ Cayley, *Collected Papers*, 8, 86. § Noether, *Acta Math.* 8 (1886), 161–92 (182).

‖ Cayley, *Collected Papers*, 8, 76; or from the formulae given in H. F. Baker, *Principles of Geometry*, 6, 38, 32, where ξ is used for γ.

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given by Cayley, of which a proof when $\tau = \delta = \delta' = 0$ is added below. When these are substituted the result, on easy reduction, is

$$-p-(r-3)(i+\tau)-(i+9\delta+\delta')$$

which is $-p$ when $i=\tau=\delta=\delta'=0$.

Cayley's formula for p' is deduced from the following formula for the rank q of the nodal curve

$$q+\gamma+2\delta+i(r-6)=2\nu+2p'-2,$$

equivalent to the statement that a correct estimate of the number of stationary points of this curve is $\gamma+2\delta+i(r-6)$.

2. To prove the formula for the genus p' of the nodal curve directly, in the most general case, we use Zeuthen's formula for the correspondence between the original curve (c), of order n and genus p, and the nodal curve (b), of order ν and genus p'. To each point of c correspond in general $r-4$ points of b, where the tangent at the point of c is met by other tangents of the same curve; and to each point of the nodal curve b correspond 2 points of c. Zeuthen's formula thus gives

$$2(2p'-2)-\zeta'=(r-4)(2p-2)-\zeta,$$

where ζ denotes the number of coincidences of the two points of the curve c, and ζ' the number of coincidences of two of a set of $r-4$ points on b. These numbers seem to be obtainable most clearly by considering the curve θ, of order r, on the quadric Ω, in space [5], which represents the tangent lines of the curve c. The tangent lines of this curve θ lie on Ω, and the tangent prime of Ω at a point of θ meets θ in $r-4$ other points. We thus have on θ a symmetrical $(r-4,\ r-4)$ correspondence, of valency 4. The ∞^1 joins of pairs of corresponding points in this correspondence, which lie on θ, constitute a unifold ϕ, in (1, 1) correspondence with the nodal curve b. Between θ and ϕ, of respective genera p and p', there is then a $(2,\ r-4)$ correspondence of valency 4. The coincidences of pairs of θ in this correspondence are then in number $\zeta=(r-4)+(r-4)+2\,.\,4p$, namely $\zeta=2r+8(p-1)$. The coincidences on ϕ are those in the direct lateral correspondence derived from the $(r-4,\ r-4)$ correspondence on θ* and are then in number $\zeta'=2s(r-1)+2p(s-\gamma^2)$, where $r=s=n-4$, $\gamma=4$; thus

$$\zeta'=2(r-4)(r-5)+2p(r-20).$$

If these values for ζ, ζ' are substituted, the formula for p' (supposing $\tau=\delta=\delta'=0$) is the result.

* H. F. Baker, *Principles of Geometry*, 6, 11; the number 2 is twice omitted in line 6 from the foot.

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