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A NOTE ON BAYESIAN INFERENCE IN ASSET PRICING

J.L. KNIGHT
University of Western Ontario

S.E. SATCHELL
Trinity College
and
University of Cambridge

In this paper the authors extend results by Harvey and Zhou (1990, *Journal of Financial Econometrics* 26, 221–254) and Kandel, McCulloch, and Stambaugh (1995, *Review of Financial Studies* 8(1), 1–53) to derive the posterior distribution of a key parameter in a Bayesian analysis of asset pricing models. It is shown that this distribution depends upon the same terms that constitute the standard asset pricing test of Jobson and Korkie (1985, *Canadian Journal of Administrative Science* 12, 114–138). Contrary to the view held by other authors, we find straightforward expressions for the posterior distribution that can be calculated without resorting to Monte Carlo methods.

I. INTRODUCTION

In a paper on Bayesian inference in asset pricing tests, based on linear regression models, Harvey and Zhou (1990) address the question of the posterior distribution of the fundamental parameter central to the W statistic proposed by Jobson and Korkie (1985) and Gibbons, Ross and Shanken (1989). Further discussion and results can be found in Kandel, McCulloch, and Stambaugh (1995). This parameter, which we shall call λ, is analyzed under the usual assumptions of a diffuse prior and where the assumptions on the data are those consistent with classic linear regression and normal errors.

Harvey and Zhou (1990), give a formula for the posterior mean of λ but state that "the (posterior) variance of λ is analytically intractable" (p. 227); they mention that

> Nevertheless, since λ is the difference between the squared Sharpe measures of the tangency portfolio and the given portfolio, we are still interested in this difference along with its distributional properties, for it may shed additional light on our understanding of the behaviour of the given portfolio. (p. 239)

The authors thank Peter Phillips and three anonymous referees for their helpful and constructive comments on an earlier draft. The first author thanks the Natural Sciences and Engineering Research Council of Canada for financial support. Address correspondence to: John Knight, Department of Economics, University of Western Ontario, London, Ontario N6A 5C2, Canada; e-mail: jknight@julian.uwo.ca.

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Similar remarks about the intractability of λ and the need to discover its properties have been echoed in a paper by Roy (1995).

Building on results proved in Kandel et al. (1995), we investigate the properties of λ. Whereas some of our results are known, our contribution is to derive an expression for the posterior distribution of λ (Proposition 2.3). We also present formulae for all the moments and an expression for the characteristic function that allows approximations to the distribution to be easily derived and consequently approximations to the probability that λ is in an interval near zero; knowledge of such a region can be used to complement the Bayesian confidence intervals on the individual alphas presented in Harvey and Zhou (1990). Although our analysis is theoretical, in that we use diffuse priors and condition on the market parameters, our results could be extended to include the work by Kandel et al. (1995). They consider the case where the efficient portfolio's parameters are also included in the analysis.

2. RESULTS

Following the notation of Harvey and Zhou (1990), we consider the regression

$$r_{it} = \alpha_i + \beta_{ip} r_{pt} + \varepsilon_{it}, \qquad i = 1,\ldots,N, \qquad t = 1,\ldots,T, \tag{1}$$

where r_{it} is an excess return and the assumptions are discussed in detail by the authors; they are the standard assumptions of seemingly unrelated regression.

In matrix terms, equation (1) becomes

$$R = X\beta + E, \tag{2}$$

where R is $(T \times N)$, X is $(T \times 2)$, β is $(2 \times N)$, and E is $(T \times N)$. Furthermore, $\text{vec}(E) \sim N(0, \Sigma \otimes I_T)$ where Σ is an $(N \times N)$ positive definite covariance matrix. The classical tests investigate

$$\begin{aligned} &H_0 : \alpha_i = 0, \qquad i = 1,\ldots,N, \\ &H_1 : \quad \text{at least one } \alpha_i \neq 0, \end{aligned} \tag{3}$$

and tests of these hypotheses depend (under H_1) on the parameter $\lambda = \alpha' \Sigma^{-1} \alpha$ that was discussed in Section 1; α is an $(N \times 1)$ vector, $\alpha' = (\alpha_1,\ldots,\alpha_N)$.

The diffuse prior assumption is written as

$$P(\beta, \Sigma) \propto |\Sigma|^{-(N+1)/2}, \tag{4}$$

and the following standard results are obtained:

$$P(\beta | \Sigma, R, X) \propto |\Sigma|^{-1} \exp\{-\tfrac{1}{2}\text{tr}[(\beta - \hat{\beta})'(\Sigma^{-1} \otimes X'X)(\beta - \hat{\beta})]\}$$

and

$$P(\Sigma | R, X) \propto |\Sigma|^{-\mu/2} \exp\{-\tfrac{1}{2}\text{tr}(\Sigma^{-1} S)\}, \tag{5}$$

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where

$$\hat{\beta} = (X'X)^{-1}X'R$$

and

$$S = (R - X\hat{\beta})'(R - X\hat{\beta})$$

and

$$\mu = T + N - 1.$$

Harvey and Zhou prove in their Appendix that

$$P(\alpha|\Sigma, R, X) \sim N(\hat{\alpha}, c\Sigma), \tag{6}$$

where c is the (1,1) element of $(X'X)^{-1}$ and $\hat{\alpha}$ is the first element in the vector $\hat{\beta}$. Their result, in fact, has already been proved (see Box and Tiao, 1973, Theorem 8.5.1, p. 461). Indeed the general result is that

$$P(\text{vec}(\beta)|\Sigma, R, X) \sim N(\text{vec}(\hat{\beta}), \Sigma \otimes (X'X)^{-1}),$$

so that the result follows immediately.

Having defined our terms we now consider the characteristic function, the moments, and the probability density function (p.d.f.) of λ. Our main result, the exact posterior distribution of λ, is given in Proposition 2.3, whereas Proposition 2.1 and Corollary 2.2 give the associated characteristic function and moments, respectively.

PROPOSITION 2.1. *The characteristic function of λ is given by*

$$\phi_\lambda(t) = (1 - 2ict)^{(T-N-2)/2}(1 - 2it(c + \hat{\alpha}'S^{-1}\hat{\alpha}))^{-(T-2)/2}.$$

Proof. See the Appendix.

Remark 1. From the expression for $\phi_\lambda(t)$ we note that the second term in the product corresponds to a characteristic function of a random variable with a distribution being $(c + \hat{\alpha}S^{-1}\hat{\alpha}) \times \chi^2_{(T-2)}$. Consequently, the existence of moments of λ will be determined by the existence of moments of a $\chi^2_{(T-2)}$ random variable. Therefore we can state immediately that all positive moments (including integer and fractional) moments of λ will exist. Also because it is easily shown that $E((\chi^2_m)^{-n})$ will exist only when $n < m/2$ we can now state that all inverse (negative) real moments exist as long as $n < (T - 2)/2$.

With the results of Proposition 2.1, we can derive analytically posterior moments of any order. We shall present results for the (posterior) mean and variance of λ.

COROLLARY 2.2. *The first two central posterior moments of λ are given by*

$$E(\lambda) = Nc + (T - 2)\hat{\alpha}'S^{-1}\hat{\alpha},$$
$$\text{Var}(\lambda) = 2c^2N + 2(T - 2)((\hat{\alpha}S^{-1}\hat{\alpha})^2 + 2c(\hat{\alpha}S^{-1}\hat{\alpha})).$$

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Proof. See the Appendix.

We now state our main result, the exact posterior distribution of λ, in the following proposition.

PROPOSITION 2.3. *The exact posterior distribution of* $\lambda = \alpha'\Sigma^{-1}\alpha$ *is given by*[1]

$$pdf(\lambda) = \frac{\lambda^{(N/2)-1}e^{-(\lambda/2c)}}{\Gamma\left(\frac{N}{2}\right)(2c)^{N/2}}\left(\frac{c}{(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right)^{(T-2)/2} \times {}_1F_1\left(\frac{T-2}{2};\frac{N}{2};\frac{\lambda\hat{\alpha}'S^{-1}\hat{\alpha}}{2c(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right),$$

where the confluent hypergeometric function ${}_1F_1(\,)$ *is defined as,*

$${}_1F_1(a;b;x) = \sum_{j=0}^{\infty}\frac{(a)_j}{(b)_j\, j!}\,x^j \quad \text{with } (a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}.$$

In the particular case when $T-2-N$ *is an even integer, say,* 2ℓ*, we have an alternative form of the density given by*

$$pdf(\lambda) = \frac{\lambda^{(N/2)-1}}{\Gamma\left(\frac{N}{2}\right)(2c)^{N/2}}\left(\frac{c}{(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right)^{(T-2)/2}\exp\left(\frac{-\lambda}{2(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right) \times \sum_{j=0}^{\ell}\binom{\ell}{j}\frac{1}{\left(\frac{N}{2}\right)_j}\left(\frac{-\lambda\hat{\alpha}'S^{-1}\hat{\alpha}}{2(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right)^j.$$

Proof. See the Appendix.

Remark 2. Examining the posterior *pdf* we see that it depends upon two terms, c and $\hat{\alpha}'S^{-1}\hat{\alpha}$. As mentioned earlier, these are components of exact classical tests. In particular, c, being the (1,1) element of the (2×2) data matrix $(X'X)^{-1}$, can be written in terms of moments of the market portfolio. If we denote r_{pt} as the return on the market for period t, then c can be calculated as

$$c = \frac{m_2}{Tm_2'}, \tag{7}$$

where $m_2' = (1/T)(\Sigma r_p^2 - Tm_1^2)$ is the sample variance, $m_2 = (1/T)\Sigma r_p^2$ is the sample second moment, and $m_1 = (1/T)\Sigma r_p$ is the sample mean. Considering

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the exact test of Jobson and Korkie (1985) (see also Gibbons et al., 1989), we see that this can be written as

$$J_1 = \left(\frac{T-N-1}{N}\right)\hat{\alpha}'S^{-1}\hat{\alpha}/c \tag{8}$$

because

$$J_1 = \left(\frac{T-N-1}{N}\right)\left(1+\frac{m_1^2}{m_2'}\right)^{-1}\hat{\alpha}'\left(\frac{S}{T}\right)^{-1}\hat{\alpha} \tag{9}$$

(see Campbell, Lo, and MacKinlay, 1996, equation (6.2.12), p. 224). If we write $m_2' = m_2 - m_1^2$ then from the definition of c in (7), equation (9) follows immediately. Roy (1995) has also calculated the *pdf* of λ numerically using data from Harvey and Zhou (1990) and 20,000 draws of $\hat{\alpha}$ from a matrix-variate t and 20,000 draws of Σ from an inverse Wishart distribution. He finds that there is no concentration of probability mass near zero for the Harvey and Zhou data.

3. CONCLUSION

This paper has derived an expression for the posterior distribution of λ and its moments. Our results have allowed us to resolve any debate about the tractability of the posterior *pdf* of λ. This quantity is easy to compute and its moments and asymptotics easy to calculate and prove. It is a reasonable criticism that our results do not reflect the priors held by any investor or econometrician. Indeed other authors, cited earlier, have attempted to use informative priors on this problem.

NOTE

1. As one of the referees pointed out, the ${}_1F_1(\cdot;\cdot;\cdot)$ function may converge very slowly for certain parameter values. In these situations it may well be worthwhile to approximate the expression using standard asymptotic expansions for these functions. The interested reader is referred to Slater (l960).

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APPENDIX

Proof of Proposition 2.1. From (6) and assuming Σ is positive definite such that $\Sigma = HH'$, H non-singular, we have

$$H^{-1}\alpha|\Sigma, R, X \sim N(H^{-1}\hat{\alpha}, cI_N)$$

and

$$\frac{H^{-1}\alpha}{\sqrt{c}}\Bigg|\Sigma, R, X \sim N\left(\frac{H^{-1}\hat{\alpha}}{\sqrt{c}}, I_N\right). \tag{A.1}$$

Consequently, because $\lambda = \alpha'\Sigma^{-1}\alpha = \alpha' H'^{-1}H^{-1}\alpha$ we have

$$\frac{\lambda}{c}\Bigg|\Sigma, R, X \sim \chi^2_N(\delta)$$

with

$$\delta = \frac{\hat{\alpha}'\Sigma^{-1}\hat{\alpha}}{c},$$

and thus

$$\lambda|\delta \sim c \times \chi^2_N(\delta) \tag{A.2}$$

and

$$E[e^{it\lambda}|\delta] = (1 - 2ict)^{-N/2} \exp\left(\frac{itc\delta}{1 - 2ict}\right).$$

Now from (5) we infer that Σ^{-1} is Wishart distributed, i.e., $\Sigma^{-1} \sim W_N(T - 2, S^{-1})$, and using Theorem 3.2.8 in Muirhead (1982) we have

$$\hat{\alpha}'\Sigma^{-1}\hat{\alpha} \sim \hat{\alpha}'S^{-1}\hat{\alpha} \times \chi^2_{(T-2)},$$

so that

$$\delta = \frac{\hat{\alpha}'\Sigma^{-1}\hat{\alpha}}{c} \sim \frac{\hat{\alpha}'S^{-1}\hat{\alpha}}{c} \times \chi^2_{(T-2)} \tag{A.3}$$

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and its associated characteristic function

$$E[e^{it\delta}] = \phi_\delta(t) = \left[1 - \frac{2it\hat{\alpha}'S^{-1}\hat{\alpha}}{c}\right]^{-(T-2)/2}.$$

Therefore

$$E[e^{it\lambda}] = E[E(e^{it\lambda}|\delta)]$$
$$= (1-2ict)^{-N/2} \times (1-2itc)^{(T-2)/2} \times (1-2it(c+\hat{\alpha}'S^{-1}\hat{\alpha}))^{-(T-2)/2},$$

and the result follows. ■

Proof of Corollary 2.2. From Proposition 2.1 we have the moment generating function (m.g.f.) of λ given by

$$M_\lambda(t) = \phi_\lambda(-it).$$

The mean and variance are readily found by taking the log of $M_\lambda(t)$ and evaluating its first and second derivatives at $t = 0$. ■

Proof of Proposition 2.3. From (A.2) we have

$$\lambda|\delta \sim c \times \chi^2_N(\delta),$$

where $\delta = \hat{\alpha}'\Sigma^{-1}\hat{\alpha}/c$. Also from (A.3) we note that

$$\delta \sim \frac{\hat{\alpha}'S^{-1}\hat{\alpha}}{c} \times \chi^2_{(T-2)}.$$

Letting $x \sim \chi^2_N(\delta)$, then

$$pdf(x) = e^{-(\delta/2)} \sum_{j=0}^{\infty} \frac{(\delta/2)^j}{j!} \frac{x^{(N/2)+j-1}e^{-(x/2)}}{\Gamma\left(\frac{N+2j}{2}\right)2^{(N+2j)/2}},$$

and thus $\lambda = c \times x$ has

$$pdf(\lambda|\delta) = e^{-(\delta/2)} \sum_{j=0}^{\infty} \frac{(\delta/2)^j}{j!} \frac{\lambda^{(N/2)+j-1}e^{-(\lambda/2c)}}{\Gamma\left(\frac{N+2j}{2}\right)(2c)^{(N+2j)/2}}$$

and

$$pdf(\lambda) = E[pdf(\lambda|\delta)].$$

Consequently, we require $E[\delta^j e^{-(\delta/2)}]$ where $pdf(\delta)$ is given by

$$pdf(\delta) = \left(\frac{c}{2(\hat{\alpha}'S^{-1}\hat{\alpha})}\right)^{(T-2)/2} \frac{\delta^{((T-2)/2)-1}}{\Gamma\left(\frac{T-2}{2}\right)} \exp\left(-\frac{\delta c}{2\hat{\alpha}'S^{-1}\hat{\alpha}}\right).$$

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That is,

$$E[\delta^j e^{-(\delta/2)}] = \left(\frac{c}{2(\hat{\alpha}'S^{-1}\hat{\alpha})}\right)^{(T-2)/2} \frac{1}{\Gamma\left(\frac{T-2}{2}\right)} \int_0^\infty \delta^{((T-2+2j)/2)-1} \times \exp\left(-\frac{\delta}{2}\left(1+\frac{c}{(\hat{\alpha}'S^{-1}\hat{\alpha})}\right)\right) d\delta$$
$$= \left(\frac{T-2}{2}\right)_j \left(\frac{c}{c+\hat{\alpha}'S^{-1}\hat{\alpha}}\right)^{(T-2)/2} \left(\frac{2\hat{\alpha}'S^{-1}\hat{\alpha}}{c+\hat{\alpha}'S^{-1}\hat{\alpha}}\right)^j.$$

Therefore the required $pdf(\lambda)$ is given by

$$pdf(\lambda) = \frac{\lambda^{(N/2)-1}e^{-(\lambda/2c)}}{\Gamma\left(\frac{N}{2}\right)(2c)^{N/2}} \left(\frac{c}{c+\hat{\alpha}'S^{-1}\hat{\alpha}}\right)^{(T-2)/2} \sum_{j=0}^{\infty} \frac{\left(\frac{T-2}{2}\right)_j}{\left(\frac{N}{2}\right)_j j!} \left(\frac{(\hat{\alpha}'S^{-1}\hat{\alpha})\lambda}{2c(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right)^j$$
$$= \frac{\lambda^{(N/2)-1}e^{-(\lambda/2c)}}{\Gamma\left(\frac{N}{2}\right)(2c)^{N/2}} \left(\frac{c}{c+\hat{\alpha}'S^{-1}\hat{\alpha}}\right)^{(T-2)/2} {}_1F_1\left(\frac{T-2}{2};\frac{N}{2};\frac{\lambda\hat{\alpha}'S^{-1}\hat{\alpha}}{2c(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right).$$

Now

$${}_1F_1\left(\frac{T-2}{2};\frac{N}{2};\frac{\lambda\hat{\alpha}'S^{-1}\hat{\alpha}}{2c(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right)$$
$$= \exp\left(\frac{\lambda\hat{\alpha}'S^{-1}\hat{\alpha}}{2c(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right) {}_1F_1\left(\frac{N}{2}-\frac{T-2}{2};\frac{N}{2};\frac{-\lambda\hat{\alpha}'S^{-1}\hat{\alpha}}{2c(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right),$$

and if $T-2-N=2\ell$, the first parameter in the ${}_1F_1(\cdot;\cdot;\cdot)$ function on the right is now a negative integer, namely, $-\ell$. This will ensure that the ${}_1F_1(\cdot;\cdot;\cdot)$ function is now a terminating series, not an infinite series, and in fact is a polynomial of degree ℓ in the argument $\lambda\hat{\alpha}'S^{-1}\hat{\alpha}/[2c(c+\hat{\alpha}'S^{-1}\hat{\alpha})]$. In particular we have

$${}_1F_1(-\ell;b;-x) = \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{x^j}{(b)_j},$$

and thus when $T-2-N=2\ell$ (i.e., an even integer) the *pdf* is now given by

$$pdf(\lambda) = \frac{\lambda^{(N/2)-1}}{\Gamma\left(\frac{N}{2}\right)(2c)^{N/2}} \left(\frac{c}{c+\hat{\alpha}'S^{-1}\hat{\alpha}}\right)^{(T-2)/2} \exp\left(\frac{-\lambda}{2(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right)$$
$$\times \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{1}{\left(\frac{N}{2}\right)_j} \left(\frac{\lambda\hat{\alpha}'S^{-1}\hat{\alpha}}{2c(c+\hat{\alpha}'S^{-1}\hat{\alpha})}\right)^j \quad \blacksquare$$

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