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A Note on Vertex List Colouring

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Let k be a positive integer and let G be a graph. Suppose a list $S(v)$ of positive integers is assigned to each vertex v, such that

(1) $|S(v)| = 2k$ for each vertex v of G, and
(2) for each vertex v, and each $c \in S(v)$, the number of neighbours w of v for which $c \in S(w)$ is at most k.

Then we prove that there exists a proper vertex colouring f of G such that $f(v) \in S(v)$ for each $v \in V(G)$. This proves a weak version of a conjecture of Reed.

In this note we study a problem on vertex list colouring of graphs. Let G be a graph, let $V(G)$ denote its vertex set, and suppose each $v \in V(G)$ is assigned a *list* $S(v)$ of positive integers. We say that a vertex colouring f of G is a *proper list colouring* of G with respect to the set $\mathscr{S} = \{S(v) : v \in V(G)\}$ of lists if

(i) f is a proper vertex colouring of G, that is, for each edge vw of G we have $f(v) \neq f(w)$, and
(ii) $f(v) \in S(v)$ for each vertex v of G.

The following conjecture on constrained list colouring of graphs was proposed by Reed in [10].

Conjecture 1. *Let a positive integer k, a graph G and a set $\mathscr{S} = \{S(v) : v \in V(G)\}$ of lists be given. Suppose that*

(1) $|S(v)| = k + 1$ *for each vertex $v \in V(G)$, and*
(2) *for each vertex $v \in V(G)$, and each $c \in S(v)$, the number of neighbours w of v for which $c \in S(w)$ is at most k.*

Then there exists a proper list colouring of G with respect to $\mathscr{S}$.

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In [10], Reed proved the weaker version of this conjecture formed by replacing $k+1$ by $2ek$ in (1). Our aim here will be to show that the conjecture is true with $2k$ instead of $k+1$. This result will follow quite easily from the following theorem.

Theorem 2. *Let k be a positive integer, let H be a graph of maximum degree at most k, and let $V(H) = V_1 \cup \cdots \cup V_n$ be a partition of the vertex set of H. Suppose that $|V_i| \geqslant 2k$ for each i. Then H has an independent set $\{v_1, \ldots, v_n\}$ where $v_i \in V_i$ for each i.*

This theorem with the quantity $2ek$ instead of $2k$ was proved by Alon [5] (see also Alon and Spencer [6, Chapter 5] for this lemma and its role in proving an asymptotic version of the Linear Arboricity Conjecture [4]). Theorem 2 is best possible in the sense that $2k$ could not be replaced by ck for any constant $c < 2$. This was first proved by Bollobás, Erdős and Szemerédi in [7], and they conjectured that $c = 2$ was the correct value. Further constructions were given later by Yuster [11], who showed that, for every p and q, there exists a graph with maximum degree $q2^{p-1}$ and vertex classes $V_1, \ldots, V_{2^p}$, each of size $q(2^p - 1)$, such that no set $\{v_1, \ldots, v_{2^p} : v_i \in V_i\}$ can be independent.

Theorem 2 is in fact a special case of the following more general theorem. Here we say that a set U of vertices in a graph is *dominated by* a set W if every $u \in U$ is adjacent to some $w \in W$.

Theorem 3. *Let H be a graph and let $V(H) = V_1 \cup \cdots \cup V_n$ be a partition of the vertex set of H. Suppose that, for every set $S \subseteq \{1, \ldots, n\}$, the set $V_S = \bigcup_{i \in S} V_i$ is not dominated by any subset of V_S of size less than $2|S| - 1$. Then H has an independent set $\{v_1, \ldots, v_n\}$, where $v_i \in V_i$ for each i.*

To see that Theorem 2 follows from Theorem 3, one simply notes that the number of vertices in a graph of maximum degree k that can be dominated by a set of size $2|S| - 2$ is at most $(2|S| - 2)k$. Thus the assumptions of Theorem 2 imply that $|V_S| \geqslant 2|S|k$, and so the assumptions of Theorem 3 hold.

Theorem 3 first appeared in [3] in terms of systems of disjoint representatives of families of hypergraphs, although its proof is essentially the same as the proof of [8]. A common generalization of Theorem 3 and the result of [2] was proved by Meshulam in [9] (Theorem 1.5, stated there in terms of cliques rather than independent sets), and was later given an alternate proof by Aharoni, Chudnovsky and Kotlov in [1].

We now use Theorem 2 to prove our result about list colouring.

Corollary 4. *Let a positive integer k, a graph G and a set $\mathscr{S} = \{S(v) : v \in V(G)\}$ of lists be given. Suppose that*

(1) *$|S(v)| = 2k$ for each vertex $v \in V(G)$, and*
(2) *for each vertex $v \in V(G)$, and each $c \in S(v)$, the number of neighbours w of v for which $c \in S(w)$ is at most k.*

Then there exists a proper list colouring f of G with respect to $\mathscr{S}$.

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Proof. We reformulate the problem as follows. Given k, G and $\mathscr{S}$, we define an auxiliary graph H by letting the vertex set $V(H)$ of H be $\{(v,c) : v \in V(G), c \in S(v)\}$, and joining vertices (v,c) and (w,c') in H if and only if $c = c'$ and $vw \in E(G)$. Observe then that (2) guarantees that the maximum degree of H is at most k, and (1) tells us that for each vertex v of G, the set $V_v = \{(v,c) : c \in S(v)\}$ is a set of size $2k$ in H.

Now, by Theorem 2, there is an independent set $\{(v,c_v) : v \in V(G)\}$ in H. Then, by the construction of H, the colouring of G defined by $f(v) = c_v$ for each $v \in V(G)$ gives a proper list vertex colouring of G as required. □

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