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BOOK REVIEWS

Basic Ergodic Theory and *Spectral Theory of Dynamical Systems*

By M. G. NADKARNI
Birkhäuser Verlag, Basel, Boston, Berlin, 1995 and 1998

These two books look like they might form a two-volume set, but this is not so. The first is an introduction to ergodic theory suitable as a text for a first course in the subject, with some of the standard results viewed in an unusual light (that of descriptive set theory) and with detailed discussion of several topics of special interest to the author. The second is a survey at an advanced level of parts of the interaction of measure-preserving and nonsingular ergodic theory with spectral theory and harmonic analysis. It assumes considerable background in all three of these subjects and is written as a guide for researchers, organizing many topics of recent and current interest and highlighting some major open questions on which further progress might be possible at this time. Selected topics could be worked through individually or as part of a seminar or advanced course. Interestingly, the second does not even refer to the first for needed results from basic ergodic theory.

The two books do have some formal aspects in common. Both are written in a numbered-paragraph style, references are given chapter by chapter and the indexes are short. The large number of topics addressed cannot possibly be covered in so small a space, so many theorems and examples are not proved completely but are stated or described, sometimes with an indication of the idea of the proof and references for the details, while others are found in exercises. Historical discussions and references, although not complete, trace the development of ideas and connections among them and point the way to open problems. In both books the writing is economical, with much filling in left to the reader. Sometimes an extended discussion develops a train of thought which is summarized in the statement of a theorem; this is a pleasant and natural writing style in which we can see the mathematics emerge, but it can make searches for results and definitions a challenge. There are some typographical errors, slips (such as an incorrect proof of the Hewitt–Savage 0, 1 Law) and organizational lapses (some material is repeated, and some skipping around is required), but not so many as to be annoying.

Basic Ergodic Theory smoothly and clearly presents the standard topics at not too high a level, building on permutations of a finite set. Entropy is mentioned only briefly. The author puts a special slant on some of the standard theorems by discussing descriptive-set-theoretic, measure-free versions where they are available. Thus the Poincaré Recurrence

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Theorem is presented in its usual form for measure-preserving transformations and also in measure-free and category forms, due in large part to Oxtoby's distillation of Poincaré's original argument. For example, given a Borel automorphism (one-to-one, onto, bimeasurable map) T of a standard Borel space $(X, \mathcal{B})$ (Borel isomorphic to an uncountable complete metric space with the σ-algebra generated by the open sets) and $A \in \mathcal{B}$, there is a wandering set $N \in \mathcal{B}$ such that for each $x \in A \setminus N$ we have $T^n x \in A$ for infinitely many positive n and infinitely many negative n.

Again, in addition to the usual statement of Birkhoff's Pointwise Ergodic Theorem and proof via the Maximal Ergodic Theorem (with Garsia's proof), Nadkarni also presents a proof along the lines of Birkhoff's original and a measure-free version due to Srivatsa: if T is a Borel automorphism of a standard Borel space $(X, \mathcal{B})$ and $E \in \mathcal{B}$, then the set of points for which the ergodic averages of the characteristic function of E do not converge is compressible. More than a curiosity, this result is used in proving Hopf's theorem on the existence of equivalent finite invariant measures for incompressible nonsingular systems. The descriptive version due to Weiss of Rokhlin's countable generator theorem is presented, and extensions due to Kechris are mentioned. The usual version of Dye's theorem appears with proof after Hajian, Ito and Kakutani, followed by the descriptive version due to Dougherty, Jackson and Kechris (with a sketch of the proof for the case when there are at most countably many invariant measures): any two free Borel automorphisms of a standard Borel space which have the same number of invariant ergodic probability measures are orbit equivalent—*without* having to discard any 'small' sets. The final chapter presents nonsingular (due to Krengel, Kubo and Dani) and descriptive (due to Wagh) versions of Ambrose's theorem on the representation of flows.

What is the value of these descriptive versions of normally measure-theoretic results? Sometimes we need to have available tools such as the Baire Category Theorem or regularity of measures, so it is good to work in standard Borel spaces. Sometimes we might not have any measure, or any natural measure, or we might be interested in a result that holds for all invariant or quasi-invariant measures. And even the most basic parts of any subject can be looked at in new ways, to seek deeper understanding, new results and new questions. *Basic Ergodic Theory* provides us with new tools and a fresh viewpoint, compared to the predominant literature.

Spectral Theory of Dynamical Systems begins with introductory sections which develop spectral theory by means of the Hahn–Hellinger Theorem and then presents very efficiently many topics of special interest to the author.

— Rank and multiplicity for measure-preserving and nonsingular automorphisms.
— Skew product automorphisms and their spectra.
— An extension to nonsingular hyperfinite actions of countable groups of the Helson–Parry Theorem (the existence of ± 1-valued cocycles $A(g, x)$ for which the unitary group $(V_g f)(x) = A(g, x) f(T_g x)$ acting on L^2 has spectral measure absolutely continuous with respect to Haar measure).
— Baire category type of various classes of nonsingular and measure-preserving automorphisms.
— Measures on the circle: convolution powers, rigidity, Dirichlet subgroups, absolute continuity and singularity of translations, ergodicity (of not necessarily

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quasi-invariant measures) with respect to countable subgroups of the circle group, with an application by Host to the higher mixing problem: the only pairwise independent joining of three measure-preserving transformations, at least one of which is weakly mixing and has maximal spectral type singular to Lebesgue measure, is the independent joining. Consequently, if a measure-preserving transformation is mixing and has maximal spectral type singular with respect to Lebesgue measure, then it is mixing of all orders.

— Groups of quasi-invariance and L^∞ eigenvalue groups, saturated subgroups.

— Systems of imprimitivity, including their concrete realizations by systems with cocycles.

Some of these topics are related and flow naturally one into the next, while the reasons why others were selected are not obvious.

Spectral theory, Fourier theory, dynamics and the study of measures on the circle come together in the final chapters, which describe what is known about the spectral theory of rank one measure-preserving and nonsingular automorphisms. Riesz products and generalized Riesz products describe the spectral measures of multiplication-composition operators generated by certain nonsingular automorphisms with unimodular cocycles, in particular the spectral types of rank one automorphisms and rank one nonsingular transformations. This also allows the computation of the group of L^∞ eigenvalues of any rank one automorphism. A final section describes the construction of Ornstein's rank one mixing transformations. Recent progress on the a.s. singularity of the spectra of these and related transformations has been made by Bourgain and others.

In *Basic Ergodic Theory* the unique viewpoint of the author illuminates the standard topics and informs the choice of more specialized results to explore. In *Spectral Theory of Dynamical Systems* the author's own research interests determine which aspects of spectral theory/harmonic analysis/ergodic theory to explain and which areas of current research to advertise. These two books compile and communicate in a unique way a lot of mathematics that is of current interest but is not otherwise so readily at hand.

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