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Uniformly positive entropy of induced transformations

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Abstract. Let (X, T) be a topological dynamical system consisting of a compact metric space X and a continuous surjective map $T : X \to X$. By using local entropy theory, we prove that (X, T) has uniformly positive entropy if and only if so does the induced system $(\mathcal{M}(X), \widetilde{T})$ on the space of Borel probability measures endowed with the weak* topology. This result can be seen as a version for the notion of uniformly positive entropy of the corresponding result for topological entropy due to Glasner and Weiss.

Key words: continuous surjective maps, dynamics, probability measures, Prohorov metric, weak* topology
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1. *Introduction*

Local entropy theory is a culmination of deep results in dynamics, ergodic theory and combinatorics. Given a dynamical system with positive entropy, it gives, in some sense, the location of where the entropy resides. We refer the reader to the survey article by Glasner and Ye [18] for general information on the subject as well as important contributions

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by many to the field. It is a powerful tool that can be applied in a variety of settings. For example, the second author and Kato [11], using local entropy theory, settled some old problems concerning indecomposable continua in dynamical systems. In this note we further show the power of this theory by giving a simple proof of the theorem stated in the abstract.

By a *topological dynamical system* (TDS) we mean a pair (X, T) consisting of a compact metric space X with metric d and a continuous surjective map $T : X \to X$. Such a TDS induces, in a natural way, the TDSs

$$(\mathcal{K}(X), \overline{T}) \quad \text{and} \quad (\mathcal{M}(X), \widetilde{T}).$$

In the first of these systems, $\mathcal{K}(X)$ denotes the *hyperspace* of all non-empty closed subsets of X endowed with the *Hausdorff metric*

$$d_{\mathrm{H}}(K_1, K_2) := \inf\{\delta > 0 : K_1 \subset K_2^{\delta} \text{ and } K_2 \subset K_1^{\delta}\},$$

where $A^{\delta} := \{x \in X : d(x, A) < \delta\}$ is the *δ-neighborhood* of $A \subset X$, and $\overline{T} : \mathcal{K}(X) \to \mathcal{K}(X)$ is the continuous surjective map given by

$$\overline{T}(K) := T(K) \quad (K \in \mathcal{K}(X)).$$

In the second system, $\mathcal{M}(X)$ denotes the space of all Borel probability measures on X endowed with the *Prohorov metric*

$$d_{\mathrm{P}}(\mu, \nu) := \inf\{\delta > 0 : \mu(A) \leq \nu(A^{\delta}) + \delta \text{ and } \nu(A) \leq \mu(A^{\delta}) + \delta \text{ for all } A \in \mathcal{B}_X\},$$

where $\mathcal{B}_X$ is the σ-algebra of all Borel subsets of X, and $\widetilde{T} : \mathcal{M}(X) \to \mathcal{M}(X)$ is the continuous surjective map given by

$$(\widetilde{T}(\mu))(A) := \mu(T^{-1}(A)) \quad (\mu \in \mathcal{M}(X), A \in \mathcal{B}_X).$$

It is well known that $\mathcal{K}(X)$ and $\mathcal{M}(X)$ are compact metric spaces and that $\overline{T}$ and $\widetilde{T}$ are homeomorphisms whenever T is a homeomorphism. Moreover,

$$d_{\mathrm{P}}(\mu, \nu) := \inf\{\delta > 0 : \mu(A) \leq \nu(A^{\delta}) + \delta \text{ for all } A \in \mathcal{B}_X\}$$

(see [8, page 72]). We refer the reader to [8, 12, 21, 22] for a study of the spaces $\mathcal{K}(X)$ and $\mathcal{M}(X)$.

An interesting line of investigation in the area of dynamical systems is the study of the relations between the dynamics of the TDS (X, T) and the dynamics of the induced TDSs $(\mathcal{K}(X), \overline{T})$ and $(\mathcal{M}(X), \widetilde{T})$. Such investigations were initiated by Bauer and Sigmund [3], and later were widely developed by several authors; see [2, 5–7, 13, 17, 19, 26, 27, 29], for instance. It is worth mentioning that there is a common thread throughout these works: the search for which dynamical properties the base system (X, T) and the induced systems $(\mathcal{K}(X), \overline{T})$ and $(\mathcal{M}(X), \widetilde{T})$ share with each other and which dynamical properties they do not share at all.

It is well known that the extension of a TDS to the hyperspace can dramatically increase the system's complexity. For instance, an example was given in [15] (see also [25]) of a zero topological entropy system (X, T) whose hyperspace extension $(\mathcal{K}(X), \overline{T})$ has positive topological entropy. As another example, it was proved in [4, 6] that for the generic

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homeomorphism h of the Cantor space $\{0, 1\}^{\mathbb{N}}$, the TDS $(\{0, 1\}^{\mathbb{N}}, h)$ has no Li–Yorke pair (in particular, it has zero topological entropy), but its hyperspace extension $(\mathcal{K}(\{0, 1\}^{\mathbb{N}}), \overline{h})$ is uniformly distributionally chaotic and has infinite topological entropy.

The situation for the extension $(\mathcal{M}(X), \widetilde{T})$ is completely different, at least from the point of view of topological entropy. Indeed, a deep and surprising result due to Glasner and Weiss [17] asserts that if (X, T) has zero topological entropy, then so does $(\mathcal{M}(X), \widetilde{T})$. A corresponding result for the notion of null system was obtained by Kerr and Li [23]: if (X, T) is null, then so is $(\mathcal{M}(X), \widetilde{T})$. Recall that (X, T) is said to be *null* if the topological sequence entropy of (X, T) is zero for any increasing sequence of natural numbers. Recently, Qiao and Zhou [28] obtained such a result for the notion of sequence entropy, which unified the above-mentioned results of Glasner and Weiss and of Kerr and Li. We observe that the converses of these results are trivially true, since (X, T) can be regarded as a subsystem of $(\mathcal{M}(X), \widetilde{T})$.

Our goal is to investigate the relationships between these systems for the notion of *uniformly positive entropy* (UPE). This notion was introduced by Blanchard [9] as a candidate for an analogue in topological dynamics for the notion of a K-process in ergodic theory. In fact, in that paper he proved that every non-trivial factor of a UPE system has positive topological entropy and, shortly after, he proved that a UPE system is disjoint from every minimal zero entropy system [10]. Although a UPE system is topologically weakly mixing but not always strongly mixing [10], Glasner and Weiss [16] proved that UPE is a necessary condition for a TDS (X, T) to have a T-invariant probability measure μ of full support whose corresponding measurable dynamical system (X, T, μ) is a K-process.

In this short note we prove that (X, T) has UPE if and only if so does $(\mathcal{M}(X), \widetilde{T})$. This result can be seen as a version of the Glasner–Weiss theorem [17] for the notion of uniformly positive entropy. In the proof of this result we will use local entropy theory; more precisely, we will use the characterization of UPE by means of the notion of an independence set given by Huang and Ye [20] (see also Kerr and Li [24]). Moreover, we will also use an important technique developed by Glasner and Weiss [17] which connects linear operators to combinatorics.

We point out that Huang and Ye [20] proved the corresponding result for the hyperspace, namely, that a TDS (X, T) has UPE if and only if $(\mathcal{K}(X), \overline{T})$ has UPE.

2. *Preliminaries*

Recall that the Prohorov metric d_{P} on $\mathcal{M}(X)$ induces the so-called *weak* topology*, that is, the topology whose basic open neighborhoods of $\mu \in \mathcal{M}(X)$ are the sets of the form

$$\mathbb{V}(\mu; f_1, \ldots, f_k; \varepsilon) := \left\{ \nu \in \mathcal{M}(X) : \left| \int_X f_i \, d\nu - \int_X f_i \, d\mu \right| < \varepsilon \text{ for } i = 1, \ldots, k \right\},$$

where $k \geq 1$, $f_1, \ldots, f_k : X \to \mathbb{R}$ are continuous functions and $\varepsilon > 0$. For each $n \in \mathbb{N}$, let

$$\mathcal{M}_n(X) := \left\{ \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \in \mathcal{M}(X) : x_1, \ldots, x_n \in X \text{ not necessarily distinct} \right\},$$

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where δ_x denotes the unit mass concentrated at the point x of X. It is classical that $\bigcup_{n\in\mathbb{N}} \mathcal{M}_n(X)$ is dense in $\mathcal{M}(X)$. Since $\mathcal{M}_n(X)$ is $\widetilde{T}$-invariant, we can consider the TDS $(\mathcal{M}_n(X), \widetilde{T})$, where we are also denoting by $\widetilde{T}$ the corresponding restricted map. It will be convenient to denote $\mathcal{M}(X)$ by $\mathcal{M}_\infty(X)$.

Let us now recall some definitions and notations from entropy theory. In what follows, all logarithms are in base 2. Let (X, T) be a TDS. Given covers $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of X, let

$$\mathcal{U}_1 \vee \cdots \vee \mathcal{U}_n := \{U_1 \cap \cdots \cap U_n : U_1 \in \mathcal{U}_1, \ldots, U_n \in \mathcal{U}_n\}.$$

The *entropy* of an open cover $\mathcal{U}$ of X is defined by

$$H(\mathcal{U}) := \log N(\mathcal{U}),$$

where $N(\mathcal{U})$ denotes the minimum cardinality of a subcover of $\mathcal{U}$. The *topological entropy of T with respect to $\mathcal{U}$* is defined by

$$h_{\text{top}}(T, \mathcal{U}) := \lim_{n\to\infty} \frac{1}{n} H(\mathcal{U}^{n-1}),$$

where $\mathcal{U}^{n-1} := \mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-(n-1)}\mathcal{U}$, and the *topological entropy* of T is given by

$$h_{\text{top}}(T) := \sup_{\mathcal{U}} h_{\text{top}}(T, \mathcal{U}),$$

where the supremum is taken over all open covers of X. The notion of topological entropy was introduced by Adler, Konheim and McAndrew [1]. It plays a fundamental role in topological dynamics and its applications. Finally, an open cover $\mathcal{U} = \{U, V\}$ of X consisting of two sets is called a *standard cover* if both U and V are non-dense in X. The TDS (X, T) is said to have *uniformly positive entropy* (UPE) if $h_{\text{top}}(T, \mathcal{U}) > 0$ for every standard cover $\mathcal{U}$ of X. The notion of UPE is due to Blanchard [9].

The following useful characterization of UPE was given by Huang and Ye [20, Theorem 7.4(i)] (see also Kerr and Li [24]).

THEOREM 0. *A TDS (X, T) has UPE if and only if every pair (U, V) of disjoint non-empty open sets in X has an independence set of positive density.*

Recall that a set $I \subset \mathbb{N}$ is said to be an *independence set* for a tuple $(A_1, A_2, \ldots, A_k)$ of subsets of X if

$$\bigcap_{j\in J} T^{-j}(A_{\sigma(j)}) \neq \emptyset$$

for every non-empty finite subset J of I and every function $\sigma : J \to \{1, 2, \ldots, k\}$. Recall also that a subset I of $\mathbb{N}$ has *positive density* if the limit

$$\lim_{n\to\infty} \frac{|I \cap \{1, \ldots, n\}|}{n}$$

exists and is non-zero, where $|K|$ denotes the cardinality of the subset K of $\mathbb{N}$.

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3. *Proof of the main result*

LEMMA 1. *The sets of the form*

$$\mathbb{W}(U_1, \ldots, U_k; \eta_1, \ldots, \eta_k) := \{\nu \in \mathcal{M}(X) : \nu(U_i) > \eta_i \text{ for } i = 1, \ldots, k\}, \quad (1)$$

where $k \geq 1$, $U_1, \ldots, U_k$ are non-empty disjoint open sets in X and $\eta_1, \ldots, \eta_k$ are positive real numbers with $\eta_1 + \cdots + \eta_k < 1$, form a basis for the weak topology on $\mathcal{M}(X)$.*

Proof. Suppose that $\mu \in \mathbb{W} := \mathbb{W}(U_1, \ldots, U_k; \eta_1, \ldots, \eta_k)$. By inner regularity, there are compact sets $C_1, \ldots, C_k$ in X such that $C_i \subset U_i$ and $\mu(C_i) > \eta_i$ for all i. Let $\delta > 0$ be so small that $C_i^\delta \subset U_i$ and $\mu(C_i) - \delta > \eta_i$ for all i. Then $d_{\mathrm{P}}(\nu, \mu) < \delta$ implies

$$\nu(U_i) \geq \nu(C_i^\delta) \geq \mu(C_i) - \delta > \eta_i \quad \text{for all } i \in \{1, \ldots, k\},$$

that is, $\nu \in \mathbb{W}$. This proves that $\mathbb{W}$ is open in $\mathcal{M}(X)$.

Now take an open ball $B_{d_{\mathrm{P}}}(\mu; \varepsilon)$ in $\mathcal{M}(X)$ and choose $\delta \in (0, \varepsilon)$. We claim that there exist disjoint open sets $U_1, \ldots, U_k$ in X such that

$$\operatorname{diam} U_i < \delta \quad \text{for all } i \in \{1, \ldots, k\}, \quad (2)$$

and

$$\mu(U_1) + \cdots + \mu(U_k) > 1 - \delta/2. \quad (3)$$

Indeed, let $\{V_1, \ldots, V_k\}$ be an open cover of X by sets with diameters $< \delta$. Let $C_0 := \varnothing$ and let $C_1, \ldots, C_k$ be compact sets in X such that

$$C_i \subset V_i \backslash (C_0 \cup \cdots \cup C_{i-1}) \quad \text{and} \quad \mu(V_i \backslash (C_1 \cup \cdots \cup C_i)) < \frac{\delta}{2k} \quad (i \in \{1, \ldots, k\}).$$

Since $C_1, \ldots, C_k$ are disjoint and since (2) and (3) hold with $C_1, \ldots, C_k$ instead of $U_1, \ldots, U_k$, it is clear that there exist $U_1, \ldots, U_k$ with the desired properties. Moreover, we may assume $\mu(U_i) > 0$ for all i. Let $Y := X \backslash (U_1 \cup \cdots \cup U_k)$. For each i, take $\eta_i > 0$ with $\mu(U_i) - (\delta/2k) < \eta_i < \mu(U_i)$. We shall prove that

$$\mu \in \mathbb{W}(U_1, \ldots, U_k; \eta_1, \ldots, \eta_k) \subset B_{d_{\mathrm{P}}}(\mu; \varepsilon).$$

It is clear that $\mu \in \mathbb{W}(U_1, \ldots, U_k; \eta_1, \ldots, \eta_k)$. Pick $\nu \in \mathbb{W}(U_1, \ldots, U_k; \eta_1, \ldots, \eta_k)$ and $A \in \mathcal{B}_X$. Let $I := \{1 \leq i \leq k : A \cap U_i \neq \varnothing\}$. Since $\mu(Y) < \delta/2$ and $A^\delta \cap U_i = U_i$ whenever $i \in I$, we get

$$\begin{aligned} \nu(A^\delta) &\geq \sum_{i \in I} \nu(A^\delta \cap U_i) = \sum_{i \in I} \nu(U_i) > \sum_{i \in I} \eta_i > \sum_{i \in I} \mu(A \cap U_i) - \frac{\delta}{2} \\ &= \mu(A) - \mu(A \cap Y) - \frac{\delta}{2} > \mu(A) - \delta. \end{aligned}$$

This proves that $d_{\mathrm{P}}(\nu, \mu) \leq \delta < \varepsilon$. □

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LEMMA 2. *Given non-empty open sets $\mathbb{U}_0, \mathbb{U}_1$ in $\mathcal{M}_n(X)$, where $1 \leq n \leq \infty$, there exist $m \in \mathbb{N}$ and non-empty open sets $U_{0,1}, \ldots, U_{0,m}, U_{1,1}, \ldots, U_{1,m}$ in X such that*

$$\frac{1}{m}\sum_{i=1}^{m} \delta_{x_{k,i}} \in \mathbb{U}_k \text{ whenever } x_{k,1} \in U_{k,1}, \ldots, x_{k,m} \in U_{k,m} \quad (k \in \{0,1\}). \tag{4}$$

Moreover, we can take $m = n$ if $n < \infty$.

Proof. By Lemma 1, there is an open set $\mathbb{W}_k$ of the form $\mathbb{W}(V_{k,1}, \ldots, V_{k,j_k}; \eta_{k,1}, \ldots, \eta_{k,j_k})$ in $\mathcal{M}(X)$ such that

$$\varnothing \neq \mathbb{W}_k \cap \mathcal{M}_n(X) \subset \mathbb{U}_k \quad (k \in \{0,1\}). \tag{5}$$

Assume $n < \infty$ and take

$$\mu_k := \frac{1}{n}\sum_{i=1}^{n} \delta_{y_{k,i}} \in \mathbb{W}_k \quad \text{for } k \in \{0,1\}. \tag{6}$$

Put $m := n$. For each $k \in \{0,1\}$ and each $i \in \{1, \ldots, n\}$, if $y_{k,i} \in V_{k,j}$ for some (necessarily unique) $j \in \{1, \ldots, j_k\}$, let $U_{k,i}$ be an open neighborhood of $y_{k,i}$ with $U_{k,i} \subset V_{k,j}$, and if $y_{k,i} \notin V_{k,1} \cup \cdots \cup V_{k,j_k}$, put $U_{k,i} := X$. We claim that (4) holds. Indeed, take $x_{k,1} \in U_{k,1}, \ldots, x_{k,n} \in U_{k,n}$. By (5), it is enough to show that

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{x_{k,i}} \in \mathbb{W}_k. \tag{7}$$

Let $r_{k,j} := |\{1 \leq i \leq n : y_{k,i} \in V_{k,j}\}|$ and $s_{k,j} := |\{1 \leq i \leq n : x_{k,i} \in V_{k,j}\}|$. Since $y_{k,i} \in V_{k,j}$ implies $x_{k,i} \in U_{k,i} \subset V_{k,j}$, we have that $r_{k,j} \leq s_{k,j}$. Thus, by (6),

$$\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{x_{k,i}}\right)(V_{k,j}) = \frac{s_{k,j}}{n} \geq \frac{r_{k,j}}{n} = \mu_k(V_{k,j}) > \eta_{k,j},$$

for every $1 \leq j \leq j_k$, which proves (7).

Now assume $n = \infty$. Since $\bigcup_{t\in\mathbb{N}} \mathcal{M}_t(X)$ is dense in $\mathcal{M}(X)$, we can take

$$\nu_k := \frac{1}{t_k}\sum_{i=1}^{t_k} \delta_{z_{k,i}} \in \mathbb{W}_k \quad \text{for } k \in \{0,1\}. \tag{8}$$

For each $i \in \{1, \ldots, t_k\}$, let $A_{k,i}$ be an open neighborhood of $z_{k,i}$ with $A_{k,i} \subset V_{k,j}$ if $z_{k,i} \in V_{k,j}$ (put $A_{k,i} := X$ if $z_{k,i} \notin V_{k,1} \cup \cdots \cup V_{k,j_k}$). Put $m := t_0 t_1$ and produce a sequence $U_{0,1}, \ldots, U_{0,m}$ (respectively, $U_{1,1}, \ldots, U_{1,m}$) by putting each $A_{0,i}$ (respectively, each $A_{1,i}$) exactly t_1 (respectively, t_0) times. By reasoning as in the previous paragraph, we obtain (4) from (5) and (8). □

We will also need the following result [17, Proposition 2.1].

LEMMA 3. *Given constants $\varepsilon > 0$ and $b > 0$, there exist constants $m_0 \in \mathbb{N}$ and $c > 0$ such that the following property holds for every $m \geq m_0$: if $\varphi : \ell_1^k \to \ell_\infty^m$ is a linear map with $\|\varphi\| \leq 1$, and if $\varphi(B_{\ell_1^k})$ contains more than 2^{bm} points that are ε-separated, then $k \geq 2^{cm}$.*

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We observe that ℓ_1^k denotes the vector space $\mathbb{R}^k$ endowed with the ℓ_1-norm, that is, $\|(r_1, \dots, r_k)\| := |r_1| + \cdots + |r_k|$, and that ℓ_∞^m denotes the vector space $\mathbb{R}^m$ endowed with the ℓ_∞-norm, that is, $\|(s_1, \dots, s_m)\| := \max\{|s_1|, \dots, |s_m|\}$. Moreover, $B_{\ell_1^k}$ denotes the closed unit ball of the Banach space ℓ_1^k.

THEOREM 4. *For every TDS (X, T), the following assertions are equivalent:*

(i) (X, T) *has UPE;*.

(ii) $(\mathcal{M}_n(X), \widetilde{T})$ *has UPE for some* $1 \le n \le \infty$*;*

(iii) $(\mathcal{M}_n(X), \widetilde{T})$ *has UPE for every* $1 \le n \le \infty$.

Proof. (ii) $\Rightarrow$ (i): Let us consider first the case $n = \infty$. Suppose that $(\mathcal{M}(X), \widetilde{T})$ has UPE. Let $\mathcal{U} = \{U_0, U_1\}$ be a standard cover of X. We have to prove that $h_{\text{top}}(T, \mathcal{U}) > 0$. For this purpose, let V_0 and V_1 be non-empty open sets in X with

$$V_0 \subset U_0 \backslash \overline{U_1},\ V_1 \subset U_1 \backslash \overline{U_0} \quad \text{and} \quad \overline{V_0} \cap \overline{V_1} = \varnothing.$$

Define

$$\mathbb{V}_0 := \{\mu \in \mathcal{M}(X) : \mu(V_0) > 0.9\} \quad \text{and} \quad \mathbb{V}_1 := \{\mu \in \mathcal{M}(X) : \mu(V_1) > 0.9\},$$

which are disjoint non-empty open sets in $\mathcal{M}(X)$. By Theorem 0, the pair $(\mathbb{V}_0, \mathbb{V}_1)$ has an independence set $I \subset \mathbb{N}$ of positive density. Let $b > 0$ and $m_1 \in \mathbb{N}$ be such that

$$\frac{|I \cap \{1, \dots, m\}|}{m} > b \quad \text{whenever } m \ge m_1.$$

Put $\varepsilon = 0.5$ and let $m_0 \in \mathbb{N}$ and $c > 0$ be the constants associated to ε and b according to Lemma 3. Fix $m \ge \max\{m_0, m_1\}$ and choose a subcover $\{A_1, A_2, \dots, A_{k_m}\}$ of $\mathcal{U}^{m-1}$ with minimum cardinality. Let

$$B_1 := A_1 \quad \text{and} \quad B_i := A_i \backslash (A_1 \cup \cdots \cup A_{i-1}) \text{ for } 2 \le i \le k_m.$$

As $\{A_1, A_2, \dots, A_{k_m}\}$ is minimal, we have that $B_i \neq \varnothing$ for every i. Let

$$M := [t_{i,j}]_{1 \le i \le k_m, 0 \le j \le m-1}$$

be a $k_m \times m$ matrix of 0s and 1s such that

$$B_i \subset U_{t_{i,0}} \cap T^{-1}(U_{t_{i,1}}) \cap T^{-2}(U_{t_{i,2}}) \cap \cdots \cap T^{-(m-1)}(U_{t_{i,m-1}}),$$

for all $1 \le i \le k_m$. Consider the linear map $\varphi : \ell_1^{k_m} \to \ell_\infty^m$ given by

$$\varphi(r_1, \dots, r_{k_m}) := [r_1 \cdots r_{k_m}]\, M.$$

Clearly, $\|\varphi\| \le 1$. Let $J := I \cap \{1, \dots, m\}$. Since I is an independence set for $(\mathbb{V}_0, \mathbb{V}_1)$, for each $\sigma : J \to \{0, 1\}$, there exists $\mu_\sigma \in \mathcal{M}(X)$ such that

$$\widetilde{T}^j(\mu_\sigma) \in \mathbb{V}_{\sigma(j)} \quad \text{for all } j \in J.$$

Let $\sigma, \sigma' : J \to \{0, 1\}$ be distinct functions and let $s \in J$ be such that $\sigma(s) \neq \sigma'(s)$. Let us assume $\sigma(s) = 1$ and $\sigma'(s) = 0$. Then

$$\mu_\sigma(T^{-s}(V_1)) > 0.9 \quad \text{and} \quad \mu_{\sigma'}(T^{-s}(V_0)) > 0.9.$$

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Since

$$T^{-s}(V_1) \subset \bigcup \{B_i : t_{i,s} = 1\} \subset T^{-s}(U_1)$$

(for the first inclusion use the fact that $T^{-s}(V_1)$ and $T^{-s}(U_0)$ are disjoint), we obtain

$$\mu(T^{-s}(V_1)) \leq \sum_{i=1}^{k_m} t_{i,s}\, \mu(B_i) \leq \mu(T^{-s}(U_1)) \quad (\mu \in \mathcal{M}(X)).$$

Hence, the sth coordinates of the points

$$\varphi(\mu_\sigma(B_1), \ldots, \mu_\sigma(B_{k_m})) \quad \text{and} \quad \varphi(\mu_{\sigma'}(B_1), \ldots, \mu_{\sigma'}(B_{k_m}))$$

are greater than 0.9 and smaller than 0.1, respectively, showing that these points are ε-separated. Since $|J| > bm$, there are more than 2^{bm} functions σ. This shows that $\varphi(B_{\ell_1^{k_m}})$ contains more than 2^{bm} points that are ε-separated. Thus, by Lemma 3, $k_m \geq 2^{cm}$. This implies that $h_{\text{top}}(T, \mathcal{U}) \geq c > 0$, as desired.

Now suppose that $(\mathcal{M}_n(X), \widetilde{T})$ has UPE for some $2 \leq n < \infty$. Given disjoint non-empty open sets U_0, U_1 in X, consider

$$\mathbb{U}_0 := \left\{\mu \in \mathcal{M}_n(X) : \mu(U_0) > \frac{n-1}{n}\right\} \quad \text{and} \quad \mathbb{U}_1 := \left\{\mu \in \mathcal{M}_n(X) : \mu(U_1) > \frac{n-1}{n}\right\},$$

which are disjoint non-empty open sets in $\mathcal{M}_n(X)$. Since $(\mathcal{M}_n(X), \widetilde{T})$ has UPE, the pair $(\mathbb{U}_0, \mathbb{U}_1)$ has an independence set $I \subset \mathbb{N}$ of positive density. Hence, given $J \subset I$ non-empty and finite, and given $\sigma : J \to \{0, 1\}$, we have that

$$\bigcap_{j \in J} \widetilde{T}^{-j}(\mathbb{U}_{\sigma(j)}) \neq \varnothing.$$

If $\mu := (1/n) \sum_{i=1}^{n} \delta_{x_i}$ belongs to the above intersection, then

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{T^j x_i} \in \mathbb{U}_{\sigma(j)} \quad \text{for all } j \in J.$$

By the definitions of $\mathbb{U}_0$ and $\mathbb{U}_1$, this implies that

$$\{T^j x_1, \ldots, T^j x_n\} \subset U_{\sigma(j)} \quad \text{for all } j \in J.$$

Thus,

$$\{x_1, \ldots, x_n\} \subset \bigcap_{j \in J} T^{-j}(U_{\sigma(j)}),$$

proving that the above intersection is non-empty. Thus, I is an independence set of positive density for the pair (U_0, U_1). By Theorem 0, (X, T) has UPE.

(i) $\Rightarrow$ (iii): Suppose that (X, T) has UPE. Fix $1 \leq n \leq \infty$ and let $\mathbb{U}_0, \mathbb{U}_1$ be disjoint non-empty open sets in $\mathcal{M}_n(X)$. By Lemma 2, there exist an integer $m \geq 1$ and non-empty open sets $U_{0,1}, \ldots, U_{0,m}, U_{1,1}, \ldots, U_{1,m}$ in X such that

$$R_m(U_{0,1} \times \cdots \times U_{0,m}) \subset \mathbb{U}_0 \quad \text{and} \quad R_m(U_{1,1} \times \cdots \times U_{1,m}) \subset \mathbb{U}_1, \tag{9}$$

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where $R_m(x_1, \ldots, x_m) := (1/m)\sum_{i=1}^{m} \delta_{x_i}$. Moreover, we may assume $m = n$ if $n < \infty$. Let $X_m := X \times \cdots \times X$ (m times) and $T_m := T \times \cdots \times T$ (m times). Note that $\widetilde{T} \circ R_m = R_m \circ T_m$. Since $\mathbb{U}_0 \cap \mathbb{U}_1 = \varnothing$, (9) implies that

$$(U_{0,1} \times \cdots \times U_{0,m}) \cap (U_{1,1} \times \cdots \times U_{1,m}) = \varnothing. \tag{10}$$

Since any finite product of UPE systems is a UPE system [14], the TDS (X_m, T_m) has UPE. Hence, by (10) and Theorem 0, there exists $I \subset \mathbb{N}$ of positive density such that

$$\bigcap_{j\in J} T_m^{-j}(U_{\sigma(j),1} \times \cdots \times U_{\sigma(j),m}) \neq \varnothing$$

for every non-empty finite subset J of I and every function $\sigma : J \to \{0, 1\}$. For such a J and such a σ, we have that

$$\begin{aligned}\bigcap_{j\in J} \widetilde{T}^{-j}(\mathbb{U}_{\sigma(j)}) &\supset \bigcap_{j\in J} \widetilde{T}^{-j}(R_m(U_{\sigma(j),1} \times \cdots \times U_{\sigma(j),m})) \\ &\supset \bigcap_{j\in J} R_m(T_m^{-j}(U_{\sigma(j),1} \times \cdots \times U_{\sigma(j),m})) \\ &\supset R_m\Bigg(\bigcap_{j\in J} T_m^{-j}(U_{\sigma(j),1} \times \cdots \times U_{\sigma(j),m})\Bigg) \neq \varnothing.\end{aligned}$$

This shows that I is an independence set of positive density for the pair $(\mathbb{U}_0, \mathbb{U}_1)$. Thus, by Theorem 0, $(\mathcal{M}_n(X), \widetilde{T})$ has UPE. □

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