When $p \neq 3$, we have

$$\left(\frac{p}{3}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = \left(\frac{-3}{p}\right) = 1,$$

the first equation by quadratic reciprocity (see [2]) and the second by the multiplicative property of the Legendre symbol. Hence $p \equiv 1 \pmod 3$.

Acknowledgement: I am grateful to the referee for pointing out the neat proof of the proposition and for improving the quadratic reciprocity argument.

References

1. T. L. Heath, *The thirteen books of Euclid's Elements* (2nd edn), Dover (1956).
2. H. Davenport, *The higher arithmetic* (8th edn), Cambridge University Press (2008).

10.1017/mag.2019.19

K. ROBIN McLEAN
Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL
e-mail: *krmclean@liv.ac.uk*

103.05 Sums of integers

The purpose of this Note is to draw attention to a new way for proving the well-known fact that $1 + 2 + \ldots + n = \frac{1}{2}n(n+1)$. Our proof involves looking at a symmetry property of a certain polynomial and also using Vieta's formula. For alternative approaches of this same result, we refer the reader to [1, 2, 3, 4] and the very recent [5].

Let n be a positive integer. The polynomial

$$p_n(x) = (x-1)(x-2)\ldots(x-n),$$

has roots $r_i = i$ where $i = 1, 2, \ldots, n$. Furthermore, p_n satisfies a symmetry relation with respect to the midpoint of its roots, $\frac{1}{2}(n+1)$, which reads $p_n\left(\frac{1}{2}(n+1)+x\right) = (-1)^n p_n\left(\frac{1}{2}(n+1)-x\right)$ and is straightforward to verify. Therefore, the polynomial $q_n(x) = p_n\left(\frac{1}{2}(n+1)+x\right)$ is even or odd, according to whether n is even or odd, so that $q_n(x)$ contains either only even powers of x or only odd powers of x. We thus have

$$q_n(x) = \sum_{k=0}^{n} a_{n-k}x^{n-k} = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + \ldots . \quad (1)$$

Now, let s_i denote the ith root of q_n, so that $\frac{1}{2}(n+1) + s_i = r_i = i$ is the corresponding root of p_n. Vieta's formula and (1) combined give us $0 = a_{n-1} = -(s_1 + s_2 + \ldots + s_n)$. From this we conclude:

$$1 + 2 + \ldots + n = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n}\left(\frac{n+1}{2} + s_i\right) = n\frac{(n+1)}{2} + \sum_{i=1}^{n} s_1 = \frac{n(n+1)}{2}.$$

https://doi.org/10.1017/mag.2019.20 Published online by Cambridge University Press

It is worth noting that a similar argument works for roots $\tilde{r}_i = a + (i-1)d$ in an arithmetic sequence and yields

$$\sum_{i=1}^{n} \big(a + (i-1)d\big) = n\,\frac{(2a + (n-1)d)}{2}.$$

(As an outline of the proof, first consider

$$\tilde{p}_n(x) = \big(x - r_1\big)\big(x - r_2\big)\ldots\big(x - r_n\big)$$

to show that the polynomial

$$\tilde{q}_n(x) = \tilde{p}_n\big(a + \big(\tfrac{1}{2}(n-1)\big)d + x\big)$$

verifies $\tilde{q}_n(-x) = (-1)^n \tilde{q}_n(x)$; now, if $\tilde{s}_i$ stands for the ith root of $\tilde{q}_n$, then $a + \frac{1}{2}(n-1)d + \tilde{s}_i = \tilde{r}_i = a + (i-1)d$, and the proof ends as before.)

Acknowledgement:

The authors thank the referee for judicious comments and suggestions that helped improve the presentation of this Note.

References

1. A. J. Crachiola, Proof without words: partial sums of an arithmetic sequence, *College Math. J.* **43** (2012) p. 321.
2. S. J. Farlow, Count the dots: $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$, *College Math. J.* **26** (1995) p. 190.
3. L. Larson, A discrete look at $1 + 2 + 3 + \ldots + n$, *College Math. J.* **16** (1985) pp. 369-382.
4. I. Richards, Proof without words: sum of integers, *Math. Mag.* **57** (1984) p. 104.
5. J. Gaspar, Proof without words: using trapezoids to compute triangular numbers, *Math. Mag.* **91** (2018) pp. 206-207.

10.1017/mag.2019.20

SAMUEL G. MORENO,
ESTHER M. GARCÍA-CABALLERO

Departamento de Matemáticas, Universidad de Jaén, s/n, 23700 Jaén, Spain

e-mail: *samuel@ujaen.es, emgarcia@ujaen.es*

103.06 Another approach to $\int_0^{\pi} \log(2\sin u)\,du$

For $x \in (0, \pi)$, the *Lobachevsky function*

$$f(x) = \int_0^x \log(2\sin u)\,du = -\lim_{\delta \to 0^+} \int_{\delta}^{x} \log(2\sin u)\,du$$

arises in the computation of volumes in hyperbolic space. Showing that $f(\pi) = 0$, so that f can be extended continuously (and π-periodically) to $\mathbb{R}$ via $f(x) = -\int_0^x \log|2\sin u|\,du$, takes some work [e.g. 1, 2]. Here is an elementary approach.

https://doi.org/10.1017/mag.2019.20 Published online by Cambridge University Press