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A SUBGROUP LATTICE OF A FINITE SOLUBLE GROUP

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Abstract

A subgroup R of a finite group G is called *weakly subnormal* in G if R is not subnormal in G but it is subnormal in every proper overgroup of R in G. In this paper, weak subnormality is used to construct a subgroup lattice of a finite soluble group containing the lattice of all subnormal subgroups. A new characterisation of Schmidt groups is also obtained: they are exactly those groups with all subgroups subnormal or weakly subnormal.

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1. Introduction

All groups considered in this paper are finite.

The study of subgroup lattices of a group plays a central role in understanding its structure because many group theoretical properties are encoded in the embedding and relationships between its subgroups. This approach connects group theory with lattice theory, universal algebra and category theory and enables us to use tools from lattice theory and combinatorics.

A *subgroup lattice* of a group G is a set $\mathcal{L}$ of subgroups of G, partially ordered by inclusion, which is closed under intersections and joins, that is, if $A, B \in \mathcal{L}$, then $A \cap B$ and $\langle A, B\rangle$ are also elements of $\mathcal{L}$. In this case, the meet of two elements of $\mathcal{L}$ is just their intersection and the join is the subgroup generated by their union.

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The simplest examples of subgroup lattices of a group G are $\mathcal{L}(G)$, the lattice of all subgroups of G, and $\mathcal{L}_n(G)$, the lattice of all normal subgroups of G. A third interesting subgroup lattice arose from Wielandt's results [3, Corollary A.14.1 and Theorem A.14.4]: the set $sn(G)$ of all subnormal subgroups of G is a subgroup lattice of G, which is in between $\mathcal{L}_n(G)$ and $\mathcal{L}(G)$.

The subnormal subgroup lattice provides an optimal framework for understanding the internal construction of a finite group and so the problem of finding subgroup lattices containing the lattice of all subnormal subgroups turns out to be significant in the structural study of groups.

One approach to this problem is connected with extensions of subnormality within the framework of formation theory (see [1, 9] and [2, Ch. 6]).

In this paper, we propose a different approach based on the weak subnormality introduced and studied in [5]. It appears in a natural way in proofs of subnormality criteria.

DEFINITION 1.1. A subgroup R of a group G is *weakly subnormal* in G if R is not subnormal in G but it is subnormal in every proper overgroup of R in G.

Obviously, every nonnormal maximal subgroup of G is weakly subnormal in G. Therefore, if G is a nonnilpotent group, then the set $wsn(G)$ of all weakly subnormal subgroups of G is nonempty.

An example of a group with nonmaximal weakly subnormal subgroups is a Schmidt group, that is, a nonnilpotent group with all proper subgroups nilpotent. Note that, according to [5, Theorem 1], the structure of a p-soluble group with a weakly subnormal p-subgroup, p a prime, is quite similar to the structure of a Schmidt group.

The main motivation for the study of weakly subnormal subgroups in [5] was the study of variations of the Baer–Suzuki theorem. Our motivation here is to use weak subnormality to construct a subgroup lattice containing the subnormal subgroup lattice in every soluble group.

Let $\mathcal{L}^\star(G)$ be the set of all maximal subgroups of G containing a weakly subnormal subgroup of G of prime order.

THEOREM 1.2. *If G is soluble, $sn(G) \cup \mathcal{L}^\star(G)$ is a subgroup lattice of G.*

The condition on maximal subgroups to contain weakly subnormal subgroups of primer order is essential in Theorem 1.2 as the following example shows.

EXAMPLE 1.3. Let $G = [Q]P$, where $Q = \langle a \rangle$ is a normal subgroup of order 5, $P = \langle z \rangle$ is cyclic of order 4 and $a^z = a^2$. Let M be a maximal subgroup of G with order 10. Obviously, $M \in sn(G)$ and $P \in wsn(G)$ is maximal in G but $M \cap P = \langle z^2 \rangle \notin sn(G)$ and $M \cap P$ is nonmaximal in G.

Theorem 1.2 is saying in this case that the set of all subnormal subgroups of G is a sublattice of the subgroup lattice of G. This example shows that the set $sn(G) \cup \mathrm{Max}(G)$ cannot be a sublattice of the subgroup lattice of G, which was clear from the very beginning. Here, $\mathrm{Max}(G)$ is the set of all maximal subgroups of G.

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The following theorem characterises Schmidt groups in terms of weakly subnormal subgroups.

THEOREM 1.4. *A nonnilpotent group G is a Schmidt group if and only if every subgroup of G is either subnormal or weakly subnormal.*

Note that if G is Schmidt group with nonnormal Sylow subgroup of prime order and $\Phi(G) \neq 1$, then

$$sn(G) \subset sn(G) \cup \mathcal{L}^{\star}(G) \subset \mathcal{L}(G) = sn(G) \cup wsn(G).$$

2. Preliminary results

We shall adhere to the notation and terminology of [3]. The results of this section provide the necessary groundwork for the proofs of our theorems.

The following statements will be used in the paper without any further reference.

(1) If H is a subnormal subgroup of a group G and K is an overgroup of H in G, then H is subnormal in K, and if N is normal in G and X is a subgroup of G with $N \leq X$, then X is subnormal in G if and only if X/N is subnormal in G/N.
(2) If H is subnormal in an overgroup K of G in G and K is subnormal in G, then H is subnormal in G.
(3) If R is a weakly subnormal subgroup of G, then R is contained in a unique maximal subgroup of G [6, Theorem 2.9].
(4) If N is normal in G and $N \leq R$, then R is weakly subnormal in G if and only if R/N is weakly subnormal in G/N.

LEMMA 2.1. *Let R be a weakly subnormal subgroup of a group G and let N be a normal subgroup of G such that RN is nilpotent. Then RN is a weakly subnormal subgroup of G.*

PROOF. Since RN is nilpotent, it follows that R is subnormal in RN. Therefore RN is a nonsubnormal proper subgroup of G. Let M be a maximal subgroup of G containing RN. Then R is subnormal in M and so is RN by [3, Theorem A.14.4]. Furthermore, M is the unique maximal subgroup of G containing RN. Let X be a proper subgroup of G containing RN. Then X is contained in M and so RN is subnormal in X. Consequently, RN is a weakly subnormal subgroup of G. □

The following lemma is a extended version of [5, Theorem 1] for soluble groups.

LEMMA 2.2. *Let R be a nilpotent weakly subnormal subgroup of a soluble group G. If $\pi(R) \cap \pi(F(G)) = \emptyset$, then the following statements hold:*

(1) *$G = QR$ for some Sylow q-subgroup G of G;*
(2) *R centralises $\Phi(Q)$;*
(3) *$Q/\Phi(Q)$ is a chief factor of G, and $R\Phi(Q)$ is the maximal subgroup of G containing R.*

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PROOF. Let M be the maximal subgroup of G containing R. Assume that XR is a proper subgroup of G for each Sylow subgroup X of $F(G)$. Then $F(G)R$ is contained in M. Moreover, $F(G)R \leq F(M)$ and so $F(G)R$ is nilpotent. Thus $R \leq C_G(F(G)) \leq F(G)$ [3, Theorem A.10.6], a contradiction. Therefore there exists a prime q and a Sylow q-subgroup Q of $F(G)$ such that $G = QR$ and (1) holds. Since $R\Phi(Q)$ is a proper subgroup of G and $\Phi(Q)$ is normal in G, it follows that $R\Phi(Q) \leq F(M)$. Hence R centralises $\Phi(G)$ and (2) holds. To prove (3), we may assume that $\Phi(Q) = 1$ since $R\Phi(Q)/\Phi(Q)$ is a weakly subnormal subgroup of $G/\Phi(Q)$ by Lemma 2.1. Then $Q = Q_1 \times \cdots \times Q_t$ is a direct product of minimal normal subgroups $Q_1, \ldots, Q_t$ of G by [3, Theorem A.11.5]. Assume $t \geq 2$. Then $Q_iR \leq F(M)$ and so R centralises Q_i for all i. Then R centralises Q and G is nilpotent, a contradiction. Therefore $t = 1$, Q is a minimal normal subgroup of G and R is a maximal subgroup of G, as required. □

If p is a prime and G is a soluble group with $O_p(G) = 1$, every weakly subnormal p-subgroup R of G satisfies $\pi(R) \cap \pi(F(G)) = \emptyset$. This gives the following result.

COROLLARY 2.3 [5, Theorem 1]. *Let p be a prime and let R be a weakly subnormal p-subgroup of a soluble group G. If $O_p(G) = 1$, then the following statements hold:*

(1) *$G = QR$ for some Sylow q-subgroup G of G;*
(2) *R centralises $\Phi(Q)$;*
(3) *$Q/\Phi(Q)$ is a chief factor of G and $R\Phi(Q)$ is the maximal subgroup of G containing R.*

The basic structure of Schmidt groups, which is described in the following lemma, was established in [4, 8].

LEMMA 2.4. *Let S be a Schmidt group. Then the following statements hold:*

(1) *$\pi(S) = \{p, q\}$;*
(2) *$S = [Q]\langle a\rangle$, where Q is a normal Sylow q-subgroup of S and $\langle a\rangle$ is a Sylow p-subgroup of S such that $\langle a^p\rangle \in Z(S)$;*
(3) *$Q/\Phi(Q)$ is a minimal normal subgroup of $S/\Phi(Q)$ and $\langle a\rangle\Phi(Q)$ is a maximal subgroup of G;*
(4) *$\Phi(S) = Z(S) = Q' \times \langle a^p\rangle$;*
(5) *if $Z(S) = 1$, then $|S| = q^m p$, where m is the exponent of q modulo p.*

A group G is said to be an *$S_{\langle q,p\rangle}$-group* if G satisfies the statements of Lemma 2.4.

3. Proof of Theorem 1.2

If $\mathcal{L}^\star(G) = \emptyset$, then $sn(G) \cup \mathcal{L}^\star(G) = sn(G)$, and the result holds. Therefore, we may assume that G contains at least one weakly subnormal subgroup of prime order. Let R be a weakly subnormal subgroup of G of prime order, p say. If A is a subgroup of G, we write $A^* = AO_p(G)/O_p(G)$.

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By Lemma 2.1, R^* is weakly subnormal in G^* and $O_p(G^*) = 1$. The first claim follows from Corollary 2.3.

Claim 1.

(a) R^* is a Sylow p-subgroup of G^* and $|R^*| = p$;
(b) $G^* = Q^*R^*$ where Q is a Sylow q-subgroup of G for some prime $q \neq p$; hence Q^* is a normal maximal subgroup of G^*;
(c) $Q^*/\Phi(Q^*)$ is a chief factor of G^*;
(d) $R^*\Phi(Q^*)$ is the maximal subgroup of G^* containing R^* and $R^*\Phi(Q^*)$ is nilpotent.

Let T be the normal subgroup of G containing $O_p(G)$ such that $T^* = \Phi(Q^*)$.

Claim 2. If H is a normal maximal subgroup of G, then $H = QO_p(G)$.

Since G is soluble, either $|G : H| = q$ or $|G : H| = p$. Assume that $|G : H| = q$. Then $O_p(G) \subseteq H$ and H^* is a normal maximal subgroup of G^* of index q. Hence $R^* \leq H^*$. By Claim 1, $H^* = R^*\Phi(Q^*)$, and so $R^*\Phi(Q^*)$ is normal in G^*, which is a contradiction. Hence $|G : H| = p$, and so $Q \subseteq H$. Assume that $H \neq QO_p(G)$ and write $X = H \cap QO_p(G)$. Then $Q \leq X$ and RX is a proper subgroup of G since X is not maximal in G. By Claim 1, RT is the unique maximal subgroup of G containing R. Hence RX is contained in RT, and so $T = QO_p(G)$, which is a contradiction.

Claim 3. If M_1 and M_2 are two different maximal subgroups of G containing weakly subnormal subgroups of prime order, then $G = \langle M_1, M_2\rangle$ and $M_1 \cap M_2 \trianglelefteq G$.

Assume that R and L are weakly subnormal subgroups of prime order contained in M_1 and M_2, respectively. Suppose that $|R| = p$ and $|L| = q$. If $p \neq q$, then $L^* \leq Q^*$ and so L^* is subnormal in G^*. If $L^* < G^*$, then L is subnormal in G, which is a contradiction. Hence $L^* = G^*$ and $R \leq O_p(G)$, which is not possible. Therefore, $|R| = |L| = p$ and $O_p(G) \subseteq M_1 \cap M_2$. By Claim 1, $M_1 = RT$ and $M_2 = LT$. Hence $M_1 \cap M_2 = T$ is a normal subgroup of G.

Claim 4. If M is a maximal subgroup of G containing a weakly subnormal subgroup of prime order and H is a normal maximal subgroup of G, then $M \cap H \trianglelefteq G$.

By Claim 2, $H = QO_p(G)$ is the unique normal maximal subgroup of G. Without loss of generality, we may assume that $M = TR$. Then $T \subseteq M \cap H$. Since, by Claim 1, M is a complement of the chief factor H/T of G, it follows that $T = M \cap H$. Thus $M \cap H \trianglelefteq G$.

Claim 5. If M is a maximal subgroup of G containing a weakly subnormal subgroup of prime order and F is a proper subnormal subgroup of G, then $M \cap F$ is subnormal in G and $\langle M, F\rangle \in \{M, G\}$.

Since F is subnormal in G and $F \neq G$, F is contained in a normal maximal subgroup. By Claim 2, $F \leq H$. Then, by Claim 4, $F \cap M \subseteq H \cap M \trianglelefteq G$. Since $F \cap M$ is subnormal in $H \cap M$ and $H \cap M \trianglelefteq G$, it follows that $M \cap F$ is subnormal in G.

Consequently, $sn(G) \cup \mathcal{L}^\star(G)$ is a sublattice of $\mathcal{L}(G)$ and the proof of the theorem is complete. □

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4. Proof of Theorem 1.4

Necessity. Let $G = [Q]P$ be an $S_{\langle q,p\rangle}$-group. Assume that Z is a proper subgroup of G.

If Z contains some Sylow p-subgroup of G, then, by Lemma 2.4, Z is nonsubnormal in G. If X is a proper subgroup of G containing Z, then Z is subnormal in X because X is nilpotent. Hence Z is weakly subnormal in G.

If Z does not contain any Sylow p-subgroup of G, then QZ is a proper subgroup of G, and so QZ is nilpotent. By Lemma 2.4, QZ is subnormal in G. Thus Z is subnormal in G.

Therefore every subgroup of G is either subnormal or weakly subnormal.

Sufficiency. Let G be a nonnilpotent group such that every subgroup of G is either subnormal or weakly subnormal. We will show that G is soluble by induction on $|G|$. Let A be a minimal normal subgroup of G. If X/A is a nonsubnormal subgroup of G/A, then X is nonsubnormal in G and so X is weakly subnormal in G. Then X/A is weakly subnormal in G/A. Hence every subgroup of G/A is either subnormal or weakly subnormal. By induction, G/A is soluble.

Assume that A is nonabelian. If M is a maximal subgroup of A, then M is not subnormal in A. Therefore M is weakly subnormal in G. Hence $G = A$ is a nonabelian simple group. If Z is a proper subgroup of G and Y is a maximal subgroup of Z, then Y cannot be subnormal in G. Thus Y is weakly subnormal in G and Y is subnormal in Z. Therefore Z is nilpotent and G is a Schmidt group. By Lemma 2.4, G is soluble, which is a contradiction. It follows that A is abelian and G is soluble.

Since G is not nilpotent, it follows that G contains a nonsubnormal Sylow p-subgroup P for some $p \in \pi(G)$. Then P is weakly subnormal in G. If L is a subgroup of G, we write $L^* = LO_p(G)/O_p(G)$. By Corollary 2.3, G^* is a $\{p, q\}$-group for some prime $q \neq p$, P^* has order p and if Q is a Sylow q-subgroup of G, then Q^* is normal in G^*, $Q^*/\Phi(Q^*)$ is a chief factor of G and P^* centralises $\Phi(Q^*)$.

Since $QO_p(G)$ is a normal maximal subgroup of G, then Q cannot be weakly subnormal in G and so Q is subnormal in G by hypothesis. This implies that Q is normal in G. In particular, $QO_p(G)$ is nilpotent.

Let M be a maximal subgroup of G. Then either $Q \leq M$ or $G = QM$ and then M contains a Sylow p-subgroup of G.

Assume that M contains Q and $M \neq QO_p(G)$. Let M_p be a Sylow p-subgroup of M. Then $M = QM_p$ and $M_p \subseteq P^x$ for some $x \in G$. Since P^x is not contained in M, it follows that M_p is contained in at least two maximal subgroups of G. Therefore, M_p must be subnormal in G and M is nilpotent.

Assume that M contains some Sylow p-subgroup of G. Then $M = M_qP^x$ for some $x \in G$. Then P^x is subnormal in M and so P^x is normal in M. Since $Q \trianglelefteq G$, $M_q = Q \cap M \trianglelefteq M$. Therefore, M is nilpotent.

Consequently, every maximal subgroup of G is nilpotent and G is a Schmidt group, as required. □

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5. A final remark

Following [7], we will say that a subgroup H of a group G is a *maximal intersection* in G if there exists a family $M_1, \ldots, M_t$ of maximal subgroups of G with $H = M_1 \cap \cdots \cap M_t$. Let $\mathcal{M}(G)$ be the subposet of the subgroup lattice $\mathcal{L}(G)$ of G consisting of G and all the maximal intersections in G. Notice that $\mathcal{M}(G)$ is a lattice in which the meet $H \wedge K$ of two elements H and K coincides with their intersection and their join $H \vee K$ is the smallest maximal intersection in G containing $\langle H, K\rangle$. The maximum element of $\mathcal{M}(G)$ is G, and the minimum element coincides with the Frattini subgroup $\Phi(G)$ of G.

The following example from [7] shows that, in general, $\mathcal{M}(G)$ is not a sublattice of the subgroup lattice $\mathcal{L}(G)$ of G.

EXAMPLE 5.1. Let $\mathbb{F}$ be the field with three elements and let $C = \langle -1\rangle$ be the multiplicative group of $\mathbb{F}$. Let $V = \mathbb{F}^3$ be a three-dimensional vector space over $\mathbb{F}$ and let $\sigma = (1, 2, 3) \in \mathrm{Sym}(3)$. The wreath product $H = Cwr\langle\sigma\rangle$ has an irreducible action on V defined as follows: if $v = (f_1, f_2, f_3) \in V$ and $h = (c_1, c_2, c_3)\sigma^i \in H$, then

$$v^h = (f_{1\sigma^{-i}}c_{1\sigma^{-i}}, f_{2\sigma^{-i}}c_{2\sigma^{-i}}, f_{3\sigma^{-i}}c_{3\sigma^{-i}}).$$

Consider the semidirect product $G = [V]H$ and let $v = (1, -1, 0) \in V$. Since H and H^v are two maximal subgroups of G,

$$K = H \cap H^v = C_H(v) = \{(1, 1, z) \mid z \in C\} \cong C_2$$

is a maximal intersection in G. Since $G/V \cong H$ and $\Phi(H) = 1$, V is also a maximal intersection in G. However, VK is not a maximal intersection in G. Indeed, if X is a maximal intersection in G containing V, then $X = VY$ with Y a maximal intersection in H. But $H \cong C_2 \times \mathrm{Alt}(4)$ and the unique subgroup of order 2 of H that can be obtained as an intersection of maximal subgroups is $\{(z, z, z) \mid z \in C\}$.

Note that the subgroup VK is subnormal in G. Hence Example 5.1 does not give an answer for the following problem.

PROBLEM 5.2. Is $sn(G) \cup \mathcal{M}(G)$ a sublattice of $\mathcal{L}(G)$ for each soluble group G?

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