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Almost commuting self-adjoint operators and measurements

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Abstract. We study the problem when an n-tuple of self-adjoint operators in an infinite-dimensional separable Hilbert space H with small commutators is close to an n-tuple of commuting self-adjoint operators on H. We give an affirmative answer to the problem when the synthetic-spectrum and the essential synthetic-spectrum are close. Examples are also exhibited that, in general, the answer to the problem when $n \geq 3$ is negative even the associated Fredholm index vanishes. This is an attempt to solve a problem proposed by David Mumford related to quantum theory and measurements.

1 Introduction

At the First International Congress of Basic Science held in 2023, David Mumford delivered the opening plenary lecture on *Consciousness, Robots, and DNA* [32]. He began by cautioning the audience that there would be no mathematics in his talk. Nevertheless, by the very end of this magnificent lecture – which culminated with the idea of DNA as a measuring instrument opening a Pandora's box – he proposed an unexpected problem in C^*-algebras: determining when almost commuting self-adjoint operators can be approximated by commuting ones.

A version of this problem proposed by Mumford may be reformulated as follows: Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be a positive integer. When is there a constant $\delta > 0$ such that the following statement holds? For any separable Hilbert space H and self-adjoint operators $T_1, T_2, \ldots, T_n$ (with $\|T_j\| \leq 1$) satisfying

$$\|T_iT_j - T_jT_i\| < \delta, \ \ 1 \leq i, j \leq n, \tag{e1.1}$$

there exist self-adjoint operators $S_1, S_2, \ldots, S_n$ on H such that

$$S_iS_j = S_jS_i \ \text{ and } \ \|S_j - T_j\| < \varepsilon, \ \ 1 \leq i, j \leq n. \tag{e1.2}$$

In the case that $n = 2$ and H is any finite-dimensional Hilbert space (no bound on the dimension) the exactly the same problem is known as von-Neumann–Kadison–Halmos problem for almost commuting self-adjoint matrices (see [12, 13]). In this setting, the problem has an affirmative solution [21] (see also [9, 14], and [30] for fixed dimensions). However, in an infinite-dimensional Hilbert space H, as Mumford

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noted in his recent book (p. 182 of [31]) – and as was previously known – the answer is negative in general (see also Example 4.6 of [9]).

The affirmative solution for almost commuting self-adjoint matrices has spurred further research in the study of weak semi-projectivity in C^*-algebras. It also has a deep impact on the Elliott program of classification of amenable C^*-algebras (see [8, 24, 25, 27, 28]). Moreover, Mumford's problem opens new avenues for applying this branch of C^*-algebra theory to previously unexpected fields, such as DNA research and artificial intelligence [31].

The first main result of this article is as follows.

Theorem 1.1. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. There exists $\delta(n, \varepsilon) > 0$ satisfying the following: Suppose that H is an infinite-dimensional separable Hilbert space, and $T_1, T_2, \ldots, T_n$ are self-adjoint linear operators on H with $\|T_i\| \leq 1$ $(1 \leq i \leq n)$ such that*

$$\|T_iT_j - T_jT_i\| < \delta, \ \ 1 \leq i, j \leq n \ \text{ and } \ d_H(X, Y) < \varepsilon/8, \tag{e1.3}$$

where $X = s\mathrm{Sp}^{\varepsilon/8}(T_1, T_2, \ldots, T_n)$ and $Y = s\mathrm{Sp}^{\varepsilon/8}_{ess}((T_1, T_2, \ldots, T_n))$. Then there are bounded self-adjoint linear operators $S_1, S_2, \ldots, S_n$ on H such that

$$S_iS_j = S_jS_i, \ \ 1 \leq i, j \leq n \ \text{ and } \ \|T_j - S_j\| < \varepsilon, \ \ 1 \leq j \leq n. \tag{e1.4}$$

(See Remark 6.5 for the case that each T_j is compact.)

Here, $s\mathrm{Sp}^{\eta}((T_1, T_2, \ldots, T_n))$ and $s\mathrm{Sp}^{\eta}_{ess}((T_1, T_2, \ldots, T_n))$ are the sets of η-synthetic-spectrum and essential η-synthetic-spectrum, respectively (see Definition 2.8). In (e 1.3), $d_H(X, Y)$ is the Hausdorff distance between the two compact subsets of $\mathbb{I}^n$, the n-dimensional unit cube. In the special case that $s\mathrm{Sp}^{\varepsilon/8}_{ess}((T_1, T_2, \ldots, T_n))$ is full, that is, it contains the whole $\mathbb{I}^n$, then the second part of (e 1.3) is automatically satisfied.

Remark 6.6 provides some justification of the second condition related to macroscopic observables and measurements, and a discussion of the Mumford problem.

More generally, let H be a Hilbert (right) module over a C^*-algebra A, let $L(H)$ be the C^*-algebra of all bounded module maps on H with adjoints, and let $F(H)$ be the linear span of rank one module maps T of the form $T(x) = z\langle y, x\rangle$ (for all $x \in H$), where $y, z \in H$. Denote by $K(H)$ the closure of $F(H)$ which is a C^*-algebra. We also have the following theorem.

Theorem 1.2. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. There exists $\delta(n, \varepsilon) > 0$ satisfying the following: Suppose that H is a countably generated Hilbert-module over a σ-unital purely infinite simple C^*-algebra A, and $T_1, T_2, \ldots, T_n \in L(H)$ are self-adjoint bounded module maps with $\|T_i\| \leq 1$ $(1 \leq i \leq n)$ such that*

$$\|T_iT_j - T_jT_i\| < \delta, \ \ 1 \leq i, j \leq n. \tag{e1.5}$$

Suppose also

(i) $K(H) = L(H)$, *or*

(ii) $K(H) \neq L(H)$, *and* $d_H(X, Y) < \varepsilon/8$, *where* $X = s\mathrm{Sp}^{\varepsilon/8}(T_1, T_2, \ldots, T_n)$ *and* $Y = s\mathrm{Sp}^{\varepsilon/8}((\pi_c(T_1), \pi_c(T_2), \ldots, \pi_c(T_n)))$, *and* $\pi_c : L(H) \to L(H)/K(H)$ *is the quotient map.*

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Then there are $S_1, S_2, \ldots, S_n \in L(H)_{s.a.}$ *such that*

$$S_iS_j = S_jS_i,\ \ 1 \le i, j \le n \ \text{ and } \ \|T_j - S_j\| < \varepsilon,\ \ 1 \le j \le n. \tag{e1.6}$$

While we are mostly interested in the case that $n > 2$, we also look the special case when $n = 2$. For a pair of self-adjoint operators, we have a more definite answer for some Hilbert modules.

Theorem 1.3. *Let $\varepsilon > 0$. There exists $\delta(\varepsilon) > 0$ satisfying the following: Suppose that (i) H is an infinite-dimensional separable Hilbert space, or (ii) H is a countably generated Hilbert module over a σ-unital purely infinite simple C^*-algebra A such that $L(H)$ has real rank zero, or (iii) H is a countably generated Hilbert module over a σ-unital simple C^*-algebra B of stable rank one such that $L(H)$ has real rank zero and $L(H)/K(H)$ is simple.*

Suppose also that $T_1, T_2 \in L(H)$ are self-adjoint bounded module maps on H with $\|T_i\| \le 1$ $(i = 1, 2)$ such that

$$\|T_1T_2 - T_2T_1\| < \delta \ \text{ and } \ \kappa_1(\lambda - \pi(T_1 + iT_2)) = 0 \tag{e1.7}$$

for all $\lambda \notin s\mathrm{Sp}^{\delta}((\pi(T_1), \pi(T_2)))$, where $\pi : L(H) \to L(H)/K(H)$ is the quotient map. Then there are self-adjoint bounded module maps $S_1, S_2 \in L(H)$ such that

$$S_1S_2 = S_2S_1 \ \text{ and } \ \|S_j - T_j\| < \varepsilon, \ \ j = 1, 2. \tag{e1.8}$$

Moreover, the condition that $L(H)$ has real rank zero can be removed if we assume that $s\mathrm{Sp}^{\delta}((\pi(T_1), \pi(T_2)))$ is connected.

The second condition in (e 1.7) means that a Fredholm index vanishes (see Definition 2.4). Note that both cases (ii) and (iii) generalize case (i). In case (ii), $K(H)$ is stable (whenever H is not a finitely generated projective Hilbert module), and $L(H)/K(H)$ is simple (analogous to the Calkin algebra). Moreover, in case (ii), $L(H)/K(H)$ retains its simplicity while $L(H)$ has real rank zero.

Case (i) in Theorem 1.3 is a more definite result than that of Theorem 1.1 for $n = 2$. In fact, we prove a slightly more general result (see Theorem 5.3 below). One should also compare (i) in Theorem 1.3 to Theorem 1.1 of [16] (see Remark 5.6). Theorem 1.1 of [16] provides a more definitive estimate of $\mathrm{dist}(a, N_f(A))$, the distance from an element to the set of normal elements with finite spectrum in C^*-algebras with real rank zero. However, when $K_1(K(H)) \neq 0$, normal elements $N = S_1 + iS_2$ may not be approximated by normal elements in $K(H)$ with finite spectrum. Thus, cases (ii) and (iii) in Theorem 1.3 cover scenarios where almost normal elements are close to normal elements that cannot be approximated normal elements with finite spectra – complementing the results in [16].

We further demonstrate that the vanishing index condition for the operator pair is necessary. Given well-known instability of the spectrum (and essential spectrum) of bounded operators under perturbations, it becomes essential to consider some version of δ-spectrum (typically larger than the spectrum). For $n \ge 2$, we introduce

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the concept of δ-synthetic-spectrum for n-tuples of self-adjoint elements in a unital C^*-algebra (see Definition 2.8). We will also show that, unfortunately, the version of Mumford's problem formulated above has a negative solution in general – even the associated Fredholm index vanishes when $n > 2$.

In quantum mechanics, one often encounters the statement, "when the commutators go to zero, we recover the classical system." If "recover" means that the observables are approximated by commuting observables, then the example in (iii) of Chapter 14 of [31] (Example 4.6 of [9]), and Propositions 5.5 and 6.4 all reveal significant topological obstructions to such claims. Nevertheless, Theorem 1.1, interpreted as in Remark 6.6, may offer an alternative perspective.

The article is organized as follows. Section 2 contains some notations used in the article, gives the definition of δ-synthetic-spectrum and δ-near-spectrum for n-tuples of self-adjoint operators, and discusses the relationship between them (see also Proposition 2.17). Section 3 is a study of n-tuples of almost commuting self-adjoint elements in a unital purely infinite simple C^*-algebra. Section 4 presents the proof of Theorems 1.1 and 1.2. In Section 5, we will discuss the case of a pair of almost commuting self-adjoint bounded module maps and provide a proof of Theorem 1.3. In Section 6, the last section, we show why the Fredholm index does not appear in the statement of Theorem 1.1. We then present an example that an n-tuple version of Theorem 1.3 does not hold in general. A final remark was added which contains the justification of the second condition in (e 1.3) and its connection to the Mumford original question.

2 Preliminaries

Definition 2.1. Let A be a C^*-algebra. Denote by $A_{s.a.}$ the set of all self-adjoint elements in A. Denote by $M(A)$ the multiplier algebra (the idealizer of A in A^{**}). If $x, y \in A$ and $\varepsilon > 0$, we write

$$x \approx_\varepsilon y, \text{ if } \|x - y\| < \varepsilon. \tag{e2.9}$$

For unital C^*-algebra A, denote by $GL(A)$ and $GL_0(A)$, the group of invertible elements and the path connected component of $GL(A)$ containing the identity of A, respectively.

Definition 2.2. Let H be an infinite-dimensional separable Hilbert space. Denote by $B(H)$ the C^*-algebra of all bounded linear operators on H and by $\mathcal{K}$ the C^*-algebra of all compact operators on H. Denote by $\pi_c : B(H) \to B(H)/\mathcal{K}$ the quotient map.

Let $T \in B(H)$. Recall that the essential spectrum of $T, \mathrm{sp}_{ess}(T) = \mathrm{sp}(\pi_c(T))$, is the spectrum of $\pi_c(T)$.

Definition 2.3. Let A be a C^*-algebra and H be a Hilbert (right) module over A (or Hilbert A-module; see [17]). Denote by $L(H)$ the C^*-algebra of all bounded module maps on H with adjoints. A rank-one module map $T \in L(H)$ is a bounded module map of the form $T(x) = z\langle y, x\rangle$ for all $x \in H$ (and fixed $y, z \in H$), where $\langle \cdot, \cdot \rangle$ is the A-valued inner product on H. Denote by $F(H)$ the linear span of rank-one module

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maps and $K(H)$ the (norm) closure of $F(H)$. Following Theorem 1 of [17], we identify $L(H)$ with $M(K(H))$.

Definition 2.4. Let A be a unital C^*-algebra, and $x \in A$ be an invertible element. Define $u_x = x(x^*x)^{-1/2}$. Note that u_x is a unitary. Define

$$\kappa_1(x) = [u_x] \in K_1(A). \tag{e2.10}$$

Then $\kappa_1(x) = 0$ if and only if $x \in GL_0(A)$, when $K_1(A) = U(A)/U_0(A)$. Let $T \in B(H)$ be such that $\pi_c(T)$ is invertible in $B(H)/\mathcal{K}$. Then

$$\mathrm{Ind}(T) = \kappa_1(\pi_c(T)) = [u_{\pi_c(T)}] \text{ in } K_1(B(H)/\mathcal{K}) \cong \mathbb{Z}. \tag{e2.11}$$

Definition 2.5. In $\mathbb{R}^n$, by $\mathrm{dist}(x, y)$, we mean the Euclidean distance between x and y, that is, $\|x - y\|_2$. Denote by $\mathbb{I}^n = \{(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n : |r_i| \le 1\}$. If $\eta > 0$ and $x \in \mathbb{I}^n$, define $B(x, \eta) = \{y \in \mathbb{I}^n : \mathrm{dist}(x, y) < \eta\}$. Let $e_0(\xi) = 1$ for all $\xi \in \mathbb{I}^n$ be the constant function, $e_i \in C(\mathbb{I}^n)$ be defined by $e_i((r_1, r_2, \ldots, r_n)) = r_i$ for $(r_1, r_2, \ldots, r_n) \in \mathbb{I}^n$, $i = 1, 2, \ldots, n$. *This notation will be used throughout this article.* Note that $C(\mathbb{I}^n)$ is generated by $\{e_i : 0 \le i \le 1\}$.

Definition 2.6. Let X be a metric space with a fixed metric. Suppose that $Y \subset X$. For any $\eta > 0$, define

$$Y_\eta = \{\xi \in X : \mathrm{dist}(\xi, Y) < \eta\}. \tag{e2.12}$$

Definition 2.7. Fix $M \ge 1$. Fix an integer $k \in \mathbb{N}$. Define

$$P_k = \{\xi = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_j = m_j/k, |m_j| \le Mk, m_j \in \mathbb{Z},\ 1 \le j \le n\}.$$

P_k has only finitely many points. Most of the time, we choose $M = 1$.

Definition 2.8. Let $n \in \mathbb{N}$ and $M > 0$. In what follows, for each $0 < \eta < 1$, we choose a fixed integer $k = k(\eta) \in \mathbb{N}$ such that $k = \inf\{l \in \mathbb{N} : (M+1)/l < \frac{\eta}{1+2\sqrt{n}}\}$. Denote $D^\eta = P_k$. We write $D^\eta = \{x_1, x_2, \ldots, x_m\}$. Then D^η is $\eta/2$-dense in $M^n = \{(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n : |t_i| \le M\}$. Moreover, $D^\eta \subset D^\delta$ if $0 < \delta < \eta$.

Let A be a unital C^*-algebra and $(a_1, a_2, \ldots, a_n)$ be an n-tuple of self-adjoint elements in A with $\|a_i\| \le M$ $(1 \le i \le n)$. Fix $\xi = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{I}^n$. Let $\theta_{\lambda_i,\eta} \in C([-M, M])$ be such that $0 \le \theta_{\lambda_i,\eta} \le 1$, $\theta_{\lambda_i,\eta}(t) = 1$, if $|t - \lambda_i| \le 3\eta/4$, $\theta_{\lambda_i,\eta}(t) = 0$ if $|t - \lambda_i| \ge \eta$, and $\theta_{\lambda_i\eta}$ is linear in $(\lambda_i - \eta, \lambda_i - 3\eta/4)$ and in $(\lambda_i + 3\eta/4, \lambda_i + \eta)$, $i = 1, 2, \ldots, n$. Define, for $(t_1, t_2, \ldots, t_n) \in M^n$,

$$\Theta_{\xi,\eta}(t_1, t_2, \ldots, t_n) = \prod_{i=1}^{n} \theta_{\lambda_i,\eta}(t_i) \text{ and} \tag{e2.13}$$

$$\Theta_{\xi,\eta}(a_1, a_2, \ldots, a_n) = \theta_{\lambda_1,\eta}(a_1)\theta_{\lambda_2,\eta}(a_2)\ldots\theta_{\lambda_n,\eta}(a_n). \tag{e2.14}$$

Note that we do not assume that $a_1, a_2, \ldots, a_n$ mutually commute and the product in (e2.14) has a fixed order.

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For $x_j \in D^\eta$, write $x_j = (x_{j,1}, x_{j,2}, \dots, x_{j,n}) \in D^\eta$. We may sometime write $\theta_{j,i,\eta} := \theta_{x_{j,i},\eta}$ and $\Theta_{j,\eta} := \Theta_{x_j,\eta}, 1 \le i \le n, j = 1, 2, \dots, m$. Set

$$sSp^\eta((a_1, a_2, \dots, a_n)) = \bigcup_{\|\Theta_{x_j,\eta}(a_1,a_2,\dots,a_n)\| \ge 1-\eta} \overline{B(x_j, \eta)}. \tag{e2.15}$$

This union of finitely many closed balls $sSp^\eta((a_1, a_2, \dots, a_n))$ is called η-synthetic-spectrum of the n-tuple $(a_1, a_2, \dots, a_n)$ which, by the definition, is compact.

In what follows, we are particularly interested in the case that $M = 1$.

If $A = B(H)$ for some infinite-dimensional separable Hilbert space, define

$$sSp^\eta_{ess}((a_1, a_2, \dots, a_n)) = sSp^\eta((\pi_c(a_1), \pi_c(a_2), \dots, \pi_c(a_n))) \tag{e2.16}$$

which will be called essential η-synthetic-spectrum. It is clear that

$$sSp^\eta_{ess}((a_1, a_2, \dots, a_n)) \subset sSp^\eta((a_1, a_2, \dots, a_n)). \tag{e2.17}$$

Definition 2.9. Let A and B be C^*-algebras. A linear map $L : A \to B$ is a c.p.c. map if it is completely positive and contractive.

Definition 2.10. Let A be a unital C^*-algebra and $a_1, a_2, \dots, a_n \in A_{s.a.}$ for some $n \in \mathbb{N}$. Let us assume that $\|a_i\| \le 1, i = 1, 2, \dots, n$. Fix $0 < \eta < 1/2$. Suppose that $X \subset \mathbb{I}^n$ is a compact subset and $L : C(X) \to A$ is a unital c.p.c. map such that

$$\text{(i)} \quad \|L(e_j|_X) - a_j\| < \eta, \ 1 \le j \le n, \tag{e2.18}$$

$$\text{(ii)} \quad \|L((e_i e_j)|_X) - L(e_i|_X)L(e_j|_X)\| < \eta, \ 1 \le i, j \le n, \text{ and} \tag{e2.19}$$

$$\text{(iii)} \quad \|L(f)\| \ge 1 - \eta \tag{e2.20}$$

for any $f \in C(X)_+$ which has value 1 on an open ball with the center $x \in X$ (for some x) and the radius η. Then we say that X is an η-near-spectrum of the n-tuple $(a_1, a_2, \dots, a_n)$. We may write

$$X = nSp^\eta((a_1, a_2, \dots, a_n))$$

for convenience. The reader is recommended to read Remark 2.11 for clarification.

If L is, additionally, a unital *homomorphism,* then we say that X is an η-spectrum of the n-tuple $(a_1, a_2, \dots, a_n)$. In this case, let $J = \ker L \subset C(X)$ (L is now a homomorphism). Then $C(X)/J \cong C(Z)$ for some compact subset $Z \subset X$ and, by (iii) above, we may assume that $Z \subset X \subset Z_\eta$ (see Definition 2.6). In other words, there is an injective homomorphism $\varphi : C(Z) \to A$ such that

$$\|\varphi(f|_Z) - L(e_j|_X)\| < \eta \text{ and } L = \varphi \circ \pi_Z, \tag{e2.21}$$

where $\pi_Y : C(X) \to C(Z)$ is the quotient (restriction) map.

Let H be an infinite-dimensional separable Hilbert space and let $T_1, T_2, \dots, T_n \in B(H)_{s.a.}$. If X is an η-spectrum of $(\pi_c(T_1), \pi_c(T_2), \dots, \pi_c(T_n))$, we say that X is an essential η-spectrum for $(T_1, T_2, \dots, T_n)$. In this case, we write

$$Sp^\eta_{ess}((a_1, a_2, \dots, a_n)) := X.$$

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Remark 2.11. Here are some clarifications:

1. It should be noted that $nSp^{\delta}((a_1, a_2, \ldots, a_n))$ is not uniquely determined and depends on the choice of the c.p.c. map. Writing $X = nSp^{\delta}((a_1, a_2, \ldots, a_n)) \neq \varnothing$ *simply means that there exists a non-empty compact subset* X *and a unital c.p.c. map* $L : C(X) \to A$ *satisfying conditions (i)–(iii) of Definition 2.10.* This convention is adopted for notational convenience. Moreover, if we write $n\mathrm{Sp}^{\eta}((a_1, a_2, \ldots, a_n)) = \varnothing$, we mean that no such X and L exist which satisfy (i)–(iii).

2. However, we will later show that δ-near spectra are *unique* up to an η-neighborhood when the n-tuple self-adjoint operators are almost commuting within δ as elaborated in Proposition 2.16.

3. On the other hand, the δ-synthetic-spectrum is always uniquely defined and – importantly – computable in the context of computations. Their relationship is described in Proposition 2.16 below (see also Proposition 2.17).

4. Let X be an η-near-spectrum and Y be a δ-near-spectrum for $(a_1, a_2, \ldots, a_n)$, respectively. Suppose that $\delta < \eta$, then, by the definition, Y is also an η-near-spectrum. In particular, if $nSp^{\delta}((a_1, a_2, \ldots, a_n)) \neq \varnothing$, then $nSp^{\eta}((a_1, a_2, \ldots, a_n)) \neq \varnothing$.

5. In practice, computing the spectrum of an operator $T_1 + iT_2$ (where $T_1, T_2 \in B(H)_{s.a.}$) may be more challenging than computing the norm of $\Theta_{j,\eta}((T_1, T_2))$.

Proposition 2.12. *Fix $n \in \mathbb{N}$. For any $\eta > 0$, there exists $\delta(n, \eta) > 0$ satisfying the following: Suppose that A is a unital C^*-algebra and $a_i \in A_{s.a.}$ with $\|a_i\| \leq 1, 1 \leq i \leq n$, such that*

$$\|a_i a_j - a_j a_i\| < \delta, \ \ 1 \leq i, j \leq n. \tag{e2.22}$$

Then

(1) $X := s\mathrm{Sp}^{\eta}((a_1, a_2, \ldots, a_n)) \neq \varnothing$, and

(2) $Y := nSp^{\eta}((a_1, a_2, \ldots, a_n)) \neq \varnothing$, that is, there exists a non-empty compact subset $Y \subset \mathbb{I}^n$ and a c.p.c. map $L : C(Y) \to A$ satisfies conditions (i)–(iii) in Definition 2.10.

Proof. Suppose that the lemma is false. Then we obtain an $\eta > 0$ and a sequence of unital C^*-algebras $\{A_k\}$ and a sequence of n-tuples $\{a_{j,k}\} \subset \{(A_k)_{s.a.}\}$ with $\|a_{j,k}\| \leq 1$ $(1 \leq j \leq n)$, $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \|a_{i,k}a_{j,k} - a_{j,k}a_{i,k}\| = 0, \ \ 1 \leq j \leq n, \tag{e2.23}$$

and, for case (1), $s\mathrm{Sp}^{\eta}((a_{1,k}, a_{2,k}, \ldots, a_{n,k})) = \varnothing, k \in \mathbb{N}$, (or, for case (2), $nSp^{\eta}((a_{1,k}, a_{2,k}, \ldots, a_{n,k})) = \varnothing, k \in \mathbb{N}$.)

Consider $B = \prod_k A_k$ and quotient map $\Pi : B \to B/\bigoplus_k A_k$. Let $s_j = \Pi(\{a_{j,k}\})$, $j = 1, 2, \ldots, n$. Then $s_i s_j = s_j s_i, 1 \leq i, j \leq n$. Let C be the C^*-subalgebra of $B/\bigoplus_k A_k$ generated by $s_0 = 1, s_1, s_2, \ldots, s_n$. Then $C = C(X_0)$ for some non-empty compact subset X_0 of $\mathbb{I}^n$. Let $\varphi : C(X_0) \to B/\bigoplus_k A_k$ be the monomorphism given by C with $\varphi(e_j|_{X_0}) = s_j$, $j = 0, 1, 2, \ldots, n$.

To show (1), consider $D^{\eta} = \{x_1, x_2, \ldots, x_m\}$ (see Definition 2.8). Fix $x \in X_0$. Choose $x_j \in D^{\eta}$ such that $\mathrm{dist}(x_j, x) < \eta/2$, then $\|\Theta_{x_j,\varepsilon}|_{X_0}\| = 1$. Hence, since φ is a monomorphism,

$$\|\varphi(\Theta_{x_j,\eta}|_{X_0})\| = 1. \tag{e2.24}$$

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Recall that $\Pi(\{\Theta_{x_j,\eta}(a_{1,k}, a_{2,k}, \dots, a_{n,k})\}_{k\in\mathbb{N}}) = \varphi(\Theta_{x_j,\eta}|_{X_0})$. Hence, there is an infinite subset $S \subset \mathbb{N}$ such that, for all $k \in S$,

$$\|\Theta_{x_j,\eta}(a_{1,k}, a_{2,k}, \dots, a_{n,k})\| \geq 1 - \eta. \tag{e2.25}$$

It follows that $X_0 \subset \bigcup_{\|\Theta_{x_j,\varepsilon}(a_{1,k},a_{2,k},\dots,a_{n,k})\|\geq 1-\eta} B(x_j, \eta)$ for all $k \in S$. In other words, $(a_{1,k}, a_{2,k}, \dots, a_{n,k})$ has η-synthetic-spectrum containing $X_0 \neq \varnothing$ for all $k \in S$. This contradicts "$s\mathrm{Sp}^{\eta}((a_{1,k}, a_{2,k}, \dots, a_{n,k})) = \varnothing, k \in \mathbb{N}$." Thus (1) follows.

For (2), we obtain, by the Choi–Effros Theorem [4], a c.p.c. map $L : C(X_0) \to B$ such that $\Pi \circ L = \varphi$. Write $L = \{L_k\}$, where each $L_k : C(X_0) \to A_k$ is a c.p.c map, $k \in \mathbb{N}$. Then

$$\lim_{k\to\infty} \|L_k(e_j|_{X_0}) - a_{j,k}\| = 0. \tag{e2.26}$$

Let $\{x_1, x_2, \dots, x_m\} \subset X_0$ be an $\eta/4$-dense subset, and $f_1, f_2, \dots, f_m \in C(X_0)_+$ with $0 \leq f_j \leq 1, f_j(x) = 1$ if $\mathrm{dist}(x, x_j) \leq \eta/4$ and $f_j(x) = 0$ if $\mathrm{dist}(x, x_j) \geq \eta/2, j = 1, 2, \dots, m$.

Since $\Pi \circ L = \varphi$ is injective, there exists a subsequence $\{k(l)\} \subset \mathbb{N}$ such that

$$\|L_{k(l)}(f_j)\| \geq (1 - \eta/4), \quad j = 1, 2, \dots, m. \tag{e2.27}$$

Let $f \in C(X_0)_+$ and $x \in X_0$ be such that $f(y) \geq 1$ for all $y \in \overline{B(x, \eta)}$. Since $\{x_1, x_2, \dots, x_l\}$ is $\eta/4$-dense in X_0, there is x_j such that $\mathrm{dist}(x, x_j) < \eta/4$. Then $f \geq f_j$. It follows that

$$\|L_{k(l)}(f)\| \geq \|L_{k(l)}(f_j)\| \geq (1 - \eta/4), \quad j = 1, 2, \dots, m. \tag{e2.28}$$

Thus, since $\{e_j|_{X_0} : 0 \leq j \leq n\}$ generates $C(X_0)$, by (e 2.26) and (e 2.28), for all large $k(l)$, X_0 is an η-near-spectrum of $(a_{1,k}, a_{2,k}, \dots, a_{n,k})$. A contradiction. ∎

Lemma 2.13. *Let X be a compact metric space and $\mathcal{G} \subset C(X)$ be a finite generating subset.*

Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exists $\delta(\mathcal{F}, \varepsilon) > 0$ satisfying the following: Suppose that A is a unital C^*-algebra and $\varphi_1, \varphi_2 : C(X) \to A$ are unital c.p.c. maps such that

$$\|\varphi_1(g) - \varphi_2(g)\| < \delta \text{ and } \|\varphi_i(gh) - \varphi_i(g)\varphi_i(h)\| < \delta \text{ for all } g, h \in \mathcal{G}, \quad i = 1, 2. \tag{e2.29}$$

Then, for all $f \in \mathcal{F}$,

$$\|\varphi_1(f) - \varphi_2(f)\| < \varepsilon. \tag{e2.30}$$

Moreover, if $X \subset \mathbb{I}^n$ and $\mathcal{G} = \{1, e_1|_X, e_2|_X, \dots, e_n|_X\}$, then, for any $\varepsilon > 0$ and any finite subset $\mathcal{H} \subset C(\mathbb{I}^1)$, we may also require that, for all $h \in \mathcal{H}$,

$$\|\varphi_1(h(e_j)|_X) - h(\varphi_1(e_j|_X))\| < \varepsilon, \quad 1 \leq j \leq n. \tag{e2.31}$$

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Proof. Suppose that the lemma is false. Then there exist $\varepsilon_0 > 0$, a finite subset $\mathcal{F} \subset C(X)$ and a sequence of unital C^*-algebras $\{A_n\}$, and two sequences of unital c.p.c. maps $\varphi_{1,n}, \varphi_{2,n} : C(X) \to A_n$ such that

$$\lim_{n\to\infty} \|\varphi_{1,n}(g) - \varphi_{2,n}(g)\| = 0 \text{ for all } g \in \mathcal{G}, \tag{e2.32}$$

$$\lim_{n\to\infty} \|\varphi_{i,n}(gh) - \varphi_{i,n}(g)\varphi_{i,n}(h)\| = 0 \text{ for all } g, h \in \mathcal{G},\ i = 1, 2, \text{ and} \tag{e2.33}$$

$$\max\{\|\varphi_{1,n}(f) - \varphi_{2,n}(f)\| : f \in \mathcal{F}\} \ge \varepsilon_0 \text{ for all } n \in \mathbb{N}. \tag{e2.34}$$

Moreover, in the case that $X \subset \mathbb{I}^n$ and $\mathcal{G} = \{e_j|_X : 0 \le j \le n\}$, there exists $\varepsilon_1 > 0$ and a finite subset $\mathcal{H}_0$ such that

$$\max\{\|\varphi_{1,n}(h(e_j)) - h(\varphi_{1,n}(e_j))\| : h \in \mathcal{H}_0, 1 \le j \le n\} \ge \varepsilon_1 \tag{e2.35}$$

for all $n \in \mathbb{N}$.

Put $B = \prod_n A_n$. Let $\Pi : B \to B/\bigoplus_n A_n$ be the quotient map. Then, since $\mathcal{G}$ is a generating set of $C(X)$, by (e 2.33), $\psi_i := \Pi \circ (\{\varphi_{i,n}\}) : C(X) \to B/\bigoplus_n A_n$ is a unital homomorphism, $i = 1, 2$, and by (e 2.32), $\psi_1 = \psi_2$. Hence,

$$\psi_1(f) = \psi_2(f) \text{ for all } f \in C(X). \tag{e2.36}$$

But this implies that, for all sufficiently large n,

$$\|\varphi_{1,n}(f) - \varphi_{2,n}(f)\| < \eta_0/2 \text{ for all } f \in \mathcal{F}. \tag{e2.37}$$

This leads a contradiction to (e 2.34). The first part of the lemma then follows.

For the second part, since ψ_1 is a unital homomorphism, for any $h \in \mathcal{H}_0$,

$$\psi_1(h(e_j|_X)) = h(\psi_1(e_j|_X)),\ 1 \le j \le n. \tag{e2.38}$$

It follows that, for all large n and $h \in \mathcal{H}_0$,

$$\|\varphi_{1,n}(h(e_j|_X)) - h(\varphi_{1,n}(e_j|_X)\| < \varepsilon_1,\ 1 \le j \le n. \tag{e2.39}$$

Another contradiction. So the lemma follows. ∎

Corollary 2.14. *Fix $n \in \mathbb{N}$ and $0 < \eta < 1$. There exists $\delta(n, \eta) > 0$ satisfying the following: Suppose that A is a unital C^*-algebra, $a_1, a_2, \dots, a_n \in A_{s.a.}$ with $\|a_i\| \le 1$ $(1 \le i \le n)$, $Y \subset \mathbb{I}^n$ is a compact subset, and $L : C(Y) \to A$ is a unital c.p.c. map such that*

$$\|L(e_ie_j|_Y) - L(e_i|_Y)L(e_j|_Y)\| < \delta \text{ and } \|L(e_i|_Y) - a_i\| < \delta,\ i, j = 1, 2, \dots, n. \tag{e2.40}$$

Then, for $D^\eta = \{x_1, x_2, \dots, x_m\}$,

$$\|L(\Theta_{x_j,\eta}|_Y) - \Theta_{x_j,\eta}(a_1, a_2, \dots, a_n)\| < \eta,\ 1 \le j \le m. \tag{e2.41}$$

Proof. Let $D^\eta = \{x_1, x_2, \dots, x_m\} \subset \mathbb{I}^n$ be as defined in Definition 2.8, where $x_j = (x_{j,1}, x_{j,2}, \dots, x_{j,n})$ $(1 \le j \le m)$. There is $\delta_1(\eta) > 0$ satisfying the following:

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for any unital C^*-algebra A and any pair of self-adjoint elements $a, b \in A_{s.a.}$ with $\|a\|, \|b\| \le 1$, if $\|a - b\| < \delta_1$, then

$$\|\theta_{j,i,\eta}(a) - \theta_{j,i,\eta}(b)\| < \eta/2(n+1), \ \ 1 \le j \le m, \tag{e2.42}$$

where $\theta_{j,i,\eta}$ is defined in Definition 2.8.

Let $\mathcal{G} = \{e_i : 1 \le i \le n\} \cup \{1\}$ and $\mathcal{H} = \{\theta_{j,i,\eta} : 1 \le i \le m, \ 1 \le j \le n\}$. For $X = \mathbb{I}^n$, choose $\delta_2(n, \eta/4) > 0$ such that (the second part of) Lemma 2.13 holds for $\mathcal{H}, \mathcal{G}$ above and $\eta/4(n+1)$ (in place of η). Choose $\delta_3 > 0$ such that, for any unital c.p.c. map $\Phi : C(Y) \to A$ (any unital C^*-algebra A), that

$$\|\Phi(e_i e_j|_Y) - \Phi(e_i|_Y)\Phi(e_j|_Y)\| < \delta_3 \ \ (1 \le i, j \le n) \tag{e2.43}$$

implies, for $1 \le j \le m$,

$$\|\Phi(\Theta_{x_j,\eta}|_Y) - \Phi(\theta_{j,1,\eta}|_Y)\Phi(\theta_{j,2,\eta}|_Y)\dots\Phi(\theta_{j,n,\eta}|_Y)\| < \eta/4. \tag{e2.44}$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.

Suppose that L and $\{a_i : 1 \le i \le n\}$ satisfy the assumption of this lemma for δ. Since $\Theta_{x_j,\eta} = \theta_{j,1,\eta}\theta_{j,2,\eta}\dots\theta_{j,n,\eta}$ and

$$\Theta_{x_j,\eta}(b_1, b_2, \dots, b_n) = \theta_{j,1,\eta}(b_1)\theta_{j,2,\eta}(b_2)\dots\theta_{j,n,\eta}(b_n) \tag{e2.45}$$

for any n-tuple $(b_1, b_2, \dots, b_n)$ of self-adjoint elements in A with $\|b_i\| \le 1$ $(1 \le i \le n)$, by the choice of δ (and δ_1) and by the second part of (e 2.40) and (e 2.42), we have, for each $1 \le j \le m$,

$$\|\Theta_{x_j,\eta}(L(e_1|_X), L(e_2|_X), \dots, L(e_n|_X)) - \Theta_{x_j,\eta}(a_1, a_2, \dots, a_n)\| < \eta/2. \tag{e2.46}$$

By the choice of δ_3, we also have

$$\|L(\Theta_{x_j,\eta}|_X) - L(\theta_{j,1,\eta}(e_1|_X))L(\theta_{j,2,\eta}(e_2|_X))\dots L(\theta_{j,n,\eta}(e_n|_X))\| < \eta/4. \tag{e2.47}$$

Define $\tilde{L} : C(\mathbb{I}^n) \to A$ by $\tilde{L}(f) = L(f|_Y)$ for all $f \in C(Y)$. By the choice δ and applying (the second part of) Lemma 2.13 to $\tilde{L}_i = \tilde{L}$, $i = 1, 2$, we obtain

$$\|\tilde{L}(\theta_{j,i,\eta}(e_i)) - \theta_{j,i,\eta}(\tilde{L}(e_i))\| < \eta/4(n+1), \ \ 1 \le i \le n, \ \ 1 \le j \le m. \tag{e2.48}$$

In other words,

$$\|L(\theta_{j,i,\eta}(e_i)|_Y) - \theta_{j,i,\eta}(L(e_i|_Y))\| < \eta/4(n+1), \ \ 1 \le i \le n, \ \ 1 \le j \le m. \tag{e2.49}$$

Note that $\theta_{j,i,\eta}(e_i)|_Y = \theta_{j,i,\eta}(e_i|_Y)$. It follows that (keeping in mind of (e 2.45))

$$\begin{aligned} &L(\theta_{j,1,\eta}(e_1|_Y))L(\theta_{j,2,\eta}(e_2|_Y))\dots L(\theta_{j,n,\eta}(e_n|_Y)) \\ &\quad \approx_{\frac{n\eta}{4(n+1)}} \Theta_{x_j,\eta}(L(e_1|_Y), L(e_2|_Y), \dots, L(e_n|_Y)). \end{aligned} \tag{e2.50}$$

Combining (e 2.50) with (e 2.47), we obtain

$$\|L(\Theta_{x_j,\eta}|_Y) - \Theta_{x_j,\eta}(L(e_1|_Y), L(e_2|_Y), \dots, L(e_n|_Y))\| < \eta/2, \ \ 1 \le j \le n. \tag{e2.51}$$

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We conclude that, by (e 2.51) and (e 2.46),

$$\|L(\Theta_{x_j,\eta}|_Y) - \Theta_{x_j,\eta}(a_1, a_2, \dots, a_n)\|$$

$$\le \|L(\Theta_{x_j,\eta}|_Y) - \Theta_{x_j,\eta}(L(e_1|_Y), L(e_2|_Y), L(e_n|_Y))\| \tag{e2.52}$$

$$+ \|\Theta_{j,\eta}(L(e_1|_Y), L(e_2|_Y), L(e_n|_Y)) - \Theta_{x_j,\eta}(a_1, a_2, \dots, a_n)\| \tag{e2.53}$$

$$< \eta/2 + \eta/2 < \eta. \tag{e2.54}$$

■

Definition 2.15. Let M be a metric space (we only consider the case that $M \subset \mathbb{R}^n$ is compact such as the case that $M = \mathbb{I}^n$). Recall that the Hausdorff distance of two compact subsets of $X, Y \subset M$ is defined by

$$d_H(X, Y) = \max\{\sup_{x\in X}\{\mathrm{dist}(x, Y)\}, \sup_{y\in Y}\{\mathrm{dist}(y, X)\}\}. \tag{e2.55}$$

Let $F(M)$ be the set of all non-empty compact subsets of M. Then $(F(M), d_H)$ is a compact metric space with the metric d_H when M is compact.

Proposition 2.16. *Fix $k \in \mathbb{N}$. For any $\eta > 0$, there exits $\delta(k, \eta) > 0$ satisfying the following:*

(1) *Suppose that A is a unital C^*-algebra and $a_1, a_2, \dots, a_k \in A_{s.a.}$ with $\|a_i\| \le 1$ $(1 \le i \le k)$ such that $(a_1, a_2, \dots, a_k)$ has a non-empty δ-near-spectrum $X = nSp^{\delta}((a_1, a_2, \dots, a_n))$. If Y is also a non-empty δ-near-spectrum of $(a_1, a_2, \dots, a_k)$, then $Z = s\mathrm{Sp}^{\eta}((a_1, a_2, \dots, a_k)) \neq \varnothing$, and*

$$d_H(X, Y) < \eta \text{ and } X, Y \subset Z \subset X_{2\eta}. \tag{e2.56}$$

(2) *Suppose, in addition to (1), A has a (closed two-sided) ideal $J \subset A$, $\pi : A \to A/J$ is the quotient map, and $\Omega_1 = s\mathrm{Sp}^{\eta}((\pi(a_1), \pi(a_2), \dots, \pi(a_n)))$ and $\Omega_2 = nSp^{\delta}((\pi(a_1), \pi(a_2), \dots, \pi(a_n))) \neq \varnothing$. Then*

$$\Omega_1 \subset Z, \ \Omega_2 \subset \Omega_1 \subset (\Omega_2)_{2\eta} \text{ and } \Omega_2 \subset \overline{X_\eta}. \tag{e2.57}$$

Proof. We may assume that $0 < \eta < 1$. We first prove (1).

Let $\{x_1, x_2, \dots, x_m\}$ be an $\eta/8$-dense subset of $\mathbb{I}^n$. Choose $f_j \in C(\mathbb{I}^n)_+$ with $0 \le f_j \le 1$, $f_j(x) = 1$ if $\mathrm{dist}(x, x_j) \le \eta/4$, and $f_j(x) = 0$, if $\mathrm{dist}(x, x_j) \ge \eta/2$, $j = 1, 2, \dots, m$. Put $\mathcal{F} = \{f_j : 1 \le j \le m\}$.

Note that $\{e_j : 0 \le j \le n\}$ is a generating set of $C(\mathbb{I}^n)$. Let $\delta_1 := \delta(\mathcal{F}, \eta/32) > 0$ be given by Lemma 2.13 (for $\eta/32$ and $\mathcal{F}$ as well as $\mathbb{I}^n$) and $\delta_2 := \delta(n, \eta/2) > 0$ be given by Corollary 2.14. Choose $\delta = \min\{\delta_1/2, \delta_2/2, \eta/33\}$.

Now suppose that X and Y are δ-near-spectra of the n-tuple $(a_1, a_2, \dots, a_n)$. Let $L_1, L_2 : C(X) \to A$ be unital c.p.c maps such that

$$\|L_1(e_ie_j|_X) - L_1(e_i|_X)L_1(e_j|_X)\| < \delta, \ 1 \le i, j \le n, \tag{e2.58}$$

$$\|L_2(e_ie_j|_Y) - L_2(e_i|_Y)L_2(e_j|_Y)\| < \delta, \ 1 \le i, j \le n, \tag{e2.59}$$

$$\|L_1(e_j|_X) - a_j\| < \delta, \ \|L_2(e_j|_Y) - a_j\| < \delta, \ 1 \le j \le n, \text{ and} \tag{e2.60}$$

$$\|L_1(f|_X)\| \ge (1 - \delta) \text{ and } \|L_2(f|_Y)\| \ge (1 - \delta) \tag{e2.61}$$

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for any $f \in C(\mathbb{I}^n)_+$ with $f(x) = 1$ for all $x \in B(\xi, \delta)$, for some $\xi \in X$ (for L_1), or $\xi \in Y$ (for L_2).

Let $\xi \in X$. We may assume that $\xi \in B(x_j, \eta/8)$. Then $B(x_j, \delta) \subset B(\xi, \eta/4)$. On the other hand, by (e 2.61), one must have, as $\eta/4 > \delta$,

$$\|L_1(f_j|_X)\| \geq (1 - \delta). \tag{e2.62}$$

Define $L_l' : C(\mathbb{I}^n) \to A$ $(l = 1, 2)$ by $L_1'(f) = L_1(f|_X)$ and $L_2'(f) = L_2(f|_Y)$ for all $f \in C(\mathbb{I}^n)$. By (e 2.59) and applying Lemma 2.13, we have, for $1 \leq j \leq m$,

$$\|L_1'(f_j) - L_2'(f_j)\| < \eta/32, \text{ or } \|L_1(f_j|_X) - L_2(f_j|_Y)\| < \eta/32. \tag{e2.63}$$

Then, by (e 2.62), we obtain, for $1 \leq j \leq m$,

$$\|L_2(f_j|_Y)\| \geq \|L_1(f_j|_X)\| - \eta/32 \geq 1 - \delta - \eta/32 > 3/4. \tag{e2.64}$$

Since $f_j(x) = 0$, if $\operatorname{dist}(x, x_j) \geq \eta/2$, we must have $\operatorname{dist}(x_j, Y) < \eta/2$. Therefore,

$$\operatorname{dist}(\xi, Y) \leq \operatorname{dist}(\xi, x_j) + \operatorname{dist}(x_j, Y) < \eta/8 + \eta/2 = 5\eta/8. \tag{e2.65}$$

This holds for all $\xi \in X$. Exchanging the role of X and Y, we obtain

$$d_H(X, Y) < \eta. \tag{e2.66}$$

Let $D^\eta = \{\xi_1, \xi_2, \ldots, \xi_N\}$ and $Z = \bigcup_{\|\Theta_{\xi_j,\eta}(a_1,a_2,\ldots,a_k)\| \geq 1-\eta} \overline{B(\xi_j, \eta)}$.

By the choice of δ and applying Corollary 2.14, we have that

$$\|L_1(\Theta_{\xi_j,\eta}|_X) - \Theta_{\xi_j,\eta}(a_1, a_2, \ldots, a_k)\| < \eta/2, \quad j = 1, 2, \ldots, N. \tag{e2.67}$$

Pick $\xi \in X$. There is $\xi_j \in D^\eta$ such that $\operatorname{dist}(\xi, \xi_j) < \eta/2$. Recall that $\Theta_{\xi_j,\eta}(y) = 1$ if $\operatorname{dist}(y, \xi_j) \leq 3\eta/4$. As $\delta < \eta/32$, we have $\eta/2 + \delta < 3\eta/4$. Hence, $\Theta_{\xi_j,\eta}(y) = 1$ for all $y \in B(\xi, \delta) \subset B(\xi_j, 3\eta/4)$. It follows that

$$\|L_1(\Theta_{\xi_j,\eta}|_X)\| \geq 1 - \delta. \tag{e2.68}$$

Thus, by (e 2.67),

$$\|\Theta_{\xi_j,\eta}(a_1, a_2, \ldots, a_k)\| \geq 1 - \delta - \eta/2 \geq 1 - \eta. \tag{e2.69}$$

This implies that

$$X \subset \bigcup_{\|\Theta_{\xi_j,\eta}(a_1,a_2,\ldots,a_k)\| \geq 1-\eta} \overline{B(\xi_j, \eta)} = s\mathrm{Sp}^\eta((a_1, a_2, \ldots, a_k)). \tag{e2.70}$$

Since $X \neq \varnothing$, we conclude that $Z \neq \varnothing$. The same argument shows that $Y \subset s\mathrm{Sp}^\eta((a_1, a_2, \ldots, a_k))$.

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Next, let $\|\Theta_{\xi_j,\eta}(a_1, a_2, \ldots, a_k)\| \geq 1-\eta$. By (e 2.67),

$$\|L_1(\Theta_{\xi_j,\eta}|_X)\| \geq 1-\eta-\eta/2 > 1/4. \tag{e2.71}$$

Hence, $\Theta_{\xi_j,\eta}|_X \neq 0$. Thus,

$$X \cap B(\xi_j, \eta) \neq \varnothing. \tag{e2.72}$$

It follows that

$$B(\xi_j, \eta) \subset X_{2\eta}. \tag{e2.73}$$

Hence,

$$Z \subset X_{2\eta}. \tag{e2.74}$$

This completes the proof of part (1).

For (2), let us keep notation above. Then $\Omega_1 \subset Z$ is immediate.

Suppose that $\Omega_2 \neq \varnothing$. Then that $\Omega_2 \subset \Omega_1 \subset (\Omega_2)_{2\eta}$ follow from part (1). It remains to show that $\Omega_2 \subset \overline{X_\eta}$.

There is a unital c.p.c. map $\Phi : C(\Omega_2) \to A/J$ such that

$$\|\Phi(e_i e_j|_{\Omega_2}) - \Phi(e_i|_{\Omega_2})\Phi(e_j|_{\Omega_2})\| < \delta, \ \ 1 \leq i, j \leq k; \tag{e2.75}$$

$$\|\Phi(e_j|_{\Omega_2}) - \pi(a_j)\| < \delta, \ \ 1 \leq j \leq k, \ \text{ and} \tag{e2.76}$$

$$\|\Phi(f)\| \geq 1-\delta \tag{e2.77}$$

for any $f \in C(\Omega_2)_+$ with $f(x) = 1$ if $x \in B(\zeta, \delta)$ for some $\zeta \in \Omega_2$.

Choose $y \in \Omega_2$. We may assume that $\operatorname{dist}(y, x_j) < \eta/8$. Hence, $B(y, \delta) \subset \operatorname{dist}(x_j, \eta/4)$. It follows from (e 2.77) that

$$\|\Phi(f_j)\| \geq 1-\delta. \tag{e2.78}$$

Let $\tilde{L}_1 : C(\mathbb{I}^n) \to A/J$ be defined $\tilde{L}_1(f) = \pi \circ L_1(f|_X)$ and $\tilde{\Phi} : C(\mathbb{I}^n) \to A/J$ by $\tilde{\Phi}(f) = \Theta(f|_{\Omega_2})$ for all $f \in C(\mathbb{I}^n)$. By the choice of δ, (e 2.59), (e 2.76), and (e 2.60), and applying Lemma 2.13 to $\pi \circ \tilde{L}_1$ and $\tilde{\Phi}$, we have

$$\|\pi \circ \tilde{L}_1(f_j) - \tilde{\Phi}(f_j)\| < \eta/32, \ \ 1 \leq j \leq m. \tag{e2.79}$$

In other words,

$$\|\pi \circ L_1(f_j|_X) - \Phi(f_j|_{\Omega_2})\| < \eta/32, \ \ 1 \leq j \leq m. \tag{e2.80}$$

Thus,

$$\|L_1(f_j|_X)\| \geq \|\pi \circ L_1(f_j|_X)\| \geq 1-\delta-\eta/32 \geq 3/4. \tag{e2.81}$$

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Hence, $f_j|_X \neq 0$. It follows that $\mathrm{dist}(x_j, X) < \eta/2$.

$$\mathrm{dist}(y, X) < \eta/2 + \eta/8 < \eta. \tag{e2.82}$$

Hence, $\Omega_2 \subset \overline{X_\eta}$. ∎

Proposition 2.17. *Let $\eta > 0$. There exists $\delta(\eta) > 0$ satisfying the following: Suppose that A is a unital C^*-algebra and $a_1, a_2 \in A_{s.a.}$ such that $\|a_i\| \leq 1$ $(i = 1, 2)$, and*

$$\|a_1a_2 - a_2a_1\| < \delta. \tag{e2.83}$$

Then $\mathrm{sp}(a_1 + ia_2) \subset s\mathrm{Sp}^\eta((a_1, a_2))$.

Proof. By Proposition 2.16, it suffices to show that $sp(a_1 + a_2) \subset n\mathrm{Sp}^\eta((a_1, a_2))$. Otherwise, one obtains $\eta_0 > 0$, a sequence of unital C^*-algebras $\{A_n\}$, a sequence of elements $a_n, b_n \in (A_n)_{s.a.}$ with $\|a_n\|, \|b_n\| \leq 1$ such that

$$\lim_{n\to\infty} \|a_nb_n - b_na_n\| = 0, \tag{e2.84}$$

and a sequence of numbers $\lambda_n \in \mathrm{sp}(a_n + ib_n)\backslash s\mathrm{Sp}^{\eta_0}((a_n, b_n))$. Put $c_n = a_n + ib_n$ and $Z_n = n\mathrm{Sp}^{\eta_0/4}((a_n, b_n))$, $n \in \mathbb{N}$. Note that $\lambda_n \in \mathbb{I}^2 \subset \mathbb{C}$ and $Z_n \subset \mathbb{I}^2$, $n \in \mathbb{N}$. Since $(F(\mathbb{I}^2), d_H)$ is compact, without loss of generality, by passing to a subsequence, we may assume that $Z_n \to Z$ for some compact subset $Z \subset \mathbb{I}^n$ in $(F(\mathbb{I}^2), d_H)$ as $n \to \infty$, and $\lambda_n \to \lambda \in \mathbb{I}^2$.

By (4) of Remark 2.11, $(n\mathrm{Sp}^{\eta_0/4}((a_n, b_n)))_{\eta_0/4} \subset n\mathrm{Sp}^{\eta_0}((a_n, b_n))$, $n \in \mathbb{N}$. It follows that

$$\mathrm{dist}(\lambda_n, Z_n) \geq \eta_0/4, \quad n \in \mathbb{N}. \tag{e2.85}$$

Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\mathrm{dist}(\lambda, Z_n) \geq \eta_0/5 \text{ for all } n \geq n_0. \tag{e2.86}$$

Let $B = \prod_n A_n$ and $\Pi : B \to B/\oplus_n A_n$ be the quotient map. Put $x = \Pi(\{c_n\})$. Then, by (e 2.84), x is normal. Let $\mathrm{sp}(x) = X$. Then $\varphi : C(X) \to B/\oplus_n A_n$ defined by $\varphi(f) = f(x)$ for $f \in C(X)$ is a unital monomorphism. By the Choi–Effros Lifting Theorem [4], there is a sequence of unital c.p.c. maps $L_n : C(X) \to A_n$ such that $\Pi \circ \{L_n\} = \varphi$. Let $\delta := \delta(2, \eta_0/4)$ be as in Proposition 2.16. Then, since φ is a homomorphism, for all large n, X is a (non-empty) δ-near-spectrum for (a_n, b_n). Applying Proposition 2.16, we may assume that, for all n,

$$X \subset Z_n. \tag{e2.87}$$

Note that, by (e 2.86) and (e 2.87), $\lambda \notin X$. Let $y = (\lambda - x)^{-1}$. Note that $\Pi(\{\lambda_n - (a_n + ib_n)\}) = \lambda - x$. Hence, there are $y_n \in A_n$ such that $(\|y_n\| \leq \|y\| + 1)$

$$\Pi(\{y_n\}) = y \text{ and } \lim_{n\to\infty} \|y_n(\lambda_n - (a_n + ib_n)) - 1\| = 0. \tag{e2.88}$$

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Hence, for all large n,

$$\lambda_n \notin \mathrm{sp}(a_n + ib_n). \tag{e2.89}$$

This is a contradiction. So the proposition follows. ∎

3 Purely infinite simple C^*-algebras

Recall that the Calkin algebra $B(H)/\mathcal{K}$ (when H is infinite-dimensional separable Hilbert space) is purely infinite and simple. The results in the section will be used in the next four sections.

Definition 3.1. An even $1/k$-brick $\mathtt{b} = \mathtt{b}^k$ with corner $\xi = (x_1, x_2, \dots, x_n) \in P_k$ (with $M = 1$ – see Definition 2.7) is a subset of $\mathbb{I}^n$ of the form

$$\{(y_1, y_2, \dots, y_n) : x_i \le y_i \le x_i + 1/k,\ x_i, x_i + 1/k \in P_k\}. \tag{e3.90}$$

For the remaining of this article, *all 1/k-bricks are assumed to be even and have their corner* $\xi \in P_k$. An open $1/k$-brick $\mathtt{b}^o$ is the interior of the $1/k$-brick $\mathtt{b}$. The boundary $\partial(\mathtt{b})$ of a $1/k$-brick $\mathtt{b}$ is defined as $\mathtt{b}\backslash\mathtt{b}^o$.

A 1/k-brick combination $\mathtt{B}^k$ is a finite union of $1/k$-bricks. Since two distinct $1/k$-bricks have distinct corners, a $1/k$-brick combination $\mathtt{B}^k$ has the following easy property: If $\mathtt{b}_1, \mathtt{b}_2$ are two distinct $1/k$-bricks of $\mathtt{B}^k$, then $\mathtt{b}_1^o \cap \mathtt{b}_2^o = \varnothing$.

We believe that Proposition 3.2 to be well-known.

Proposition 3.2. *If Y is a union of finitely many compact convex subsets in $\mathbb{R}^n$ (for some integer $n \in \mathbb{N}$), then $K_i(C(Y))$ is finitely generated, $i = 0, 1$.*

Proof. We prove this by induction on the number of convex sets.

If Y is convex, then Y is contractive. Hence, $K_0(C(Y)) = \mathbb{Z}$ and $K_1(C(Y)) = \{0\}$.

Suppose that the proposition is proved for m many closed convex sets, $m \ge 1$.

Now suppose that Y is a union of $m+1$ many compact convex sets in $\mathbb{R}^n$. Write $Y = \cup_{i=1}^{m+1} Y_i$, where each Y_i is a compact convex subset of $\mathbb{R}^n$.

Put $X_1 = \cup_{i=1}^{m} Y_i$. Then $Y_{m+1} \cap X_1 = \cup_{i=1}^{m}(Y_i \cap Y_{m+1})$ which is a union of m many compact convex subsets. By the induction assumption, $K_i(C(X_1 \cap Y_{m+1}))$ is finitely generated ($i = 0, 1$).

Then

$$C(Y) = \{(f, g) : (f, g) \in C(X_1) \oplus C(Y_{m+1}) : f|_{X_1 \cap Y_{m+1}} = g|_{X_1 \cap Y_{m+1}}\} \tag{e3.91}$$

is a pull back. By the Mayer–Vietoris sequence in K-theory for C^*-algebras (see [15]), we obtain the following commutative diagram:

(e3.92)

$$\begin{array}{ccccc} K_0(C(Y)) & \longrightarrow & K_0(C(X_1)) \oplus K_0(C(Y_{m+1})) & \to & K_0(C(X_1 \cap Y_{m+1})) \\ \uparrow & & & & \downarrow \\ K_1(C(X_1 \cap Y_{m+1})) & \leftarrow & K_1(C(X_1)) \oplus K_1(C(Y_{m+1})) & \leftarrow & K_1(C(Y)). \end{array}$$

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Since we have established that $K_i(C(X_1))$, $K_i(C(Y_{m+1}))$, and $K_i(C(X_1 \cap Y_{m+1}))$ are all finitely generated abelian groups, we conclude that $K_i(C(Y))$ is also finitely generated. This ends the induction. ∎

Definition 3.3. Let X be a compact subset of $\mathbb{I}^n$. Define $\mathtt{B}_X^k$ to be the union of all even $1/k$-bricks (in $\mathbb{I}^n$ with corners at P_k – see Definition 2.7) which has a non-empty intersection with X.

We will use the following easy facts:

(i) $X \subset \mathtt{B}_X^k$; (ii) if $\mathfrak{b}^k$ is one of the bricks in $\mathtt{B}_X^k$, then $\mathfrak{b}^k \cap X \neq \varnothing$; (iii) for any $\zeta \in \mathtt{B}_X^k$, $\mathrm{dist}(\zeta, X) \leq \sqrt{n}/k$; and (iv) $K_i(C(\mathtt{B}_X^k)$ is finitely generated (this follows from Proposition 3.2).

In what follows, $\mathtt{B}_X^k$ is called the $1/k$-brick cover of X. (The introduction of even $1/k$-bricks is for convenience not essential.)

Definition 3.4. Let X be a compact metric space. We say that X has property (F), if $X = \sqcup_{i=1}^m X_i$ is a disjoint union of finitely many path connected compact metric spaces X_i such that $K_i(C(X_j))$ is finitely generated ($i = 0, 1$), $j = 1, 2, \ldots, m$. Choose a base point $x_i \in X_i$. Write $Y_i = X_i \backslash \{x_i\}$, $i = 1, 2, \ldots, m$. Denote by $\Omega(X) = \sqcup_{i=1}^m Y_i$.

Let B be a C^*-algebra and $\varphi : C(X) \to B$ be a homomorphism. Denote by $\varphi^\omega = \varphi|_{C_0(\Omega(X))} : C_0(\Omega(X)) \to B$ the restriction.

Every finite CW complex X has property (F). If X is a $1/k$-brick combination in $\mathbb{I}^n$, then (by Proposition 3.2) X also has property (F).

Recall that if $\psi : C(X) \to B$ has finite-dimensional range, then there are mutually orthogonal projections $p_1, p_2, \ldots, p_l \in B$ and $x_1, x_2, \ldots, x_l \in X$ such that

$$\psi(f) = \sum_{i=1}^{l} f(x_i) p_i \ \text{ for all } \ f \in C(X). \tag{e3.93}$$

Theorem 3.5. *Let X be a compact metric space with property (F). Suppose that B is a unital purely infinite simple C^*-algebra and $\varphi : C(X) \to B$ is a unital injective homomorphism such that*

$$[\varphi^\omega] = 0 \ \text{ in } \ KK(C_0(\Omega(X)), B) \tag{e3.94}$$

(see Definition 3.4 for φ^ω). Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F}$, there exists a set of finitely many mutually orthogonal projections $p_1, p_2, \ldots, p_k$ with $\sum_{i=1}^k p_i = 1$ and $x_1, x_2, \ldots, x_k \in X$ such that

$$\|\psi(f) - \varphi(f)\| < \varepsilon \ \text{ for all } \ f \in \mathcal{F}, \tag{e3.95}$$

where $\psi(f) = \sum_{j=1}^k f(x_j) p_j$ for all $f \in C(X)$. Moreover, if X is connected, we may require that either $[p_j] = 0$, or $[p_j] = [1_B]$, in $K_0(B)$, $j = 1, 2, \ldots, k$.

Proof. Since X has property (F), we may write $C(X) = C(\sqcup_{i=1}^m X_i) = \oplus_{i=1}^m C(X_i)$, where each X_i is path connected and $K_j(C(X_i))$ is finitely generated ($j = 0, 1$), $1 \leq i \leq m$. Let $d_i \in C(X)$ be such that $d_i|_{X_i} = 1$ and $d_i(x) = 0$ if $x \notin X_i, 1 \leq i \leq m$. Put $E_i = \varphi(d_i)$ ($1 \leq i \leq m$). By considering $\varphi|_{C(X_i)} \to E_i B E_i$ for each i, we reduce the

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general case to the case that X has only one path connected component. Note that, if originally, X is not connected, then we do not need to consider the "Moreover" part of the theorem. If X is connected, then we need to point out that $\varphi(1_{C(X)}) = 1_A$.

So now we assume that X is connected. Put $\Omega := \Omega(X) = X\backslash\{x\}$ for some $x \in X$. Then we have a splitting short exact sequence

$$0 \to C_0(\Omega) \to C(X) \to \mathbb{C} \to 0.$$

Recall that $K_i(C(X))$ is finitely generated ($i = 0, 1$). It follows that, for any unital C^*-algebra B,

$$KK(C(X), B) = KK(C_0(\Omega), B) \oplus KK(\mathbb{C}, B). \tag{e3.96}$$

Define $\varphi_1 : C(X) \to B$ by $\varphi_1(f) = f(x) \cdot 1_B$ for all $f \in C(X)$. Then one computes that, by (e 3.94),

$$[\varphi_1] = [\varphi] \text{ in } KK(C(X), B). \tag{e3.97}$$

Fix $\varepsilon > 0$ and finite subset $\mathcal{F} \subset C(X)$. Applying Theorem A of [6], we obtain an integer k, a unitary $u \in M_{k+1}(B)$, and $x_1, x_2, \dots, x_k \in X$ such that, for all $f \in \mathcal{F}$,

$$\begin{aligned} &\|\mathrm{diag}(\varphi(f), f(x_1), f(x_2), \dots, f(x_k)) \\ &\qquad - u^* \mathrm{diag}(\varphi_1(f), f(x_1), f(x_2), \dots, f(x_k))u\| < \varepsilon/3. \end{aligned} \tag{e3.98}$$

Define $\varphi_0 : C(X) \to M_k(B)$ by $\varphi_0(f) = \mathrm{diag}(f(x_1), f(x_2), \dots, f(x_k))$ for all $f \in C(X)$. Then we may rewrite $\varphi_0(f) = \sum_{i=1}^k f(x_i)p_i$ for all $f \in C(X)$, where $\{p_1, p_2, \dots, p_k\}$ is a set of mutually orthogonal projections such that $\sum_{i=1}^k p_i = 1_{M_k}$. Choose mutually orthogonal projections $p_{0,1}, p_{0,2}, \dots, p_{0,k} \in M_{k+1}(B)$ such that $p_0 1_{M_k} = 1_{M_k} p_0 = 0$, where $p_0 = \sum_{i=1}^k p_{0,i}$, and $[p_{0,i} + p_i] = 0$ in $K_0(B)$ (recall that B is purely infinite simple). Define $\varphi_{00} : C(X) \to (1_{M_k} + p_0)M_{k+1}(B)(1_{M_k} + p_0) \subset M_{k+1}(B)$ by $\varphi_{00}(f) = \sum_{i=1}^k f(x_i)(p_i + p_{0,i})$ for all $f \in C(X)$. By replacing φ_0 by φ_{00} and u by $u \oplus 1_B$, we may assume, without loss of generality, that $[p_i] = 0$, $i = 1, 2, \dots, k$, and

$$\|\mathrm{diag}(\varphi(f), \varphi_0(f)) - u^* \mathrm{diag}(\varphi_1(f), \varphi_0(f))u\| < \varepsilon/3 \text{ for all } f \in \mathcal{F}. \tag{e3.99}$$

(But then u is a unitary in $M_{k+2}(B)$.)

Recall that, by Zhang's theorem [37], B has real rank zero. Since φ is injective, by applying Lemma 4.1 of [8], we obtain a set of mutually orthogonal nonzero projections $q_1, q_2, \dots, q_k \in B$ with $q = \sum_{i=1}^k q_i < 1_B$ such that

$$\left\| \varphi(f) - \left((1-q)\varphi(f)(1-q) + \sum_{i=1}^k f(x_i) q_i \right) \right\| < \varepsilon/3 \text{ for all } f \in \mathcal{F}. \tag{e3.100}$$

There are nonzero subprojections $q_i' \le q_i$ such that $[q_i'] = 0$ in $K_0(B)$, $i = 1, 2, \dots, k$ (recall again that B is purely infinite simple). We may then write

$$\left\| \varphi(f) - \left(\gamma(f) + \sum_{i=1}^k f(x_i) q_i' \right) \right\| < \varepsilon/3 \text{ for all } f \in \mathcal{F}, \tag{e3.101}$$

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where $\gamma(f) = (1-q)\varphi(f)(1-q) + \sum_{i=1}^k f(x_i)(q_i - q_i')$ for all $f \in C(X)$. There are $w_i \in M_{2k+3}(B)$ such that

$$w_i^* w_i = q_i' \text{ and } w_i w_i^* = q_i' \oplus p_i, \quad i = 1, 2, \dots, k. \tag{e3.102}$$

Let $w = (1-q) + \sum_{i=1}^k (q_i - q_i') + \sum_{i=1}^k w_i$. Then $w^*w = 1_B$ and $ww^* = 1_{M_{k+1}} + \sum_{i=1}^k p_i$. Moreover,

$$w^*(\mathrm{diag}\left(\gamma(f) + \sum_{i=1}^k f(x_i)q_i', \varphi_0(f)\right)w = \gamma(f) + \sum_{i=1}^k f(x_i)q_i' \tag{e3.103}$$

Hence,

$$\|\varphi(f) - w^*(\mathrm{diag}\left(\gamma(f) + \sum_{i=1}^k f(x_i)q_i', \varphi_0(f)\right)w\| < \varepsilon/3 \text{ for all } f \in \mathcal{F}. \tag{e3.104}$$

Define, for all $f \in C(X)$,

(e3.105)

$$\psi(f) = w^*u^*(\mathrm{diag}(\varphi_1(f), \varphi_0(f))uw = f(x)w^*u^*1_A u^* w + \sum_{i=1}^k f(x_i)w^*u^*p_i uw.$$

Then ψ is a unital homomorphism from $C(X)$ to B with finite-dimensional range and, by (e 3.99), (e 3.101), and (e 3.104), for all $f \in \mathcal{F}$,

$$\begin{aligned}
\|\varphi(f) - \psi(f)\| &\le \|\varphi(f) - w^*(\mathrm{diag}(\gamma(f) + \sum_{i=1}^k f(x_i)q_i', \varphi_0(f)))w\| \\
&+ \|w^*(\mathrm{diag}(\gamma(f) + \sum_{i=1}^k f(x_i)q_i', \varphi_0(f)))w - w^*u^*\mathrm{diag}(\varphi_1(f), \varphi_0(f))uw\| \\
&< \varepsilon/3 + \|\mathrm{diag}(\gamma(f) + \sum_{i=1}^k f(x_i)q_i', \varphi_0(f)) - u^*\mathrm{diag}(\varphi_1(f), \varphi_0(f))u\| \\
&< \varepsilon/3 + \varepsilon/3 + \|\mathrm{diag}(\varphi(f), \varphi_0(f)) - u^*\mathrm{diag}(\varphi_1(f), \varphi_0(f))u\| < 2\varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Finally, since $[p_i] = 0$ in $K_0(B)$ as we arranged earlier, by (e 3.105), the "Moreover" part of the statement also holds. ∎

Let $X \subset \mathbb{I}^n$ be a compact subset. We use the usual Euclidean metric. Recall that, for any $\eta > 0$,

$$X_\eta = \{\xi \in \mathbb{I}^n : \mathrm{dist}(\xi, X) < \eta\}. \tag{e3.106}$$

Theorem 3.6. *Fix $n \in \mathbb{N}$. For any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset C(\mathbb{I}^n)$, there exists $\delta(n, \varepsilon) > 0$ satisfying the following.*

Let X be a compact subset of $\mathbb{I}^n$. Suppose that B is a unital purely infinite simple C^*-algebra and $\varphi : C(X) \to B$ is a unital injective homomorphism. If Y is any compact

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subset with property (F) such that $X \subset Y \subset X_\delta$, then there is a unital injective homomorphism $\psi : C(Y) \to B$ such that

(e3.107) $\|\varphi(f|_X) - \psi(f|_Y)\| < \varepsilon$ *for all* $f \in \mathcal{F}$, $[\psi] = [\varphi \circ \pi_X]$ *in* $KK(C(Y), B)$,

where $\pi_X : C(Y) \to C(X)$ is the quotient map by restriction ($\pi_X(g) = g|_X$ for all $g \in C(Y)$).

Proof. Let $\varepsilon > 0$ and finite subset $\mathcal{F} \subset C(\mathbb{I}^n)$ be given. Choose $0 < \delta < \min\{\varepsilon/2, 1/8\}$ such that, for all $g \in \mathcal{F}$,

(e3.108) $$|g(\xi) - g(\xi')| < \varepsilon/4 \text{ for all } \xi, \xi' \in \mathbb{I}^n \text{ with } \operatorname{dist}(\xi, \xi') < 2\delta.$$

Suppose that Y is a compact subset of $\mathbb{I}^n$ such that $X \subset Y \subset X_\delta$ which has property (F). Since Y is compact, there is $\delta_1 > 0$ with $0 < \delta_1 < \delta$ such that

(e3.109) $$Y \subset X_{\delta_1}.$$

Put $\eta_0 = \frac{\delta - \delta_1}{4} > 0$ and $r = \delta_1 + \eta_0$. Then $0 < r = \delta_1 + \eta_0 < \delta_1 + 2\eta_0 < \delta$. There is a finite η_0-net $\{\xi_1, \xi_2, \dots, \xi_m\} \subset X$ such that $\cup_{i=1}^m B(\xi_i, \eta_0) \supset X$, where $B(\xi_i, \eta_0) = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \xi_i) < \eta_0\}$. Then $\cup_{i=1}^m B(\xi_i, r) \supset Y$. Note that we assume that $\xi_i \neq \xi_j$, if $i \neq j$.

Choose a nonzero projection $p \in B$ with $[p] = 0$ in $K_0(B)$ and $1 - p \neq 0$. Since pBp is a purely infinite simple C^*-algebra, there is a unital embedding $\Phi : O_2 \to pBp$. Applying Theorem 2.8 of [18], one obtains a unital embedding $\psi_0 : C(Y) \to \Phi(O_2) \subset pBp$. Note $[\psi_0] = 0$ in $KK(C(Y), pBp)$. There is a partial isometry $w \in B$ such that $w^*w = 1 - p$ and $ww^* = 1$. Define $\psi_1 : C(Y) \to B$ by $\psi_1(f) = w^*\varphi(f|_X)w \oplus \psi_0(f)$ for all $f \in C(Y)$. Then ψ_1 is a unital injective homomorphism.

Choose mutually orthogonal nonzero projections $p_1, p_2, \dots, p_m \in \Phi(O_2) \subset pBp$ such that $\sum_{i=1}^m p_i = p$. Define $\varphi_0 : C(X) \to \Phi(O_2) \subset pBp$ by $\varphi_0(f) = \sum_{i=1}^m f(\xi_i)p_i$. Then $[\varphi_0] = 0$ in $KK(C(X), B)$. Define $\varphi_1 : C(X) \to B$ by $\varphi_1(f) = w^*\varphi(f)w + \varphi_0(f)$ for all $f \in C(X)$. Then φ_1 is injective. One computes that

(e3.110) $$[\varphi] = [\varphi_1] \text{ in } KK(C(X), B).$$

It follows from Theorem 1.7 of [6] that there is a unitary $u \in B$ such that

(e3.111) $$\|\varphi(f|_X) - u^*\varphi_1(f|_X)u\| < \varepsilon/4 \text{ for all } f \in \mathcal{F}.$$

Applying Theorem 3.5, there is a unital homomorphism $\psi_{00} : C(Y) \to \Phi(O_2) \subset pBp$ with finite-dimensional range such that

(e3.112) $$\|\psi_0(g|_Y) - \psi_{00}(g|_Y)\| < \varepsilon/4 \text{ for all } g \in \mathcal{F}.$$

We may write $\psi_{00}(g) = \sum_{j=1}^K g(y_j)d_i$ for all $g \in C(Y)$, where $y_j \in Y$ and $d_1, d_2, \dots, d_K$ are mutually orthogonal nonzero projections in $\Phi(O_2)$ with

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$p = \sum_{j=1}^{K} d_j$. By choosing a better approximation if necessary, without loss of generality, we may assume that $\{y_1, y_2, \dots, y_K\}$ is η_0-dense in $Y, K \ge m$. Recall that $\{\xi_1, \xi_2, \dots, \xi_n\}$ is η_0-dense in X and r-dense in Y. For each i, choose exactly one $y_{(i,1)} \subset \{y_1, y_2, \dots, y_K\}$ such that $\mathrm{dist}(y_{(i,1)}, \xi_i) < \eta_0$. Then, we may assume $\cup_{i=1}^{m}\{y_{(i,j)} : 1 \le j \le k(i)\} = \{y_1, y_2, \dots, y_K\}$, where $\{y_{(i,j)} : 1 \le j \le k(i), 1 \le i \le m\}$ is a set of distinct points such that

$$\mathrm{dist}(y_{(i,j)}, \xi_i) < \delta_1 + 2\eta_0 < \delta, \ \ 1 \le j \le k(i), \ \ 1 \le i \le n. \tag{e3.113}$$

Note that we view $(i, j) \in \{1, 2, \dots, K\}$. Then set $q_i' = \sum_{j=1}^{k(i)} d_{(i,j)}$. Put $\psi_{00}' : C(Y) \to \Phi(O_2) \subset pBp$, where $\psi_{00}'(g) = \sum_{i=1}^{m} g(\xi_i) q_i'$ for all $g \in C(Y)$.

By the choice of δ, we have, for all $g \in \mathcal{F}$,

$$\|\psi_{00}(g|_Y) - \psi_{00}'(g|_Y)\| < \varepsilon/4. \tag{e3.114}$$

Recall that $p_i, q_i' \in \Phi(O_2)$. Hence, $[p_i] = [q_i'] = 0$ in $K_0(B)$. There are partial isometries $w_i \in pB(u^* pu)$ such that $w_i^* w_i = u^* p_i u$ and $w_i w_i^* = q_i'$, $i = 1, 2, \dots, m$. Define $v = (1 - p)u \oplus \sum_{i=1}^{m} w_i$. Then

$$v^* v = u^*(1 - p)u \oplus \sum_{i=1}^{m} u^* p_i u = u^*(1 - p)u + u^* pu = 1 \ \text{ and} \tag{e3.115}$$

$$vv^* = (1 - p)uu^*(1 - p) \oplus \sum_{i=1}^{m} q_i' = (1 - p) + \sum_{i=1}^{K} d_i = 1. \tag{e3.116}$$

So v is a unitary in B. Note that

$$v^* \psi_{00}'(g) v = v^* \left(\sum_{i=1}^{n} g(\xi_i) q_i' \right) v = \sum_{i=1}^{n} g(\xi_i) u^* p_i u = u^* \varphi_0(g|_X) u \ \text{ for all } \ g \in C(Y). \tag{e3.117}$$

Then, combining with (e 3.112) and (e 3.114), for all $g \in \mathcal{F}$,

$$\begin{aligned} \|v^* \psi_0(g|_Y) v - u^* \varphi_0(g|_X) u\| &= \|v^* \psi_0(g|_Y) v - v^* \psi_{00}(g|_Y) v\| \\ &\quad + \|v^* \psi_{00}(g|_Y) v - v^* \psi_{00}'(g|_Y) v\| \\ &< \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \end{aligned} \tag{e3.118}$$

Define $\psi : C(Y) \to B$ by

$$\psi(g) = v^*(w^* \varphi(g|_X) w + \psi_0(g|_Y)) v \ \text{ for all } \ g \in C(Y). \tag{e3.119}$$

Then ψ is a unital injective homomorphism. Moreover, for all $g \in \mathcal{F}$, by (e 3.111), the definition of φ_1 and v above and by (e 3.118),

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$$\begin{aligned}\|\varphi(g|_X)-\psi(g|_Y)\| &= \|\varphi(g|_X)-u^*\varphi_1(g|_X)u\|+\|u^*\varphi_1(g|_X)u-\psi(g|_Y)\|\\ &< \varepsilon/4+\|u^*w^*\varphi(g|_X)wu\oplus u^*\varphi_0(g|_X)u-v^*(w^*\varphi(g|_X)w\oplus\psi_0(g|_Y))v\|\\ &= \varepsilon/4+\|u^*w^*\varphi(g|_X)wu\oplus u^*\varphi_0(g|_X)u-v^*\\ &\times((1-p)w^*\varphi(g|_X)w(1-p)\oplus\psi_0(g|_Y))v\|\\ &= \varepsilon/4+\|u^*w^*\varphi(g|_X)wu\oplus u^*\varphi_0(g|_X)\\ &\quad u-u^*w^*\varphi(g|_X)wu\oplus v^*\psi_0(g|_Y))v\|\\ &= \varepsilon/4+\|u^*\varphi_0(g|_X)u-v^*\psi_0(g|_Y)v\|<\varepsilon/4+\varepsilon/2<\varepsilon. \qquad \text{(e3.120)}\end{aligned}$$

To see the second formula in (e 3.107), we may write, from (e 3.119), since ψ_0 factors through $\Phi(O_2)$,

$$[\psi]=[\varphi\circ\pi_X]+[\psi_0]=[\varphi\circ\pi_X] \text{ in } KK(C(Y),B). \qquad \text{(e3.121)}$$ ∎

Corollary 3.7. *Fix an integer $n\in\mathbb{N}$ and $1>\varepsilon>0$.*

Suppose that X is a compact subset of $\mathbb{I}^n$, B is a unital purely infinite simple C^*-algebra, and $\varphi: C(X)\to B$ is a unital injective homomorphism. If Y is any compact subset with property (F) such that $X\subset Y\subset X_\varepsilon$, then there is a unital injective homomorphism $\psi: C(Y)\to B$ such that

$$\|\varphi(e_j|_X)-\psi(e_j|_X)\|<\varepsilon \text{ and } [\psi]=[\varphi\circ\pi_X] \text{ in } KK(C(Y),B), \qquad \text{(e3.122)}$$

where $\pi_X: C(Y)\to C(X)$ is the quotient map by restriction ($\pi(g)=g|_X$ for all $g\in C(Y)$).

Proof. At the beginning of the proof of Theorem 3.6, choose $\mathcal{F}=\{e_j: 1\le j\le n\}$. Then, with $\delta:=\varepsilon/8$, when $\operatorname{dist}(\xi,\xi')<2\delta$,

$$|e_j(\xi)-e_j(\xi')|<\varepsilon/4. \qquad \text{(e3.123)}$$

Then the rest of the proof of Theorem 3.6 applies (with $\delta=\varepsilon/8$). ∎

In the proof of Theorem 3.6, we may choose Y to be a $1/k$-brick combination with $\sqrt{n}/2k<\delta$ (the same δ in the proof of Theorem 3.6). Thus, we may state a variation of Theorem 3.6 as follows.

Corollary 3.8. *Let $n\in\mathbb{N}$. Then, for any $\varepsilon>0$ and any finite subset $\mathcal{F}\subset C(\mathbb{I}^n)$, there exists $0<\eta<\varepsilon/2$ satisfying the following: Suppose that B is a purely infinite simple C^*-algebra and $\varphi: C(X)\to B$ is a unital injective homomorphism, where $X\subset\mathbb{I}^n$ is a compact subset. For any integer $k\in\mathbb{N}$ with $\sqrt{n}/k<\eta$, there exists a unital injective homomorphism $\psi: C(\mathrm{B}_X^k)\to B$ such that*

$$\|\varphi(f|_X)-\psi(f)\|<\varepsilon \text{ for all } f\in\mathcal{G} \text{ and } [\psi]=[\varphi\circ\pi] \text{ in } KK(C(\mathrm{B}_X^k),B), \qquad \text{(e3.124)}$$

where $\pi: C(\mathrm{B}_X^k)\to C(X)$ is the quotient map by restriction, $\mathcal{G}\subset C(\mathrm{B}_X^k)$ is a finite subset such that $\mathcal{F}\subset\{g|_X: g\in\mathcal{G}\}$ (and where B_X^k is a $1/k$-brick cover of X).

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Theorem 3.9. *Let $\varepsilon > 0$ and $n \in \mathbb{N}$. There is $\delta = \delta(n, \varepsilon) > 0$ satisfying the following: If A is a unital purely infinite simple C^*-algebra and $s_1, s_2, \dots, s_n \in A_{s.a.}$ with $\|s_j\| \le 1$ $(1 \le j \le n)$ such that*

$$\|s_i s_j - s_j s_i\| < \delta, \quad i, j = 1, 2, \dots, n, \tag{e3.125}$$

then there is integer $k \in \mathbb{N}$ with $2\sqrt{n}/k < \delta$ and $1/k \le \eta < \delta$, and a $1/k$-brick combination $X \subset I^n$ and a unital injective homomorphism $\varphi : C(X) \to A$ such that

$$\|\varphi(e_j|_X) - s_j\| < \varepsilon, \quad j = 1, 2, \dots, n. \tag{e3.126}$$

Proof. Let $\varepsilon > 0$. It follows from Corollary 1.20 of [22] that there exists $\delta_1 > 0$ such that when $\|s_i s_j - s_j s_i\| < \delta_1$, there are $a_1, a_2, \dots, a_n \in A_{s.a.}$ such that

$$a_i a_j = a_j a_i \text{ and } \|s_i - a_i\| < \varepsilon/2 \text{ for all } 1 \le i, j \le n. \tag{e3.127}$$

Let C be the commutative C^*-subalgebra generated by $a_1, a_2, \dots, a_n$. Then there exists a unital homomorphism $\varphi_0 : C(\mathbb{I}^n) \to C$ such that $\varphi_0(e_j) = a_j$, $j = 1, 2, \dots, n$. Hence, $C \cong C(X_0)$ for some compact subset of $\mathbb{I}^n$. Denote $\varphi_1 : C(X_0) \to C$ the isomorphism given above. We may assume that $\varphi_1(e_j|X_0) = a_j$, $j = 1, 2, \dots, n$.

Put $\delta = \min\{\varepsilon/2, \delta_1/2\}$. Choose any $k \in \mathbb{N}$ with $2\sqrt{n}/k < \delta$ and $1/k \le \eta < \delta$. By Corollary 3.8, we obtain a $1/k$-brick combination $X \subset \mathbb{I}^n$ and a unital injective homomorphism $\varphi : C(X) \to B$ such that

$$\|\varphi(e_j|_X) - a_j\| < \varepsilon/2, \quad j = 1, 2, \dots, n. \tag{e3.128}$$

Hence,

$$\|\varphi(e_j|_X) - s_j\| < \varepsilon, \quad j = 1, 2, \dots, n. \tag{e3.129}$$

■

4 Multiple self-adjoint operators

The main purpose of this section is to prove Theorem 4.6.

The following lemma is a known (see the proof of Proposition 3.15 of [3]).

Lemma 4.1. *Let A be a unital C^*-algebra, and J be an ideal of A which is a σ-unital C^*-algebra of real rank zero. Suppose that $q \in A/J$ is a projection such that $[q] \in \pi_{*0}(K_0(A))$, where $\pi : A \to A/J$ is the quotient map. Then there is a projection $p \in A$ such that $\pi(p) = q$.*

Lemma 4.2. *Fix $n \in \mathbb{N}$. For any $\varepsilon > 0$, there exists $\delta(n, \varepsilon) > 0$ satisfying the following: Suppose that A is a unital C^*-algebra with an ideal J such that:*

1. *J is separable and has real rank zero;*
2. *the quotient A/J is purely infinite and simple;*
3. *$T_1, T_2, \dots, T_n \in A_{s.a.}$ with $\|T_i\| \le 1$ $(1 \le i \le n)$;*
4. *the n-tuple has a δ-near-spectrum X and $(\pi(T_1), \pi(T_2), \dots, \pi(T_n))$ has a δ-spectrum Y, where $\pi : A \to A/J$ is the quotient map. Then, there are mutually*

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orthogonal nonzero projections $p_1, p_2, \dots, p_m \in A\backslash J$, *a* δ*-dense subset* $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ *in* Y, *and a c.p.c. map* $L : C(X) \to (1-p)A(1-p)$, *where* $p = \sum_{k=1}^m p_k$, *such that*

$$\|\sum_{k=1}^m e_j(\lambda_k)p_k + L(e_j|_X) - T_j\| < \varepsilon, \tag{e4.130}$$

$$\|L(e_ie_j|_X) - L(e_i|_X)L(e_j|_X)\| < \varepsilon, \ \ 1 \le i, j \le n, \tag{e4.131}$$

and $1 - p \notin J$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/128$. Since X is a δ-near-spectrum of $(T_1, T_2, \dots, T_n)$, by Definition 2.10, there is a unital c.p.c. map $\Phi : C(X) \to A$ such that

$$\|\Phi(e_j|_X) - T_j\| < \delta \ \text{ and } \ \|\Phi(e_ie_j|_X) - \Phi(e_i|_X)\Phi(e_j|_X)\| < \delta, \ \ i, j \in \{1, 2, \dots, n\}. \tag{e4.132}$$

Since Y is a δ-spectrum of $(\pi(T_1), \pi(T_2), \dots, \pi(T_n))$, there exists, by Definition 2.10 (near (e 2.21)), a compact subset $Z \subset \mathbb{I}^n$ such that $Z \subset Y \subset Z_\delta$ and a unital monomorphism $\varphi : C(Z) \to A/J$ such that

$$\|\varphi(e_j|_Z) - \pi(T_j)\| < \delta, \ \ j = 1, 2, \dots, n. \tag{e4.133}$$

Let $e' \in A/J$ be a nonzero projection such that $1 - e' \neq 0$. Choose a partial isometry $w \in A/J$ such that

$$w^*w = 1 \ \text{ and } \ ww^* = e \le e'. \tag{e4.134}$$

It follows that $[1-e] = 0$ in $K_0(A/J)$. Let $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ be a $\delta/2$-dense subset of Z which is also δ-dense in Y. Let $q_1, q_2, \dots, q_m$ be mutually orthogonal nonzero projections in $(1-e)(A/J)(1-e)$ such that $[q_i] = 0$ in $K_0(A/J)$. Define $\psi : C(Z) \to A/J$ by

$$\psi(f) = \sum_{k=1}^m f(\lambda_k)q_k + w\varphi(f)w^* \ \text{ for all } \ f \in C(Z). \tag{e4.135}$$

Note that $[\varphi] = [\psi]$ in $KK(C(Z), A/J)$. Then, since A/J is purely infinite and simple, by Theorem 1.7 of [6], there exists a unitary $u \in A/J$ such that

$$\|u\psi(e_j|_Z)u^* - \varphi(e_j|_Z)\| < \delta/16 \ \text{ for all } \ j = 1, 2, \dots, n. \tag{e4.136}$$

Let $q_k' = uq_ku^*$, $k = 1, 2, \dots, n$, and

$$\varphi_0 : C(Z) \to ueu^*(A/J)ueu^* \ \text{ by } \ \varphi_0(f) = uw\varphi(f)w^*u^* \ \text{ for all } \ f \in C(Z).$$

Then we may write

$$u\psi(f)u^* = \sum_{k=1}^m f(\lambda_k)q_k' + \varphi_0(f) \ \text{ for all } \ f \in C(Z). \tag{e4.137}$$

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Since $[q_k'] = 0$ in $K_0(A/J)$, by applying Lemma 4.1, we obtain projections $P_k \in A$ such that

$$\pi(P_k) = q_k', \quad k = 1, 2, \ldots, m, \quad \pi(1 - \sum_{k=1}^{m} P_k) = ueu^* \text{ and} \tag{e4.138}$$

$$P_k P_{k'} = P_{k'} P_k = 0, \quad \text{if } k \neq k'. \tag{e4.139}$$

Let $P = \sum_{k=1}^{m} P_k$. Then $P \in A\backslash J$ is a projection. By the Choi–Effros lifting Theorem [4], there is a c.p.c. map $\Psi : C(Y) \to (1-P)A(1-P)$ such that $\pi \circ \Psi = \varphi_0 \circ \pi_Z$, where $\pi_Z : C(Y) \to C(Z)$ is the quotient map.

Then, by (e 4.132), (e 4.133), and (e 4.136), there are $h_j \in J$, $j = 1, 2, \ldots, n$, such that

$$\|\Phi(e_j|_X) - \left(\sum_{k=1}^{m} e_j(\lambda_k)P_k + \Psi(e_j|_Y)\right) - h_j\| < 2\delta + \delta/16, \quad j = 1, 2, \ldots, n. \tag{e4.140}$$

Let $\{d_{k,l}\}$ be an approximate identity for $P_k J P_k$, $k = 1, 2, \ldots, m$, and $\{d_{0,l}\}$ be an approximate identity for $(1-P)J(1-P)$ consisting of projections. Put

$$d_l = d_{0,l} + \sum_{k=1}^{m} d_{k,l} \text{ and } d_l' = \sum_{k=1}^{m} d_{k,l}, \quad l \in \mathbb{N}.$$

Then $\{d_l\}$ is an approximate identity for J, and $\{d_l'\}$ is an approximate identity for PJP. Choose l such that

$$\|(1-d_l)h_j\| < \delta/64 \text{ and } \|h_j(1-d_l)\| < \delta/64, \quad j = 1, 2, \ldots, n. \tag{e4.141}$$

Hence, since $P - d_l' \leq 1 - d_l$,

$$\|(P-d_l')h_j\| < \delta/64 \text{ and } \|h_j(P-d_l')\| < \delta/64, \quad j = 1, 2, \ldots, n. \tag{e4.142}$$

Put $p_k = P_k - d_{k,l}$, $k = 1, 2, \ldots, m$. Then, by (e 4.140), for $j = 1, 2, \ldots, n$,

$$(P-d_l')\Phi(e_j|_X) \approx_{33\delta/16} (P-d_l')\left(\sum_{k=1}^{m} e_j(\lambda_k)P_k + \Psi(e_j|_Y) + h_j\right) \tag{e4.143}$$

$$\approx_{\varepsilon/64} \sum_{k=1}^{m} e_j(\lambda_k)p_k. \tag{e4.144}$$

Similarly, for $j = 1, 2, \ldots, n$,

$$\Phi(e_j|_X)(P-d_l') \approx_{33\delta/16} \left(\sum_{k=1}^{m} e_j(\lambda_k)P_k + \Psi(e_j|_Y) + h_j\right)(P-d_l') \tag{e4.145}$$

$$\approx_{\delta/64} \sum_{k=1}^{m} e_j(\lambda_k)p_k. \tag{e4.146}$$

Put $p = P - d_l'$. Then

$$\|p\Phi(e_j|_X) - \sum_{k=1}^{m} e_j(\lambda_k)p_k\| < 33\delta/16 + \delta/64, \tag{e4.147}$$

$$\|p\Phi(e_j|_X)p - \sum_{k=1}^{m} e_j(\lambda_k)p_k\| < 33\delta/16 + \delta/64 \text{ and} \tag{e4.148}$$

$$\|p\Phi(e_j|_X) - \Phi(e_j|_X)p\| < 33\delta/8 + \delta/32, \quad j = 1, 2, \ldots, n. \tag{e4.149}$$

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Put $L(f) = (1-p)\Phi(f)(1-p)$ for $f \in C(X)$. Then it is a c.p.c. map. Moreover,

$$\Phi(e_j|_X) = p\Phi(e_j|_X) + (1-p)\Phi(e_j|_X) \tag{e4.150}$$

$$\approx_{33\delta/8+3\delta/64} \sum_{k=1}^{m} e_j(\lambda_k)p_k + (1-p)\Phi(e_j|X)(1-p). \tag{e4.151}$$

Hence, using the inequality above and (e 4.132),

$$\begin{aligned} &\|\sum_{k=1}^{m} e_j(\lambda_k)p_k + L(e_j|_X) - T_j\| \\ &\quad \le \|\sum_{k=1}^{m} e_j(\lambda_k)p_k + L(e_j|_X) - \Phi(e_j|_X)\| + \|\Phi(e_j|_X) - T_j\| \\ &\quad < 33\delta/8 + 3\delta/64 + \delta < \varepsilon, \quad j = 1, 2, \dots, n. \end{aligned} \tag{e4.152}$$

To arrange $(1-p) \notin J$, we may split p_1. We write $p_1 = p_1' + p_1''$, where $p_1', p_1'' \in p_1Ap_1$ are two mutually orthogonal projections and both are not in J as $p_1 \notin J$. Then define $L' : C(X) \to (1-p+p_1'')A(1-p+p_1'')$ by $L'(f) = f(\lambda_1)p_1'' + L(f)$ for all $f \in C(X)$. We then replace L with L' and replace p_1 with p_1'.

Finally, by (e 4.149) and (e 4.132),

$$\begin{aligned} L(e_ie_j|_X) - L(e_i|_X)L(e_j|_X) &= (1-p)\Phi(e_ie_j|_X)(1-p) \\ &\quad - (1-p)\Phi(e_i|_X)(1-p)\Phi(e_j|_X)(1-p) \\ &\approx_{33\delta/8+\delta/32} (1-p)(\Phi(e_i|_X) - \Phi(e_i|_X)\Phi(e_j|_X))(1-p) \approx_\delta 0. \end{aligned}$$

Thus (e 4.131) also holds. ∎

The following proposition is a list of easy facts. We refer to Definition 2.1 of [11] for the definition of these properties. It is certainly known that $M(A \otimes \mathcal{K})$ has these properties and some of these may also stated in various places. We put here for our convenience.

Proposition 4.3. *Let A be a σ-unital C^*-algebra. Then $M(A \otimes \mathcal{K})$ has the following property:*

(1) *K_0-r-cancellation with $r(n) = 1$ for all $n \in \mathbb{N}$;*
(2) *K_1-r-cancellation with $r(n) = 1$ for all $n \in \mathbb{N}$;*
(3) *K_1-stable rank 1;*
(4) *K_0-stable rank 1;*
(5) *stable exponential length $b = 3\pi$;*
(6) *stable exponential rank 4;*
(7) *K_0-divisible rank $T(n,m) = 1$ (for $(n,m) \in \mathbb{N}^2$);*
(8) *K_1-divisible rank $T(n,m) = 1$ (for $(n,m) \in \mathbb{N}^2$);*
(9) *exponential length divisible rank $E(r,n) = 3\pi$ (for all $r \in \mathbb{R}_+, n \in \mathbb{N}$).*

Proof. First, by [5], $K_i(M(A \otimes \mathcal{K})) = \{0\}, i = 0, 1$. We also have, for any $k \in \mathbb{N}, M_k(M(A \otimes \mathcal{K})) \cong M(A \otimes \mathcal{K})$. For any projection $p \in M(A \otimes \mathcal{K})$, we have $p \oplus 1_{M(A\otimes\mathcal{K})} \sim 1_{M(A\otimes\mathcal{K})}$. This immediately implies (1), (4), and (7).

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It follows from Theorem 1.1 of [38] that $\mathrm{cel}(M(A\otimes\mathcal{K}))\le 3\pi$ and $\mathrm{cer}(M(A\otimes\mathcal{K}))\le 4$. Therefore, (2), (3), (5), (6), (8), and (9) hold. ∎

Definition 4.4. Let A be a unital C^*-algebra. Let $r, b:\mathbb{N}\to\mathbb{N}$ and $T:\mathbb{N}^2\to\mathbb{N}$ be maps, and R, $s>0$. We say that A satisfies condition (R,r,b,T,s) if A has K_i-r-cancellation, $i=0,1$, K_1-stable rank s, K_0-divisible rank T, $\mathrm{cer}(M_m(A))\le R$ for all $m\in\mathbb{N}$, and $\mathrm{cel}(M_m(A))\le b(m)$ for all $m\in\mathbb{N}$.

If A is a unital purely infinite simple C^*-algebra, then A satisfies condition (R,r,b,T,s) for $R=2$, $s=1$, any r and T and $b=2\pi$. If A has stable rank one and real rank zero, then A satisfies condition (R,r,b,T,s) for $R=2$, $s=1$, any r, T, and $b=2\pi$.

We will apply the following theorem which is a variation of Theorem 3.1 of [11].

Theorem 4.5. *Fix $n\in\mathbb{N}$. Let $r, b:\mathbb{N}\to\mathbb{N}$, $T:\mathbb{N}^2\to\mathbb{N}$ be maps, and $R>0$, $s>0$.*

For any $\varepsilon>0$, there exist a positive number $\delta>0$ and an integer $l>0$ satisfying the following: Suppose that A is a unital C^*-algebra such that:

1. *A satisfies condition (R,r,b,T,s);*
2. *$K_i(A\otimes C(Y))=0$, $i=0,1$, for any compact metric space Y.*

Let X be a compact subset of $\mathbb{I}^n$, and let $L_1, L_2: C(X)\to A$ be unital c.p.c. maps such that

$$\|L_k(e_ie_j|_X)-L_k(e_i|_X)L_k(e_j|_X)\|<\delta,\ \ 1\le i,j\le n,\ \ k=1,2. \tag{e4.153}$$

Then, there are a unitary $u\in M_{l+1}(A)$ (for some $l\in\mathbb{N}$) and a homomorphism $\sigma: C(X)\to M_l(A)$ with finite-dimensional image, such that

$$\|u^*\mathrm{diag}(L_1(e_j|_X),\sigma(e_j|_X))u-\mathrm{diag}(L_2(e_j|_X),\sigma(e_j|_X))\|<\varepsilon,\ \ 1\le j\le n. \tag{e4.154}$$

Proof. Let us first fix a compact subset X. Let $\varepsilon>0$ and $\mathcal{F}=\{e_j|_X: 0\le j\le n\}$.

Since we assume that $K_i(A\otimes C(Y))=0$ for any compact metric space Y, $i=0,1$, for any finite subset $\mathcal{P}$ as in Theorem 3.1 of [11], we have that $[L]|_{\mathcal{P}}=0$ for any sufficiently multiplicative c.p.c. maps $L: C(X)\to A$. Then, by Proposition 4.3 and applying Theorem 3.1 of [11], we obtain δ_0 and a finite subset $\mathcal{G}_0\subset C(X)$ such that, if

$$\|L_i(fg)-L_i(f)L_i(g)\|<\delta_0 \ \text{ for all } \ f,g\in\mathcal{G}_0,\ \ i=1,2, \tag{e4.155}$$

there exists an integer $l>0$, a unital homomorphism $\sigma: C(X)\to M_l(A)$ with finite-dimensional range, and a unitary $u\in M_{l+1}(A)$ such that

$$\|u^*\mathrm{diag}(L_1(e_j|_X),\sigma(e_j|_X))u-\mathrm{diag}(L_2(e_j|_X),\sigma(e_j|_X))\|<\varepsilon/2,\ \ 1\le j\le n. \tag{e4.156}$$

(Note as mentioned above, that $[L_i]|_{\mathcal{P}}=0$, whenever it makes sense, since $K_i(A\otimes C(Y))=0$ for any compact metric space Y, $i=0,1$.)

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However, since $\{e_j : 0 \le j \le n\}$ generates $C(X)$ as C^*-algebra, one obtains $\delta(n,\varepsilon) > 0$ such that when (e 4.153) holds, then

$$\|L_i(fg) - L_i(f)L_i(g)\| < \delta_0 \text{ for all } f, g \in \mathcal{G}_0, \quad i = 1, 2. \tag{e4.157}$$

So Theorem 3.1 of [11] applies and the theorem would follow if we fix X first.

To find a common δ and l, independent of $X \subset \mathbb{I}^n$, we note that there are finitely many compact subsets $X_1, X_2, \dots, X_m$ of $\mathbb{I}^n$ such that, for any non-empty compact subset X, there is $s \in \{1, 2, \dots, m\}$ such that

$$d_H(X, X_s) < \varepsilon/6 \tag{e4.158}$$

(see Definition 2.15). Let $\Omega_s = \{x \in \mathbb{I}^n : \text{dist}(x, X_s) \le \varepsilon/6\}, s = 1, 2, \dots, m$. Then, for any non-empty compact subset X of $\mathbb{I}^n$, there is $s \in \{1, 2, \dots, m\}$ such that

$$d_H(X, \Omega_s) \le \varepsilon/3 \text{ and } X \subset \Omega_s. \tag{e4.159}$$

When $\varepsilon > 0$ is given, by the first part of the proof, we obtain $\delta_s > 0$ and $l_s \in \mathbb{N}$ such that the conclusion of the first part holds for $\Omega_s, s = 1, 2, \dots, m$. Choose $\delta = \min\{\delta_1, \delta_2, \dots, \delta_m\}$ and $l = \max\{l_1, l_2, \dots, l_m\}$.

Now suppose $X \subset \mathbb{I}^n$ and $L_1, L_2 : C(X) \to A$ are unital c.p.c. maps satisfying the assumption (e 4.153). Suppose that

$$d_H(X, \Omega_s) \le \varepsilon/3 \text{ and } X \subset \Omega_s. \tag{e4.160}$$

Define $\tilde{L}_k : C(\Omega_s) \to A$ by $\tilde{L}_k(e_j|_{\Omega_s}) = L_k(e_j|_X)$ (this makes sense as $X \subset \Omega_s$), $j = 1, 2, \dots, n, k = 1, 2$. Therefore,

$$\|\tilde{L}_k(e_i e_j|_{\Omega_s}) - \tilde{L}_k(e_i|_{\Omega_s})\tilde{L}_k(e_j|_{\Omega_s})\| < \delta, \ \ i, j \in \{1, 2, \dots, n\} \tag{e4.161}$$

and $k = 1, 2$. Hence, with the choice of δ, we obtain homomorphism $\sigma' : C(\Omega_s) \to M_l(A)$ with finite-dimensional range and a unitary $u \in M_{l+1}(A)$ such that, for $j = 1, 2, \dots, n$,

$$\|u^* \text{diag}(L_1(e_j|_{\Omega_s}), \sigma'(e_j|_{\Omega_s}))u - \text{diag}(L_2(e_j|_{\Omega_s}), \sigma'(e_j|_{\Omega_s}))\| < \varepsilon/2. \tag{e4.162}$$

We may write $\sigma'(e_j|_{\Omega_s}) = \sum_{k=1}^K e_j(\xi_k)p_k$, where $\xi_k \in \Omega_s$ $(k = 1, 2, \dots, K)$ and $\{p_1, p_2, \dots, p_K\}$ is a set of mutually orthogonal projections in $M_l(A)$. Since $d_H(X, \Omega_s) \le \varepsilon/4$, for each ξ_k, we may choose $x(\xi_k) \in X$ such that

$$\|\xi_k - x(\xi_k)\|_2 = \text{dist}(\xi_k, x(\xi_k)) \le \varepsilon/3, \ 1 \le k \le K. \tag{e4.163}$$

Define $\sigma : C(X) \to M_l(A)$ by

$$\sigma(f) = \sum_{k=1}^K f(x(\xi_k))p_k \text{ for all } f \in C(X). \tag{e4.164}$$

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Note that

$$\|\sigma(e_j|_X) - \sigma'(e_j|_{\Omega_s})\| \le \varepsilon/3,\ \ 1 \le j \le n. \tag{e4.165}$$

Hence, by (e 4.162), for $1 \le j \le n$,

$$\|u^*\mathrm{diag}(L_1(e_j|_X), \sigma(e_j|_X))u - \mathrm{diag}(L_2(e_j|_X), \sigma(e_j|_X))\| \le \varepsilon/2 + \varepsilon/3 < \varepsilon. \tag{e4.166}$$

■

Theorem 4.6. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $R \in \mathbb{R}_+\backslash\{0\}$, $r, b : \mathbb{N} \to \mathbb{N}$ and $T : \mathbb{N}^2 \to \mathbb{N}$ be maps and $s > 0$. There exists $\delta(n, \varepsilon) > 0$ satisfying the following:*

Suppose that A is a unital C^*-algebra such that:

1. A satisfies condition (R, r, b, T, s);

2. $K_i(A \otimes C(Z)) = 0$ for any compact metric space Z ($i = 0, 1$);

3. $J \subset A$ is an essential ideal which has real rank zero such that A/J is purely infinite simple;

4. for any $l \in \mathbb{N}$ and any nonzero projection $e \in M_l(A)\backslash M_l(J)$, we have $e \sim 1$.

Suppose further that $T_1, T_2, \dots, T_n \in A_{s.a.}$ with $\|T_i\| \le 1$ ($1 \le i \le n$) satisfy:

$$\|T_iT_j - T_jT_i\| < \delta,\ \ 1 \le i, j \le n \ \text{ and } \ d_H(X, Y) < \varepsilon/8 \tag{e4.167}$$

(recall Definition 2.15), where $X = s\mathrm{Sp}^{\varepsilon/8}(T_1, T_2, \dots, T_n)$, $Y = s\mathrm{Sp}^{\varepsilon/8}((\pi(T_1), \pi(T_2), \dots, \pi(T_n))$, and where $\pi : A \to A/J$ is the quotient map.

Then there are $S_1, S_2, \dots, S_n \in A_{s.a.}$ such that

$$S_iS_j = S_jS_i,\ \ 1 \le i, j \le n \ \text{ and } \ \|T_j - S_j\| < \varepsilon,\ \ 1 \le j \le n. \tag{e4.168}$$

Proof. Fix $n \in \mathbb{N}$. Let $\varepsilon > 0$. We will apply Theorem 4.5. Let $\delta_1 > 0$ (in place of δ) be given by Theorem 4.5 for $\varepsilon/16$.

Put $\varepsilon_0 = \min\{\varepsilon/128, \delta_1/4\}$. Let $\delta_2' > 0$ (in place of δ) be given by Lemma 4.2 for ε_0 (in place of ε). Choose $\delta_2 = \min\{\delta_2', \varepsilon_0/2\}$. Put $\varepsilon_1 = \min\{\varepsilon_0/2, \delta_2/4\}$.

Let $\delta_3 > 0$ be given by Proposition 2.16 for ε_1 (in place of η). Choose $\varepsilon_2 = \min\{\varepsilon_1/2, \delta_3/2\}$. Let $\delta_4 > 0$ be given by Proposition 2.12 for ε_2 (in place of η) and let $\delta_5 > 0$ be given by Theorem 3.9 for ε_2.

Let $\delta := \min\{\delta_1/2, \delta_2/2, \delta_4, \delta_5/2, \varepsilon_2\}$.

Now suppose that $T_1, T_2, \dots, T_n \in A_{s.a.}$ with $\|T_i\| \le 1$ ($1 \le i \le n$) such that

$$\|T_iT_j - T_jT_i\| < \delta \ \text{ for all } \ 1 \le j \le n \ \text{ and } \ d_H(X, Y) < \varepsilon/8. \tag{e4.169}$$

By the choice of δ (and δ_4, δ_5) and, applying Proposition 2.12, we obtain non-empty sets $X = s\mathrm{Sp}^{\varepsilon/8}((T_1, T_2, \dots, T_n))$ and $Y = s\mathrm{Sp}^{\varepsilon/8}((\pi(T_1), \pi(T_2), \dots, \pi(T_n))$, and the n-tuple $(T_1, T_2, \dots, T_n)$ has an ε_2-near-spectrum X_1 and, by Theorem 3.9, $(\pi(T_1), \pi(T_2), \dots, \pi(T_n))$ has an ε_2-spectrum Y_1. Moreover (by the choice of δ_3), since $\varepsilon_2 = \min\{\varepsilon_1/2, \delta_3/2\}$, we have (by applying Proposition 2.16)

$$Y_1 \subset \overline{(X_1)_{\varepsilon_1}},\ \ X_1 \subset X \subset \overline{(X_1)_{2\varepsilon_1}},\ \text{ and } \ Y_1 \subset Y \subset \overline{(Y_1)_{2\varepsilon_1}}. \tag{e4.170}$$

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By the choice of δ_2 (and $\varepsilon_2 < \delta_2'$), applying Lemma 4.2, we obtain mutually orthogonal nonzero projections $p_1, p_2, \dots, p_m \in A\backslash J$, a δ_2-dense subset $\{\lambda_1, \lambda_2, \dots, \lambda_m\} \subset Y_1$, and a c.p.c. map $L : C(X_1) \to (1-p)A(1-p)$ such that (with $p = \sum_{i=1}^m p_i$)

$$\|\sum_{i=1}^m e_j(\lambda_i)p_i + L(e_j|_{X_1}) - T_j\| < \varepsilon_0 < \delta_1 \text{ and} \tag{e4.171}$$

$$\|L(e_ie_j|_X) - L(e_i|_X)L(e_j|_X)\| < \varepsilon_0, \tag{e4.172}$$

$j = 1, 2, \dots, n$, and $1 - p \notin J$. Put

$$\Phi_1(f) = \sum_{i=1}^m f(\lambda_i)p_i + L(f|_{X_1}) \text{ for all } f \in C(\mathbb{I}^n).$$

Note that, since $(1-p) \sim 1$, $(1-p)A(1-p) \cong A$. Choose $\xi_0 \in X_1$ and define $\Phi_0 : C(X_1) \to (1-p)A(1-p)$ by $\Phi_0(f) = f(\xi_0)(1-p)$.

By the choice of δ_1 and (e 4.172), as $(1-p)A(1-p) \cong A$, and applying Theorem 4.5, we obtain a unital homomorphism $\sigma : C(X_1) \to M_l((1-p)A(1-p))$ (for some integer $l \in \mathbb{N}$) with finite-dimensional range and a unitary $u_1 \in M_{l+1}((1-p)A(1-p))$ such that

$$\|u_1^*\mathrm{diag}(\Phi_0(e_j|_{X_1}), \sigma(e_j|_{X_1}))u_1 - \mathrm{diag}(L(e_j|_{X_1}), \sigma(e_j|_{X_1}))\| < \varepsilon/16, \ 1 \le j \le n. \tag{e4.173}$$

Since σ has a finite-dimensional range, there are mutually orthogonal projections $q_1, q_2, \dots, q_L \in M_l((1-p)A(1-p))$ (for some integer $L \ge 1$) such that

$$\sigma(e_j|_{X_1}) = \sum_{k=1}^L e_j(\xi_k)q_k, \ 1 \le j \le n, \tag{e4.174}$$

where $\xi_k \in X_1, k = 1, 2, \dots, L$. Put $q = \sum_{i=1}^L q_i$. Note that, without loss of generality, by replacing l by $l+1$ and adding a projection $q_i'' \in M_l((1-p)A(1-p))\backslash M_l((1-p)J(1-p))$ to each q_i, if necessarily, we may assume that $q_i \notin M_l((1-p)A(1-p))\backslash M_l((1-p)J(1-p))$, $i = 1, 2, \dots, L$.

Define $\Phi_2 : C(\mathbb{I}^n) \to M_{l+1}((1-p)A(1-p))$ by

$$\Phi_2(f) = u_1^*\mathrm{diag}(\Phi_0(f|_{X_1}), \sigma(f|_{X_1}))u_1 \text{ for all } f \in C(\mathbb{I}^n). \tag{e4.175}$$

Let $\{e_{i,j}\}_{1\le i,j\le l+1}$ be a chosen matrix unit for $M_{l+1}(A)$. In what follows, we identify $1-p$ with $(1-p)\otimes e_{1,1}$ and $M_l(A)$ above with $EM_{l+1}(A)E$, where $E = \sum_{i=2}^{l+1} 1 \otimes e_{i,i}$. In particular, we view $q \in EM_{l+1}(A)E$.

Recall that $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ is δ_2-dense in Y_1. By the assumption that $d_H(X, Y) < \varepsilon/8$ and (e 4.170), we conclude that $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ is $\delta_2 + 2\varepsilon_1 + \varepsilon/8$-dense in X_1. Note that

$$\delta_2 + 2\varepsilon_1 + \varepsilon/8 < \varepsilon/256 + \varepsilon/128 + \varepsilon/8 = 35\varepsilon/256. \tag{e4.176}$$

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Hence, $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is $35\varepsilon/256$-dense in X_1. Let

$$F_1 = \{\xi_1, \xi_2, \ldots, \xi_L\} \cap B(\lambda_1, 35\varepsilon/256), \tag{e4.177}$$

$$F_2 = (\{\xi_1, \xi_2, \ldots, \xi_L\} \backslash F_1) \cap B(\lambda_2, 35\varepsilon/256). \tag{e4.178}$$

By induction, since $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is $35\varepsilon/256$-dense in X_1, we obtain mutually disjoint subsets $F_1, F_2, \ldots, F_{m'}$ such that $\sqcup_{i=1}^{m'} F_i = \{\xi_1, \xi_2, \ldots, \xi_L\}$, and $F_j \subset B(\lambda_j, 35\varepsilon/256)$, $j = 1, 2, \ldots, m'$, where $m' \le m$.

Put $q_k' = \sum_{\xi_j \in S_k} q_j$, $k = 1, 2, \ldots, m'$. Define

$$\sigma_1(e_j|_X) = \sum_{k=1}^{m'} e_j(\lambda_k) q_k'. \tag{e4.179}$$

Then

$$\|\sigma_1(e_j|_X) - \sigma(e_j|_X)\| < 35\varepsilon/256, \ \ 1 \le j \le n. \tag{e4.180}$$

Recall that, $e \sim 1$ for any nonzero projection $e \in M_{l+1}(A) \backslash M_{l+1}(J)$. Thus, in $M_{l+1}(A)$, viewing $q \in EM_{l+1}(A)E$, we have a partial isometry w such that $qwp = w$,

$$w^* q_k' w = p_k, \ \ k = 1, 2, \ldots, m' \ \text{ and } \ w_1^* \sigma_1(e_j|_X) w_1 = \sum_{k=1}^{m'} e_j(\lambda_k) p_k, \ \ 1 \le j \le n.$$

Then put $w_1 = (1 - p) + w$. Then

$$w_1^* w_1 = (1 - p) + \sum_{k=1}^{m'} p_k \ \ (\le (1 - p) + p \le 1 \otimes e_{1,1}). \tag{e4.181}$$

It follows that

$$\Phi_1(e_j|_X) = w_1^* \mathrm{diag}(L(e_j|_X), \sigma_1(e_j|_X)) w_1 \oplus \sum_{k=m'}^{m} e_j(\lambda_k) p_k \tag{e4.182}$$

$$= L(e_j|_X) + \sum_{k=1}^{m} e_j(\lambda_k) p_k, \ \ 1 \le j \le n. \tag{e4.183}$$

Put

$$\psi_0(f) = \sum_{k=m'}^{m} f(\lambda_k) p_k \ \text{ and } \ \Phi_3(f) = w_1^* \mathrm{diag}(L(f|_{X_1}), \sigma(f|_{X_1})) w_1 \tag{e4.184}$$

for all $f \in C(\mathbb{I}^n)$. Then, by (e 4.173), (e 4.175), (e 4.180), (e 4.183), and (e 4.171), for $1 \le j \le n$,

$$w_1^* \Phi_2(e_j) w_1 \oplus \psi_0(e_j) \approx_{\varepsilon/16} \Phi_3(e_j) \oplus \psi_0(e_j) \approx_{35\varepsilon/256} \Phi_1(e_j|_{X_1}) \approx_{\varepsilon_0} T_j. \tag{e4.185}$$

Put

$$S_j = w_1^* \Phi_2(e_j) w_1 \oplus \psi_0(e_j), \ \ 1 \le j \le n. \tag{e4.186}$$

Then, since $\varepsilon/16 + 35\varepsilon/256 + \varepsilon_0 < \varepsilon$, by (e 4.185),

$$\|S_j - T_j\| < \varepsilon, \ \ 1 \le j \le n. \tag{e4.187}$$

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Since Φ_0 is a homomorphism so is Φ_2. Hence,

(e4.188) $$S_iS_j = S_jS_i,\ \ 1 \le i, j \le n.$$

■

Proof of Theorem 1.1:

Proof. Let $A = B(H)$ and $J = \mathcal{K}$. By [5], $K_i(B(H) \otimes C(Y)) = 0$ for all compact metric space Y, $i = 0, 1$. By Proposition 4.3, we may apply Theorem 4.6. ■

4.7. Proof of Theorem 1.2

Proof. Let B be a σ-unital purely infinite simple C^*-algebra and H be a countably generated Hilbert B-module. Let $H_B = \{\{b_n\} : \sum_{n=1}^{\infty} b_n^* b_n \text{ converges in norm}\}$. Then, by Kasparov's absorbing theorem (Theorem 2 of [17]), H is an orthogonal summand of H_B and hence $K(H)$ is isomorphic to a hereditary C^*-subalgebra of $B \otimes \mathcal{K}$ which is purely infinite simple. If $K(H)$ is unital, by Theorem 1 of [17], $L(H) = K(H)$. Then the conclusion of Theorem 1.2 follows from Theorem 3.9 (without referring to synthetic spectra).

Otherwise, since H is countably generated, $C := K(H)$ is non-unital but σ-unital (see Proposition 3.2 of [2]). By Theorem 1 of [17], $L(H) \cong M(K(H)) = M(C)$. Note that $B \cong B \otimes \mathcal{K}$ (see Remark 2.5 (c) of [29]) which has real rank zero by [35]. It follows from Theorem 3.3 of [36] (see also Remark 2.5 (b) of [29]) that $M(B)/B$ is purely infinite simple, and every projection in $M(B)\backslash B$ is equivalent to $1_{M(B)}$. Thus, by Proposition 4.3, Theorem 4.6 applies. ■

5 A pair of almost commuting self-adjoint operators on Hilbert modules

Definition 5.1 (Definition 3.1 of [9]) For a unital C^*-algebra A, denote by $R(A)$ the set of elements $x \in A$ with the property that for no ideal I of A is $x + I$ one-sided and not two-sided invertible in A/I. Recall that A is said to have property (IR) if all elements in $R(A)$ belong to the norm closure of $GL(A)$, the group of invertible elements.

A non-unital C^*-algebra A has property (IR), if the C^*-algebra obtained by adjoining a unit to A has property (IR).

It follows from Lemma 4 of [23] that, if A has property (IR), then for any nonzero projection $p \in A$, pAp also has (IR).

Every C^*-algebra of stable rank one has property (IR), and every purely infinite simple C^*-algebra has property (IR).

Lemma 5.2. *Let $\varepsilon > 0$. There exists $\delta(\varepsilon) > 0$ satisfying the following: Let B be a unital purely infinite simple C^*-algebra, $a_1, a_2 \in B$ be self-adjoint elements such that $\|a_i\| \le 1$, $i = 1, 2$, and*

(e5.189) $$\|a_1a_2 - a_1a_2\| < \delta.$$

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Then there exists a pair of commuting self-adjoint elements $s_1, s_2 \in B$ such that

$$\|s_i - a_i\| < \varepsilon \text{ and } \kappa_1(\lambda - (s_1 + is_2)) = \kappa_1(\lambda - (a_1 + ia_2)) \tag{e5.190}$$

(recall Definition 2.4), whenever $\mathrm{dist}(\lambda, \mathrm{sp}(s_1 + is_2)) \geq \varepsilon$.

(Note that the identity in (e 5.190) also implies that $\lambda \notin \mathrm{sp}(a_1 + ia_2)$.)

Proof. It follows from Theorem 3.9 that exists $\delta > 0$ such that, whenever

$$\|a_1a_2 - a_2a_1\| < \delta, \tag{e5.191}$$

there exists a compact subset $X \subset \mathbb{I}^2$ and a unital injective homomorphism $\varphi : C(X) \to B$ such that

$$\|\varphi(e_i|_X) - a_i\| < \varepsilon/2, \quad i = 1, 2. \tag{e5.192}$$

Put $s_i = \varphi(e_i|_X)$, $i = 1, 2$, and $b = s_1 + is_2$. Then b is a normal element in B and $\mathrm{sp}(b) = X$. Moreover,

$$\|b - (a_1 + ia_2)\| < \varepsilon. \tag{e5.193}$$

Suppose that $\lambda \in \mathbb{C}$ such that $\mathrm{dist}(\lambda, X) \geq \varepsilon$. Then

$$\|(\lambda - b)^{-1}\| = \frac{1}{\mathrm{dist}(\lambda, X)} \leq 1/\varepsilon. \tag{e5.194}$$

Hence (also by (e 5.193)),

$$\|1 - (\lambda - b)^{-1}(\lambda - (a_1 + ia_2))\| < 1. \tag{e5.195}$$

It follows that $(\lambda - b)^{-1}(\lambda - (a_1 + ia_2)) \in GL_0(B)$. Hence, $(\lambda - b)$ and $(\lambda - (a_1 + ia_2))$ are in the same path connected component of $GL(B)$. It follows that $\kappa_1(\lambda - b) = \kappa_1(\lambda - (a_1 + ia_2))$. ∎

Theorem 5.3. *Let $\varepsilon > 0$. There exists $\delta(\varepsilon) > 0$ satisfying the following: Suppose that A is a unital C^*-algebra with an essential ideal J such that:*

1. *J is σ-unital, has real rank zero, and satisfies property (IR);*
2. *the quotient A/J is purely infinite simple.*

Let $T_1, T_2 \in A_{s.a.}$ with $\|T_i\| \leq 1$ $(i = 1, 2)$ such that:

$$\|T_1T_2 - T_2T_1\| < \delta \text{ and } \kappa_1(\lambda - (\pi(T_1 + iT_2))) = 0 \tag{e5.196}$$

for all $\lambda \notin s\mathrm{Sp}^{\delta}(\pi(T_1 + iT_2))$, where $\pi : A \to A/J$ is the quotient map.

Then, if either:

- *A has real rank zero, or*
- *$s\mathrm{Sp}^{\delta}(\pi(T_1 + iT_2))$ is path connected,*

there are $S_1, S_2 \in A_{s.a}$ such that

$$S_1S_2 = S_2S_1 \text{ and } \|S_i - T_i\| < \varepsilon, \quad i = 1, 2. \tag{e5.197}$$

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Proof. Fix $0 < \varepsilon < 1$. Let $\delta_1(\varepsilon/32) > 0$ be given by Theorem 4.4 of [9] for $\varepsilon/32$ in place of ε.

Choose $\varepsilon_1 = \min\{\varepsilon/64, \delta_1/64\}$. Let $\delta_2(\varepsilon_1) > 0$ be given by Proposition 2.16 for ε_1 (in place of η) and $k = 2$. Choose $\varepsilon_2 = \min\{\varepsilon_1/32, \delta_2/32\}$.

Let $\delta_3 := \delta(\varepsilon_2/2) > 0$ be given by Theorem 5.2 for $\varepsilon_2/2$ (in place of ε). Put $\varepsilon_3 := \min\{\delta_3, \varepsilon_2/4\} > 0$.

Let $\delta_4 > 0$ be given by Proposition 2.17 for ε_1 (in place of η).

Choose $\delta = \min\{\delta_4, \delta_3, \varepsilon_2/4, \delta_2/2, \varepsilon_1/4\} > 0$.

Now assume that $T_1, T_2 \in A_{s.a.}$ satisfy the assumption for δ. By Proposition 2.17,

$$Z := s\mathrm{Sp}^{\varepsilon_1}((\pi(T_1), \pi(T_2)) \neq \varnothing.$$

Recall that A/J is a purely infinite simple C^*-algebra. Therefore, by the choice of δ and applying Theorem 5.2, we obtain a compact subset $X \subset \mathbb{I}^2$ and a unital injective homomorphism $\varphi : C(X) \to A/J$ such that

$$\|\bar{s}_j - \pi(T_j)\| < \varepsilon_2/2, \quad j = 1, 2, \quad \text{and} \quad \kappa_1(\lambda - t) = \kappa_1(\lambda - \pi(T_1 + iT_2)) \tag{e5.198}$$

for those $\lambda \in \mathbb{I}^2$ for which $\mathrm{dist}(\lambda, X) \geq \varepsilon_2/2$, where $\bar{s}_j = \varphi(e_j|_X)$, $j = 1, 2$, and $t = \bar{s}_1 + i\bar{s}_2$. In particular, the pair $(\pi(T_1), \pi(T_2))$ has an $\varepsilon_2/2$-spectrum X. We may view it as an $nSp^{\delta_2}(\pi(T_1), \pi(T_2))$. On the other hand, by the choice of δ_2 and applying Proposition 2.16, we have

$$X \subset Z \subset X_{2\varepsilon_1}. \tag{e5.199}$$

We claim: Suppose that $\Omega \subset \mathbb{C}\backslash X$ is a bounded path connected component and that there is $\xi \in \Omega$ such that $\mathrm{dist}(\xi, X) \geq 2\varepsilon_1$. Then, for any $\lambda \in \Omega$,

$$\kappa_1(\lambda - (\bar{s}_1 + i\bar{s}_2)) = 0. \tag{e5.200}$$

To see the claim, we note that, by (e 5.199), $\xi \notin Z$. Then, by (e 5.198) and by the assumption (e 5.196),

$$\kappa_1(\xi - (\bar{s}_1 + i\bar{s}_2)) = \kappa_1(\xi - (\pi_c(T_1) + i\pi_c(T_2))) = 0. \tag{e5.201}$$

It follows that, for any $\lambda \in \Omega$,

$$\kappa_1(\lambda - (\bar{s}_1 + i\bar{s}_2)) = \kappa_1(\xi - (\bar{s}_1 + i\bar{s}_2)) = 0 \tag{e5.202}$$

and the claim follows.

Choose $k \in \mathbb{N}$ such that $2\sqrt{2}/k < \varepsilon_2/16$.

Note that, if Z is path connected, then $X_{2\varepsilon_1} = \cup_{x\in X} B(x, 2\varepsilon_1)$ is also path connected.[1] Let $\overline{X_{2\varepsilon_1}} = \{z \in \mathbb{I}^2 : \mathrm{dist}(z, X) \leq 2\varepsilon_1\}$. Choose $Y := \mathrm{B}^k_{\overline{X_{2\varepsilon_1}}}$ the $1/k$-brick cover of $\overline{X}_{2\varepsilon_1}$ (see Definition 3.3). Note that if $\overline{X}_{2\varepsilon_1}$ is path connected then Y is also connected.

Put $\varepsilon_4 = 2\varepsilon_1 + 2\sqrt{2}/k < 2\varepsilon_1 + \varepsilon_1/128$. Then $X_{\varepsilon_1} \subset Y \subset X_{\varepsilon_4}$.

[1] If $y_i \in B(x_i, 2\varepsilon_1)$, for some $x_i \in X \subset Z$, $i = 1, 2$, then there is a path in $Z \subset X_{2\varepsilon_1}$ connecting x_1 to x_2, and there is path connecting y_i to x_i in $B(x_i, 2\varepsilon_1)$, $i = 1, 2$.

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By Corollary 3.7, there is a unital injective homomorphism $\psi : C(Y) \to A/J$ such that

$$\|\psi(e_j|_Y) - \bar{s}_j\| < \varepsilon_4, \quad j = 1, 2, \quad \text{and} \quad \psi_{*1} = \varphi_{*1} \circ \pi^X_{*1}, \tag{e5.203}$$

where $\pi^X : C(Y) \to C(X)$ is the quotient map by restriction. Put $s_j = \psi(e_j|_Y)$, $j = 1, 2$, and $r = s_1 + is_2$. Then $\mathrm{sp}(r) = Y$.

Denote by Ω_∞ the unbounded path connected component of $\mathbb{C}\backslash X$. Let $\Omega' \subset \mathbb{C}\backslash Y$ be a bounded path component. Then there are two cases:

(i) $\Omega' \subset \Omega_\infty$;

(ii) if $z \in \Omega'$, then $\mathrm{dist}(z, X) \geq 2\varepsilon_2$.

To see this, let us assume it is not case (i). If $z \in \Omega'$, then z is in a bounded path connected open component $\Omega' \subset \mathbb{C}\backslash X$. But since $X_{2\varepsilon_2} \subset X_{\varepsilon_1} \subset Y$, one must have $z \notin X_{2\varepsilon_2}$. Hence, $\mathrm{dist}(z, X) \geq 2\varepsilon_2$.

Next, we will show that $\psi_{*1} = 0$. To see this, let $\lambda \in \Omega' \subset \mathbb{C}\backslash Y$, where Ω' is a bounded path connected component of $\mathbb{C}\backslash Y$. Put

$$u = (\lambda - (e_1 + ie_2)|_Y)\big([(\lambda - (e_1 + ie_2)|_Y)^*(\lambda - (e_1 + ie_2)|_Y)]^{-1/2}\big) \in C(Y). \tag{e5.204}$$

If $\Omega' \subset \Omega_\infty$ (i.e., it is case (i)), then, for $\lambda \in \Omega'$,

$$\pi^X_{*1}([u]) = [u|_X] = 0 \ \text{ in } \ K_1(C(X)). \tag{e5.205}$$

Hence, by the second part of (e 5.203),

$$\psi_{*1}([u]) = \varphi_{*1} \circ \pi^X_{*1}([u]) = 0. \tag{e5.206}$$

If $\xi \in \Omega'$ and $\mathrm{dist}(\lambda, X) \geq 2\varepsilon_2$ (i.e., in case (ii)), by the second part of (e 5.203) and the claim above, one computes that

$$\psi_{*1}([u]) = \varphi_{*1}([u|_X]) = \kappa_1(\lambda - (\bar{s}_1 + i\bar{s}_2)) = 0. \tag{e5.207}$$

Hence,

$$\psi_{*1} = 0. \tag{e5.208}$$

Since $Y \subset \mathbb{C}$ and $K_0(C(Y))$ is finitely generated, $K_0(C(Y))$ is free. Therefore, $[\psi^w] = 0$ in $KK(C(Y), A/J)$. Then, by Theorem 3.5, there is a normal element $r_0 = t_1 + it_2 \in A/J$ with $t_1, t_2 \in (A/J)_{s.a.}$ which has finitely many points in the spectrum contained in Y such that

$$\|r - r_0\| < \varepsilon_2/16. \tag{e5.209}$$

Then, by (e 5.198), (e 5.203), and (e 5.209),

$$\|\pi(T_1 + iT_2) - r_0\| \leq \|\pi(T_1 + iT_2) - (\bar{s}_1 + i\bar{s}_2)\| + \|(\bar{s}_1 + i\bar{s}_2) - r\| + \|r - r_0\| \tag{e5.210}$$

$$< \varepsilon_2/2 + \varepsilon_4 + \varepsilon_2/16 \leq \varepsilon_2/2 + 2\varepsilon_1 + \varepsilon_1/128 + \varepsilon_2/16 < 3\varepsilon_1. \tag{e5.211}$$

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We may write $r_0 = \sum_{l=1}^m \lambda_i p_i$, where $\lambda_i \in Y$ and $p_1, p_2, \dots, p_m$ are mutually orthogonal nonzero projections in A/J and $\sum_{k=1}^m p_k = 1_{A/J}$. There are two cases: (1) A has real rank zero, then, by Theorem 3.14 of [3], every projection in A/J lifts and (2) if Z is path connected, by the "Moreover" part of Theorem 3.5, we may arrange such that $[p_k] = 0$ or $[p_k] = [1_{A/J}]$ in $K_0(A/J)$. Since A is unital, $[p_k] \in \pi_{*0}(K_0(A))$. Thus, by Lemma 4.1, in both cases, there are mutually orthogonal projections $P_1, P_2, \dots, P_m \in A$ such that $\pi(P_j) = p_j$, $j = 1, 2, \dots, m$. There exists $H \in \mathcal{K}$ such that

$$\|(T_1 + iT_2) + H - \sum_{j=1}^m \lambda_j P_j\| < \varepsilon_1. \tag{e5.212}$$

Then, there are projections $Q_j \in P_j \mathcal{K} P_j$, $j = 1, 2, \dots, m$, such that, with $Q = \sum_{j=1}^m Q_j$,

$$\|QH - HQ\| < \varepsilon_1/16 \text{ and } \|(1-Q)H\| < \varepsilon_1/16. \tag{e5.213}$$

Note that

$$Q(\sum_{j=1}^m \lambda_j P_j) = \sum_{j=1}^m \lambda_j Q_j = (\sum_{j=1}^m \lambda_j P_j)Q. \tag{e5.214}$$

Moreover, by the second part of (e 5.213) and (e 5.212), and then, by the first part of (e 5.213), (e 5.212), and (e 5.214), we obtain

$$\|(1-Q)(T_1 + iT_2)(1-Q) - \sum_{j=1}^m \lambda_j (P_j - Q_j)\| < \varepsilon_1/16 + \varepsilon_1 = 17\varepsilon_1/16 \tag{e5.215}$$

(e5.216)
$$\text{and } Q(T_1 + iT_2) - (T_1 + iT_2)Q \approx_{\varepsilon_1/16} Q(T_1 + iT_2 + H) - (T_1 + iT_2 + H)Q$$

$$\approx_{2\varepsilon_1} Q(\sum_{j=1}^m \lambda_j P_j) - (\sum_{j=1}^m \lambda_j P_j)Q = 0. \tag{e5.217}$$

Hence,

(e5.218)
$$\|(T_1 + iT_2) - ((1-Q)(T_1 + iT_2)(1-Q) + Q(T_1 + iT_2)Q)\| < 2(2 + 1/6)\varepsilon_1.$$

It follows from (e 5.216) and (e 5.217), and by the assumption on T_1 and T_2 that

$$Q(T_1 + iT_2)QQ(T_1 - iT_2)Q \approx_{(2+1/6)\varepsilon_1} Q(T_1 + iT_2)(T_1 - iT_2)Q \tag{e5.219}$$

$$\approx_{2\delta} Q(T_1 - iT_2)(T_1 + iT_2)Q. \tag{e5.220}$$

Note that $(2 + 1/16)\varepsilon_1 + 2\delta < 3\varepsilon_1 + \varepsilon_2 < \delta_1$. Note also that QJQ has property (IR). By the choice of δ_1 and applying Theorem 4.4 of [9], there is a normal element $L \in QJQ$ such that

$$\|Q(T_1 + iT_2)Q - L\| < \varepsilon/16. \tag{e5.221}$$

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Put $N = \sum_{j=1}^{m} \lambda_j(P_j - Q_j) + L$. Then N is normal and, by (e 5.218) and (e 5.215),

$$\|(T_1 + iT_2) - N\| < 2(2 + 1/6)\varepsilon_1 + 17\varepsilon_1/16 + \varepsilon/16 < \varepsilon/8. \tag{e5.222}$$

Put $S_1 = (1/2)(N + N^*)$ and $S_2 = (1/2i)(N - N^*)$. Then, since N is normal, $S_1S_2 = S_2S_1$. Moreover,

$$\|S_j - T_j\| < \varepsilon, \quad j = 1, 2. \tag{e5.223}$$ ∎

The proof of Theorem 1.3:

Proof. For (i), choose $A = B(H)$ and $J = \mathcal{K}$. Note that $\mathcal{K}$ has real rank zero and stable rank one and has property (IR). So Theorem 5.3 applies.

For (ii), let $A = L(H)$ and $J = K(H)$. If $K(H)$ is unital, then $L(H) = K(H)$. As in 4.7, this case follows from Theorem 3.9 (regardless of Fredholm index).

Also in the proof of Theorem 1.2 (see 4.7), when $K(H)$ is not unital, it is a σ-unital purely infinite simple C^*-algebra which has real rank zero and has property (IR). The fact that A/J is purely infinite is also discussed in 4.7 (the proof of Theorem 1.2). Thus, Theorem 5.3 applies.

For (iii), let $A = L(H)$ and $J = K(H)$. As in 4.7 (the proof of Theorem 1.2), by Kasparov's theorems, $L(H) = M(K(H))$ and $K(H)$ is a σ-unital hereditary C^*-subalgebra of a σ-unital simple C^*-algebra of real rank zero and stable rank one. Hence, $J = K(H)$ is also a simple C^*-algebra of real rank zero and stable rank one, whence J has property (IR). If $K(H)$ is unital, then $J = K(H) = L(H)$. Therefore, the conclusion follows from Theorem 4.4 of [9]. Otherwise, by the assumption, $M(J)/J$ is simple. It follows that J has continuous scale (see [26]). Hence, by Corollary 3.3 of [26], A/J is a purely infinite simple C^*-algebra. Thus, Theorem 5.3 applies. ∎

Corollary 5.4. *For $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ satisfying the following: Suppose that H is an infinite-dimensional separable Hilbert space and $T_1, T_2 \in B(H)_{s.a.}$ with $\|T_i\| \le 1$ $(i = 1, 2)$ such that*

$$\|T_1T_2 - T_2T_1\| < \delta \text{ and } \mathrm{Ind}(\lambda - (T_1 + iT_2)) = 0 \tag{e5.224}$$

for all $\lambda \notin \overline{(\mathrm{sp}_{ess}(T_1 + iT_2))_\delta}$. Then there are $S_1, S_2 \in B(H)_{s.a.}$ such that

$$S_1S_2 = S_2S_1 \text{ and } \|S_i - T_i\| < \varepsilon, \quad i = 1, 2. \tag{e5.225}$$

Proof. Fix $\varepsilon > 0$. Let η be as in Theorem 1.3 for ε. We may assume that $\eta < \varepsilon$. Let $\delta := \delta(\delta_1/4)$ be in Proposition 2.17 for $\delta_1/4$ (in place of η). Then, if $T_1, T_2 \in B(H)_{s.a.}$ satisfy the assumption (e 5.224), then, by the proof of Proposition 2.17 and (4) of Remark 2.11,

$$\mathrm{sp}_{ess}(T_1 + iT_2) \subset n\mathrm{Sp}_{ess}^{\delta_1/4}((T_1, T_2)) \subset (n\mathrm{Sp}_{ess}^{\delta_1/4}(T_1, T_2))_{\delta_1/4} \subset n\mathrm{Sp}_{ess}^{\delta_1}((T_1, T_2)) \subset s\mathrm{Sp}^{\eta}(T_1, T_2). \tag{e5.226}$$

Then, for any $\lambda \notin s\mathrm{Sp}_{ess}^{\delta_1}((T_1, T_2))$, $\mathrm{Ind}(\lambda - (T_1 + iT_2)) = 0$. So Theorem 1.3 applies.

The next proposition provides a short way to see the second condition in (e 5.196) is necessary. We also provide a direct proof. ∎

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Proposition 5.5. *Let $1 > \varepsilon > 0$. There is $\delta > 0$ satisfying the following: Suppose that $T_1, T_2 \in B(H)_{s.a.}$ and there are $S_1, S_2 \in B(H)_{s.a.}$ such that*

$$\|T_j - S_j\| < \delta, \quad j = 1, 2, \quad \text{and} \quad S_1 S_2 = S_2 S_1. \tag{e5.227}$$

Then, for any $\lambda \notin s\mathrm{Sp}^{\varepsilon}_{ess}(T_1 + iT_2)$,

$$\mathrm{Ind}(\lambda - (T_1 + iT_2)) = 0. \tag{e5.228}$$

Note that (e 5.228) also implies that $\lambda - (T_1 + iT_2)$ is invertible.

Proof. Let $\delta_1 := \delta(2, \varepsilon/4)$ be given by Proposition 2.16 for $\varepsilon/4$ (in place of η). Choose $\delta > 0$. Suppose that T_j and S_j are in the statement of the theorem, $j = 1, 2$. Put $L = T_1 + iT_2$ and $N = S_1 + iS_2$. Then

$$\|L - N\| < 2\delta. \tag{e5.229}$$

Since N is normal, by choosing sufficiently small δ, $X := \mathrm{sp}_{ess}(N)$ is an essential δ_1-near-spectrum of and L and $X_{\delta_1/2}$ is a δ_1-near spectrum of L. We may also assume that $\delta < \delta_1/16$. By Proposition 2.16, we have

$$X \subset X_{\delta_1/2} \subset Z, \tag{e5.230}$$

where $Z = sSp^{\varepsilon}((\pi_c(T_1), \pi_c(T_2)))$.

Since N is normal, $\mathrm{Ind}(\lambda - N) = 0$ for all $\lambda \notin X$. Note, for any $\lambda \in \mathbb{C}$,

$$\|(\lambda - \pi_c(L)) - (\lambda - \pi_c(N))\| < 2\delta < \delta_1/8. \tag{e5.231}$$

Now if $\lambda \notin sSp^{\varepsilon}_{ess}((T_1 + iT_2))$, then, by (e 5.230), $\mathrm{dist}(\lambda, \pi_C(N)) \geq \delta_1/2$. Since N is normal,

$$\|(\lambda - \pi_c(N))^{-1}\| = \frac{1}{\mathrm{dist}(\lambda, \pi_c(N))}. \tag{e5.232}$$

Hence, by (e 5.231),

$$\|(\lambda - \pi_c(L))(\lambda - \pi_c(N))^{-1} - 1\| < \frac{2\delta}{\delta_1/2} < \frac{1}{4} < 1. \tag{e5.233}$$

This implies that $(\lambda - \pi_c(L))(\lambda - \pi_c(N))^{-1}$ is invertible and

$$(\lambda - \pi_c(L))(\lambda - \pi_c(N))^{-1} \in GL_0(B(H)/\mathcal{K}), \text{ or } \kappa_1((\lambda - \pi_c(L))(\lambda - \pi_c(N))^{-1}) = 0.$$

Since $\mathrm{Ind}(\lambda - N) = 0$, we conclude that

$$\mathrm{Ind}((\lambda - \pi_c(L)) = 0.$$

■

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Remark 5.6. Let us compare the results in this section and Theorem 1.1 of [16].

While there is significant overlap between the results in this section and Theorem 1.1 of [16], the proofs are quite different. Theorem 1.1 of [16] provides a quantitative estimate for the distance from $T_1 + iT_2$ to the set of normal elements with finite spectrum (though the explicit computation of $d_1(T_1 + iT_2)$ may not be straightforward). In contrast, Theorem 5.3 here is a more conceptual result. In the special case of Theorem 1.3 (i), the condition $d_1(T_1 + iT_2) = 0$ (defined in [16]) is equivalent to $\mathrm{Ind}(\lambda - (T_1 + iT_2)) = 0$ for all $\lambda \notin \mathrm{sp}(T_1 + iT_2)$.

• *Key differences and limitations:*

However, when $K_1(A) \neq \{0\}$, or $K_1(J) \neq \{0\}$, a normal element $x \in A$ (as in Theorem 5.3) may not be approximable by normal elements with finite spectrum even if $\kappa_1(\lambda - \pi(x)) = 0$ for all $\lambda \notin \mathrm{sp}(\pi(x))$. In such cases, $d_1(x)$ (from Theorem 1.1 of [16]) can be large. For instance, if x is a unitary with $[x] \neq 0$ in $K_1(A)$, then $d_1(x) = 1$, rendering Theorem 1.1 of [16] inapplicable (see Example 5.7 for details).

Moreover, if a unital C^*-algebra A has property (IR), then by Theorem 4.4 of [9], any pair of almost commuting self-adjoint elements T_1 and T_2 is close to a commuting pair – regardless of the size of $d_1(T_1 + iT_2)$ (see also Example 5.7 below).

• *Computational aspects of δ-synthetic spectrum:*

From a computational standpoint, the δ-synthetic spectrum X is tractable: it consists of finitely many balls, and its complement has only finitely many bounded connected components. Since the index remains constant in each such component, it suffices to test one point per bounded components. This avoids the complication of infinitely many "holes" in the spectrum.

• *Further points:*

– The second condition in (e5.196) does not imply $d_1(T_1 + iT_2) = 0$.

– More importantly, it should be noted that Theorems 1.1 and 1.2 here primarily address cases where $n > 2$, which lie outside the scope of [16].

Example 5.7. Let B_1 be a unital separable purely infinite simple C^*-algebra with $K_0(B_1) = 0$ and $K_1(B_1) \neq \{0\}$ and J be a non-unital separable simple C^*-algebra with $K_1(J) \neq \{0\}$. We assume that either J is purely infinite simple, or J is a separable simple C^*-algebra of real rank zero and stable rank one. Consider any unital C^*-algebra A which is given by the following essential extension:

$$0 \to J \to A \to B_1 \to 0.$$

Let $\pi : A \to B_1$ be the quotient map. Note that $\pi(A) \cong A/J \cong B_1$ is purely infinite, J has property (IR). Since $K_0(B_1) = 0$, every projection in $B_1 = A/J$ lifts to a projection in A (see Lemma 4.1). It follows from Corollary 3.16 of [3] that A has real rank zero. Hence, Theorem 5.3 applies in this case. However, from the six-term exact sequence in K-theory, one computes that $K_1(A) \neq \{0\}$. Denote by $N(A)$ the set of normal elements of A and $N_f(A)$ the set of those elements in $N(A)$ which has finite spectrum, respectively. Then the closure $\overline{N_f(A)} \neq N(A)$.

Let $p \in J$ be a nonzero projection. Since pJp is either purely infinite or stable rank one, $K_1(pAp) = K_1(pJp) \cong K_1(J) \neq \{0\}$. Thus, there is a unitary $u_0 \in pJp$ with $[u_0] \neq \{0\}$. Let $x = (1/4)(1 - p) + u_0$. Then x is a normal element with

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$\pi(x) = (1/4)1_{A/J}$. Hence, $\kappa_1(\lambda - \pi(x)) = 0$ for any $\lambda \notin \mathrm{sp}(\pi(x)) = \{1/4\}$ (so Theorem 1.3 (iii) applies).

Then $\mathrm{sp}(x) = \mathbb{T} \cup \{1/4\}$. Choose $d = 1/32$. Put $X = \{z \in \mathbb{C} : \mathrm{dist}(z, \mathrm{sp}(x)) \le d\}$. Suppose that $y \in A$ with $\|x - y\| < d/2$. Then, for any $\lambda \notin X$,

$$\|1 - (\lambda - x)^{-1}(\lambda - y)\| < \frac{d/2}{d} = 1/2. \tag{e5.234}$$

It follows that $\lambda \notin \mathrm{sp}(y)$. In other words, $\mathrm{sp}(y) \subset X$.

Let D be the disk in $\mathbb{C}$ with center at 0 and radius $3/2$. Then $X \subset D$. Choose $g \in C([0,3/2])_+$ such that $g(r) = 1$, if $r \in [1-d, 3/2], 0 \le g(r) \le 1$, if $1/4 + d < r \le 1, g(r) = 0$ if $r \in [0, 1/4 + d]$. Define $f \in C(D)$ such that $f(re^{2\pi i\theta}) = g(r)e^{2\pi i\theta}$ ($\theta \in [0,1)$). Note that $f(X) \subset \mathbb{T} \cup \{0\}$ and $f(0) = 0$. Then, there exists $0 < \delta_1 < 1/2$ such that, for any pair of normal elements y_1, y_2 with $\|y_i\| \le 3/2$ and with $\|y_1 - y_2\| < \delta_1$, then

$$\|f(y_1) - f(y_2)\| < 1/16. \tag{e5.235}$$

Put $\delta = \min\{d, \delta_1\} > 0$.

Claim: $\mathrm{dist}(x, N_f(A)) \ge \delta$ but $\mathrm{dist}(x, N(A)) = 0$.

In fact, suppose otherwise there is $y \in N_f$ such that

$$\|x - y\| < \delta. \tag{e5.236}$$

Then, since $\delta \le d, \mathrm{sp}(y) \subset X$, and

$$\|f(x) - f(y)\| < 1/16. \tag{e5.237}$$

Hence, $f(x) = u_0$ and $\mathrm{sp}(f(y)) \subset \mathbb{T} \cup \{0\}$. Therefore, $f(y)$ is a normal partial isometry. Moreover,

$$|p - f(y)f(y)^*\| \le \|u_0u_0^* - f(y)u_0^*\| + \|f(y)u_0^* - f(y)f(y)^*\| < 1/8. \tag{e5.238}$$

There exists a unitary $w \in A$ such that

$$\|w - 1\| < 1/4 \text{ and } w^*f(y)f(y)^*w = p. \tag{e5.239}$$

Also

$$\begin{aligned}\|u_0 - w^*f(y)w\| &\le \|u_0 - f(y)\| + \|f(y) - w^*f(y)\| + \|w^*f(y) - w^*f(y)w\| \\ &< 1/16 + 1/4 + 1/4 < 1.\end{aligned}$$

Note that $w^*f(y)w$ is a unitary in pJp with finite spectrum, hence $w^*f(y)w \in U_0(pJp)$. This contradict the fact that $[u_0] \ne 0$ in pJp, which proves the claim.

In other words, Theorem 1.1 in [16] could not be applied to determine $\mathrm{dist}(s_1 + is_2, N(A))$, when s_1 and s_2 are almost commuting (see also Theorem in [23]).

6 Epilogue

In this final section, we examine the connection between the vanishing Fredholm index condition and the closeness of the (synthetic) spectrum to the essential

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(synthetic) spectrum (Corollaries 6.2 and 6.3). We also present an example (Proposition 6.4) where a higher-dimensional analog of Theorem 1.3 fails – highlighting the obstructions and subtleties behind this failure. Finally, we conclude with a remark on Mumford's original question regarding quantum theory and measurements, relating it to the second condition in Theorem 1.1.

Let us first state the following corollary of Theorem 4.6.

Corollary 6.1. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. There exists $\delta(n, \varepsilon) > 0$ satisfying the following: Suppose that $T_1, T_2, \dots, T_n \in B(H)_{s.a.}$ with $\|T_i\| \leq 1$ $(1 \leq i \leq 1)$ such that*

$$\|T_iT_j - T_jT_i\| < \delta, \ \ 1 \leq i, j \leq n \tag{e6.240}$$

and $Y = s\mathrm{Sp}^{\varepsilon/4}_{ess}((T_1, T_2, \dots, T_n)) \supset \mathbb{I}^n$. Then there are $S_1, S_2, \dots, S_n \in B(H)_{s.a.}$ such that

$$S_iS_j = S_jS_i, \ \ 1 \leq i, j \leq n \ \text{ and } \ \|T_j - S_j\| < \varepsilon, \ \ 1 \leq j \leq n. \tag{e6.241}$$

Proof. This follows from Theorem 4.6 and the fact that

$$s\mathrm{Sp}^{\varepsilon/4}_{ess}((T_1, T_2, \dots, T_n)) \subset s\mathrm{Sp}^{\varepsilon/4}((a_1, a_2, \dots, a_n)) \subset \mathbb{I}^n. \qquad \blacksquare$$

The next corollary shows that, when $n = 2$, the second condition in (e 4.167) is stronger than that of vanishing index in Theorem 1.3.

Corollary 6.2. *For any $\eta > 0$, there exists $\delta > 0$ satisfying the following: Let A and H be in all cases (i)–(iii) of Theorem 1.3. Suppose that $T_1, T_2 \in L(H)_{s.a.}$,*

$$\|T_1T_2 - T_2T_1\| < \delta \ \text{ and } \ d_H(s\mathrm{Sp}^{\eta}((T_1, T_2)), s\mathrm{Sp}^{\eta}((\pi(T_1), \pi(T_2)))) < \eta. \tag{e6.242}$$

Then, for any $\lambda \notin s\mathrm{Sp}^{4\eta}((\pi(T_1), \pi(T_2)))$,

$$\kappa_1(\lambda - (\pi(T_1 + iT_2))) = 0. \tag{e6.243}$$

Proof. Let $\delta_1 := \delta(\eta) > 0$ be given by Proposition 2.16 for η. Let $\delta_2 := \delta(\eta/4) > 0$ be given by Proposition 2.17 for $\delta_1/2$.

Then we choose $\delta_3 > 0$ such that the conclusion of Proposition 2.12 holds for $\delta_2/2$. Choose $\delta = \delta_3$. Suppose that (e6.242) holds. Put $L = T_1 + iT_2$. Thus (applying Proposition 2.12) $Z := s\mathrm{Sp}^{\delta_1}((T_1, T_2))$, $Z_e := s\mathrm{Sp}^{\delta_1}(\pi(T_1), \pi(T_2))$, $X := nSp^{\delta_2}((T_1, T_2))$, and $X_e := nSp^{\delta_2}(\pi(T_1), \pi(T_2))$ are all non-empty sets.

By applying the proof of Proposition 2.17, we have

$$\mathrm{sp}(T_1 + iT_2) \subset n\mathrm{Sp}^{\eta/4}((T_1, T_2)) \ \text{ and } \ \mathrm{sp}(\pi(T_1 + iT_2)) \subset n\mathrm{Sp}^{\eta/4}((\pi(T_1), \pi(T_2))). \tag{e6.244}$$

Applying Proposition 2.16, we obtain

$$\mathrm{sp}(T_1 + iT_2) \subset (n\mathrm{Sp}^{\delta_1/2}((T_1, T_2))_{\delta_1/2} \subset n\mathrm{Sp}^{\delta_1}((T_1, T_2)) \subset \mathrm{Sp}^{\eta}((T_1, T_2)). \tag{e6.245}$$

Since

$$d_H(s\mathrm{Sp}^{\eta}((T_1, T_2)), s\mathrm{Sp}^{\eta}(\pi(T_1), \pi(T_2))) < \eta, \tag{e6.246}$$

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one has

$$\mathrm{sp}(T_1 + iT_2) \subset (s\mathrm{Sp}^{\eta}(\pi(T_1), \pi(T_2)))_{\eta} \tag{e6.247}$$

So, if $\lambda \notin (s\mathrm{Sp}^{\eta}((\pi(T_1), \pi(T_2)))_{\eta}$, then $\lambda \notin \mathrm{sp}(T_1 + iT_2)$. By [5], in cases (i) and (ii), since A is stable, $K_1(L(H)) = K_1(M(A)) = \{0\}$. For case (iii), $K_1(L(H)) = K_1(M(A)) = 0$ follows from Lemma 3.3 of [20] (note that, proved in [19], if C has real rank zero, then $\mathrm{cer}(C) \le 1 + \varepsilon \le 2$). Therefore,

$$\kappa_1(\lambda + (T_1 + iT_2)) = 0. \tag{e6.248}$$

Hence,

$$\kappa_1(\lambda - \pi(T_1 + iT_2)) = 0 \text{ in } K_1(L(H)/K(H)). \tag{e6.249}$$

■

In general, we have the following statement, which follows immediately from Theorem 4.6. Following the same spirit as the proof of Corollary 6.2, one could alternatively give a direct proof without invoking Theorem 4.6. However, since this approach is already demonstrated in Corollary 6.2, we avoid introducing additional KK-theoretic technicalities – particularly to spare readers less familiar with KK-theory.

Corollary 6.3. *Let $n \in \mathbb{N}$. For any $\varepsilon > 0$, there exists $\delta(n, \varepsilon) > 0$ satisfying the following: Suppose that $T_1, T_2, \dots, T_n \in B(H)_{s.a.}$ with $\|T_i\| \le 1$ $(i = 1, 2)$ such that*

$$\|T_i T_j - T_j T_i\| < \delta, \ 1 \le i, j \le n, \text{ and } d_H(X, Y) < \varepsilon/8, \tag{e6.250}$$

where $X = s\mathrm{Sp}^{\varepsilon/8}((T_1, T_2, \dots, T_n))$ and $Y = s\mathrm{Sp}^{\varepsilon/8}_{ess}((T_1, T_2, \dots, T_n))$. Then there exists a $1/k$-brick combination $\Omega \subset \mathbb{I}^n$ and a unital monomorphism $\varphi : C(\Omega) \to B(H)/\mathcal{K}$ such that

$$\|\varphi(e_j|_{\Omega}) - \pi_c(T_j)\| < \varepsilon, \ 1 \le j \le n, \text{ and} \tag{e6.251}$$

$$[\varphi^{\omega}] = 0 \text{ in } KK(C(\Omega), B(H)/\mathcal{K}). \tag{e6.252}$$

Proof. Applying Theorem 4.6, one obtains commuting n-tuple of self-adjoint operators $S_1, S_2, \dots, S_n \in B(H)$. Let $C = C(\Omega)$ be the C^*-subalgebra of $B(H)$ generated by $1, S_1, S_2, \dots, S_n$. Define $\psi : C(\Omega) \to B(H)$ to be the embedding. Since $K_i(B(H) \otimes C(Y)) = 0$ for any compact metric space Y, $i = 0, 1$, we conclude that $KK([\psi]) = 0$. Consequently, $[\varphi^{\omega}] = 0$, where $\varphi = \pi_c \circ \psi$. ■

We now present a counterexample demonstrating that the n-dimensional analog of Theorem 1.3 (i) is false when $n \ge 3$. Note that, in the next proposition, the triple (T_1, T_2, T_3) has only a single point in the essential η-spectrum. Hence, the KK-theory obstruction vanishes in $B(H)/\mathcal{K}$. This example also underscores the significance of Theorems 1.1 and 1.2.

Proposition 6.4. *There exists $0 < \varepsilon_0 < 1$ and a sequence of triples of self-adjoint operators $T_{n,1}, T_{n,2}, T_{n,3} \in B(H)$ such that $\|T_{n,j}\| \le 1$ (for all n),*

$$\lim_{n\to\infty} \|T_{n,i} T_{n,j} - T_{n,j} T_{n,i}\| = 0, \ \pi_c(T_j) = 1_{B(H)/\mathcal{K}}, \ 1 \le i, j \le 3, \text{ and}$$

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$$\inf\{\max_{1\le j\le 3}\{\|T_{n,j}-S_{n,j}\| : S_{n,j}\in B(H)_{s.a.} \text{ and } S_{n,j}S_{n,i}=S_{n,i}S_{n,j},\ 1\le i,j\le 3\}\}\ge \varepsilon_0,$$

where infimum is taken from all possible commuting triples $\{S_{n,j}: 1\le j\le 3\}$ *of self-adjoint operators and all* $n\in\mathbb{N}$.

Proof. Let $\mathbb{D}^3$ be the three-dimensional unit ball in $\mathbb{I}^3$. By Theorem 4.1 of [10], there exists a sequence of contractive linear maps $\Lambda_n : C(\mathbb{D}^3)\to M_{n^3}$ and $a>0$ which satisfy the following:

$$\lim_{n\to\infty}\|\Lambda_n(fg)-\Lambda_n(f)\Lambda_n(g)\|=0 \text{ for all } f,g\in C(\mathbb{D}^3) \text{ and} \tag{e6.253}$$

$$\inf\{\sup\{\|\Lambda_n(e_j|_{\mathbb{D}^3})-\varphi_n(e_j|_{\mathbb{D}^3})\|: 0\le j\le 3\}\}\ge a, \tag{e6.254}$$

where the infimum is taken among all homomorphism s $\varphi_n : C(\mathbb{D}^3)\to M_{n^3}$.

Let $B_1=\prod_n M_{n^3}$ and $\Pi_1 : B_1\to B_1/\bigoplus_n M_{n^3}$ be the quotient map. Then $\psi=\Pi_1\circ\{\Lambda_n\} : C(\mathbb{D}^3)\to B_1/B_1/\bigoplus_n M_{n^3}$ is a homomorphism. Theorem 4.1 of [10] allows each Λ_n to be σ_n-injective (see Definition 0.3 in [10]) with $\sigma_n\to 0$. This feature implies that ψ is injective. Let $q=\psi(1)$. Then there exists a sequence of projections $p_n\in M_{n^3}$ such that $\Pi_1(\{p_n\})=q$. Put $D_n=p_nM_{n^3}p_n$, $n\in\mathbb{N}$. Note that

$$\lim_{n\to\infty}\|\Lambda_n(1)-p_n\|=0. \tag{e6.255}$$

Hence,

$$\lim_{n\to\infty}\|p_n\Lambda_n(1)p_n-p_n\|=0. \tag{e6.256}$$

There are $d_n\in D_n$ such that, for all sufficiently large n,

$$d_np_n\Lambda_n(1)p_nd_n=p_n \text{ and } \lim_{n\to\infty}\|d_n-p_n\|=0. \tag{e6.257}$$

Therefore, by replacing Λ_n with $d_np_n\Lambda_np_nd_n$, we may assume that Λ_n maps $C(\mathbb{D}^3)$ into D_n and each Λ_n is unital.

Choose $\lambda_0=(1,1,1)\in\mathbb{I}^3$. Identify M_{n^3} with a C^*-subalgebra of $\mathcal{K}\subset B(H)$ such that $1_{M_{n^3}}$ becomes a projection of rank n^3. In particular, $p_n\in B(H)$ as a finite rank ($\le n^3$) projection. Put $X=\mathbb{D}^3\sqcup\{(1,1,1)\}$. Define $\Phi_n : C(X)\to B(H)$ by

$$\Phi_n(f)=f((1,1,1))(1-p_n)+\Lambda_n(f|_{\mathbb{D}^3}) \text{ for all } f\in C(\mathbb{I}^3). \tag{e6.258}$$

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Choose (with $e_0 = 1$)

$$T_{n,j} = \Phi_n(e_j) = (1 - p_n) + \Lambda_n(e_j|_{\mathbb{D}^3}), \quad j = 0, 1, 2, 3. \tag{e6.259}$$

So $T_{n,j} \in B(H)_{s.a.}$ and $\|T_{n,j}\| \leq 1, 0 \leq j \leq 3$. Moreover,

$$\lim_{n\to\infty} \|T_{n,i}T_{n,j} - T_{n,j}T_{n,i}\| = 0, \quad 1 \leq i, j \leq 3. \tag{e6.260}$$

Furthermore,

$$\pi_c(T_{n,j}) = \pi_c(e_j((1,1,1))(1 - p_n)) = e_j((1,1,1))1_{B(H)/\mathcal{K}} = 1_{B(H)/\mathcal{K}}, \quad 1 \leq j \leq 3, \tag{e6.261}$$

where $\pi_c : B(H) \to B(H)/\mathcal{K}$ is the quotient map.

Let $B = \prod(\{B(H)\})$ and $\Pi : B \to B/\bigoplus(\{B(H)\})$ be the quotient map. Define $\tilde{\varphi} : C(X) \to B/\bigoplus(\{B(H)\})$ by $\tilde{\varphi}(f) = f((1,1,1))(1 - q) + \varphi(f|_{\mathbb{D}^3})$ for $f \in C(X)$.

If there is a sequence of commuting triples of self-adjoint operators $S_{n,j} \in B(H)$ such that

$$\lim_{n\to\infty} \|S_{n,j} - T_{n,j}\| = 0, \quad 1 \leq j \leq 3, \tag{e6.262}$$

then one obtains a sequence of unital homomorphism s $\psi_n : C(\mathbb{I}^3) \to B(H)$ such that

$$\lim_{n\to\infty} \|\psi_n(e_j) - T_{n,j}\| = 0, \quad 1 \leq j \leq 3. \tag{e6.263}$$

Hence,

$$\lim_{n\to\infty} \|\psi_n(f) - \Phi_n(f|_X)\| = 0 \text{ for all } f \in C(\mathbb{I}^3). \tag{e6.264}$$

Then $\Pi \circ \{\psi_n\} = \tilde{\varphi} \circ r$, where $r : C(\mathbb{I}^3) \to C(X)$ is defined by $r(f) = f|_X$ for $f \in C(\mathbb{I}^3)$.

Choose $f_0 \in C(\mathbb{I}^3)$ such that $0 \leq f_0 \leq 1, f_0(\xi) = 0$, if $\text{dist}(\xi, \mathbb{D}^3) \geq 1/32$ and $f_n(\xi) = 1$, if $\text{dist}(\xi, \mathbb{D}^3) < 1/64$. Note that $f_0((1,1,1)) = 0$ and

$$\Phi_n(f_0|_X) = p_n \text{ and } \Phi_n(f_0 e_j|_X) = \Lambda_n(e_j|_X), \ 1 \leq j \leq 3, \ n \in \mathbb{N}. \tag{e6.265}$$

Put $c_{n,j} = \psi_n(f_0 e_j), 1 \leq j \leq 3$ and $c_{n,0} = p_n, n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \|c_{n,j} - \Lambda_n(e_j|_{\mathbb{D}^3})\| = 0 \text{ for all } 1 \leq j \leq 3. \tag{e6.266}$$

Recall that

$$\sqrt{\|e_1^2|_{\mathbb{D}^3} + e_2^2|_{\mathbb{D}^3} + e_3^2|_{\mathbb{D}^3}\|} \leq 1. \tag{e6.267}$$

Put $\bar{c}_{n,j} = \frac{c_{n,j}}{\|\sum_{j=1}^3 c_{n,j}^2\|^{1/2}}, 1 \leq j \leq 3, \bar{c}_{n,0} = p_n$, and $n \in \mathbb{N}$. It follows that

$$\lim_{n\to\infty} \|\bar{c}_{n,j} - \Lambda_n(e_j|_{\mathbb{D}^3})\| = 0. \tag{e6.268}$$

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Let C_n be the (commutative) C^*-subalgebra of $B(H)$ generated by $\{\bar{c}_{n,j}, 0 \le j \le 3\}$. Then there is an isomorphism $h_n : C_n \cong C(Y_n)$ for some compact subset $Y_n \subset \mathbb{D}^3$ such that $h_n^{-1}(e_j|_{Y_n}) = \bar{c}_{n,j}, 0 \le j \le 3$, and $n \in \mathbb{N}$. Define $H_n : C(\mathbb{D}^3) \to C \subset B(H)$ by $H_n(f) = h_n^{-1}(f|_{Y_n})$ for $f \in C(\mathbb{D}^3)$. Since $\{e_j|_{\mathbb{D}^3} : 0 \le j \le 3\}$ generates $C(\mathbb{D}^3)$, we have

$$\lim_{n\to\infty} \|H_n(f) - \Lambda_n(f)\| = 0 \text{ for all } f \in C(\mathbb{D}^3). \tag{e6.269}$$

Let $E_n = H_n(1_{C(\mathbb{D}^3)})$, $n \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} \|E_n - p_n\| = 0. \tag{e6.270}$$

Thus, for all large n, there exist unitaries $w_n \in B(H)$ such that

$$w_n^* p_n w_n = E_n, w_n E_n w_n^* = p_n \text{ and } \lim_{n\to\infty} \|w_n - 1\| = 0. \tag{e6.271}$$

Define $\Psi_n : C(\mathbb{D}^3) \to p_n B(H) p_n$ by $\Psi_n(f) = w_n H_n(f) w_n^*$ for $f \in C(\mathbb{D}^3)$. Then

$$\lim_{n\to\infty} \|\Psi_n(f) - \Lambda_n(f)\| = 0 \text{ for all } f \in C(\mathbb{D}^3). \tag{e6.272}$$

This contradicts (e 6.254). Therefore, there is no homomorphism s ψ_n such that (e 6.264) holds. In other words, there is $\varepsilon_0 > 0$ such that

$$\inf\{\sup\{\|\psi_n(e_j) - T_{n,j}\| : 1 \le j \le 3\}\} \ge \varepsilon_0, \tag{e6.273}$$

where the infimum is taken among all homomorphism s $\psi_n : C(\mathbb{I}^3) \to B(H)$. Combining this with (e 6.261), the proposition follows. ∎

Remark 6.5. By Lin's theorem for almost commuting self-adjoint matrices ([21], see also [9]), when $n = 2$, the answer to the version of the Mumford problem ((e1.1) and (e1.2)) is affirmative if each T_j is compact. However, if $n > 2$, the answer is negative even each T_j is compact in general, by Theorem 4.2 of [10], also, for $n = 3$, an earlier counterexample was given by Davidson (Theorem 2.3 of [7], and for a pair of unitaries, see [34]).

Remark 6.6. This remark was added to the original version of the article after we had the opportunity to study Chapter 14 of David Mumford's insightful new book [31], kindly provided by the author.

Some terminologies and motivation are drawn from this work.

In quantum theory, macroscopic observables may be modeled as self-adjoint operators $T_1, T_2, \ldots, T_n$ on an infinite-dimensional separable Hilbert space H. Let A be the unital C^*-subalgebra generated by $\{T_j : 1 \le j \le n\}$ and 1 (assuming $\|T_j\| \le 1$ for convenience). Suppose that $\{e_m\}$ is an orthonormal basis of H, and suppose that, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m \ge N$, $\|T_j e_m\| < \varepsilon$. In this case, these observables collapse uniformly and any measurements outside (orthogonal to) a finite-dimensional subspace vanish, meaning an external observer would detect nothing. We exclude this case by assuming none of T_j's are compact (see also Remark 6.5).

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Given a state defined by a unit vector $x \in H$, the expected value of observable T is $\exp_T(x) = \langle Tx, x\rangle$ (see p. 178 of [31]). Mumford introduces *Approximately Macroscopically Unique (AMU) states* as "near eigenvectors": for a small $\sigma > 0$,

$$\mathrm{AMU}(\{T_j : 1 \le j \le n, \sigma\}) = \{x \in H : \|x\| = 1, \mathrm{sd}_{T_j}(x) < \sigma,\ 1 \le j \le n\},$$

where $\mathrm{sd}_{T_j}(x) = \|(T_j - \exp_{T_j}(x)I)x\|$ (see Chapter 14 of [31]). Mumford asked whether AMU exist. As noted in [31], for AMU to be non-empty, it is natural to require the commutators $\|[T_i, T_j]\|$ to be small. If the question posed in the introduction (see (e1.1) and (e1.2)) had an affirmative answer, then the quantum system could be approximated by a classical one, ensuring the existence of AMU states. However, as observed by Mumford, small commutators alone are insufficient in general. (Proposition 6.4 further shows the complexity of the problem.)

In an infinite-dimensional separable Hilbert space H, a self-adjoint operator, such as T_j, may lack eigenvalues but admit approximate eigenvalues. However, if one allows non-normal states of $B(H)$, or that of $A + \mathcal{K}$ (where $\mathcal{K}$ is the ideal of compact operators), then T_j does have eigenstates and eigenvalues in the atomic representation of $B(H)$, or $A + \mathcal{K}$ (see, e.g., Proposition 4.3.10 of [33]). In particular, one should consider those states of $B(H)$ which vanish on $\mathcal{K}$.

For a point $\xi = (t_1, t_2, \ldots, t_n)$ in the unit cube of $\mathbb{R}^n$ to be a joint approximate eigenvalue, we may test it using the function $\Theta_{\xi,\eta}$ defined in Definition 2.8. In order to have $\|\Theta_{\xi,\eta}(T_1, T_2, \ldots, T_n)\| \ge 1 - \eta$ (for small η), one only needs one vector state $\varphi(\cdot) = \langle \cdot\, x_1, x_1\rangle$ (with a unit vector $x_1 \in H$) such that

$$\langle \Theta_{\xi,\eta}(T_1, T_2, \ldots, T_n)x_1, x_1\rangle \ge 1 - \eta. \tag{e6.274}$$

One may expect this can be repeatedly measured. Suppose that $y_1, \ldots, y_{l_1}$ are unit vectors and H_1 is a finite-dimensional subspace spanned by $x_1, y_1, \ldots, y_{l_1}$. Suppose that there is another unit vector $x_2 \in H_1^{\perp}$ such that (e6.274) also holds for x_2 (in place of x_1), and this persists. Then we obtain a sequence of $\{x_m\}$ $(x_{m+1} \perp H_m)$ such that (e6.274) holds for each x_m (in place of x_1). Note that unit vector states given by $\{x_m\}$ have limit points which give states φ of $A + \mathcal{K}$ vanishing on $\mathcal{K}$ and $\varphi(\Theta_{\xi,\eta}(T_1, T_2, \ldots, T_n)) \ge 1 - \eta$.

Let $\pi_c : B(H) \to B(H)/\mathcal{K}$ be the quotient map and ψ be a state of the corona algebra $B(H)/\mathcal{K}$ (or that of $(A + \mathcal{K})/\mathcal{K}$). Then the composition $\psi \circ \pi_c$ provides a quantum state which may be better suitable to this setting. Indeed, one may expect at least for one such quantum state $\psi \circ \pi_c$ satisfying

$$\psi \circ \pi_c(\Theta_{\xi,\eta}(T_1, T_2, \ldots, T_n)) \ge 1 - \eta. \tag{e6.275}$$

Such measurements, given by states of the form $\psi \circ \pi_c$, may be called *outside measurements*. Via the Gelfand–Naimark construction, this state $\psi \circ \pi_c$ corresponds to a vector state y in the atomic representation space H_a, yielding

$$\langle \Theta_{\xi,\eta}(T_1, T_2, \ldots, T_n)y, y\rangle \ge 1 - \eta. \tag{e6.276}$$

If inequality (e6.275) holds whenever (e6.274) does, then the second condition in (e1.3) always satisfies (in fact, in this case $X = Y$).

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Let us temporarily propose the following terminology:

– A unit vector $x \in H$ satisfying (e6.274) is a *synthetic-AMU state*, with ξ a *synthetic-joint-expected value.*

– A state of the form $\psi \circ \pi_c$ satisfying (e6.275) is an *essential synthetic-AMU state*, with ξ *an essential synthetic-joint-expected value.*

Note that any AMU state (with sufficiently small σ) is an (η)-synthetic-AMU state. Moreover, by Proposition 2.12, the set of synthetic-AMU states is non-empty, when $\|[T_i, T_j]\|$ is small (independent of T_j). The second condition in Theorem 1.1 then may be reformulated as:

(C): *All synthetic-joint-expected values lie in a sufficiently small neighborhood of the set of essential synthetic-joint-expected values.*

This seems to suggest a physical constraint on macroscopic systems: *local measurement should not deviate significantly from outside measurements* (or a long-term measurements along an orthonormal basis of the Hilbert space). Under this constraint, when commutators tend to zero, by Theorem 1.1, the quantum system becomes approximable by a classical commutative one.

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