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Nonconforming Mixed Finite Element Method for Time-dependent Maxwell's Equations with ABC

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Abstract. In this paper, a nonconforming mixed finite element method (FEM) is presented to approximate time-dependent Maxwell's equations in a three-dimensional bounded domain with absorbing boundary conditions (ABC). By employing traditional variational formula, instead of adding penalty terms, we show that the discrete scheme is robust. Meanwhile, with the help of the element's typical properties and derivative transfer skills, the convergence analysis and error estimates for semi-discrete and backward Euler fully-discrete schemes are given, respectively. Numerical tests show the validity of the proposed method.

AMS subject classifications: 65N30, 65N15, 35J25
Key words: Maxwell's equations, Absorbing boundary condition, Nonconforming mixed FEM, Semi-discrete and fully discrete schemes.

1. Introduction

The Maxwells equations are the fundamental equations for understanding most electromagnetic and optical phenomena. The study of theoretical and numerical solutions of Maxwell's systems has been an attractive issue in computational mathematics [2, 7].

As we know, the numerical solutions of C^0-conforming vector nodal FEMs have an interesting history [11, 12] used to Maxwell's equations. Unfortunately, any C^0-conforming FEM will fail if the solutions do not belong to $H^1(\Omega)$, especially when the domain Ω is non-convex. This kind of wrong behavior has been overcome by H. Y. Duan in [13]. The critical idea is that the local L^2 projected program is utilized and two so-called mesh-dependent stabilization terms are added to the bilinear form and right-hand side respectively. It is inevitable that the computing cost is increased and the error estimations are very complicated during the process. Therefore, $H(curl)-$conforming edge element method is meeting and becoming popular.

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In the last thirty years, $H(curl)-$conforming edge elements have been invented to solve time-dependent Maxwell's equations, see e.g., [1–3, 7, 9, 18]. Monk [2] studied the equations by using Nédélec's conforming mixed FEMs approaching of time-dependet Maxwell's systems with perfect electric conductor (PEC) conditions, and derived the interpolation error estimation and Céa lemma, as well as the error estimations in $H(curl)$ norm. Furthermore, some wonderful results were obtained by Li et al. [17–19], in which both semi-discrete and fully-discrete FE schemes were constructed, and optimal order error estimates in the broken energy norm were proved. In [1], the interior penalty discontinuous Galerkin (DG) methods for the time-dependent Maxwell's equations in cold plasma were also developed. Some convergence analysis of an adaptive edge element method for Maxwell's equations have been explored in [21, 22] and references therein.

Since 1980's, nonconforming FEMs have been widely used in numerical solutions of partial differential equations. The first constructive theoretical and numerical analysis for Maxwell's equations can be found in [14–16], in which the Crouzeix-Raviart type nonconforming FE approximating to two dimensional $curl$-$curl$ and $grad$-div system was studied. And numerical experiments indicated that the traditional weak formula could not lead to a convergence scheme even if the mesh is refined. Therefore, the traditional weak formula was modified in [14–16] by adding penalty terms, which involves the tangent and normal jumps. The error estimates and numerical experiments showed that the modified form works well. The so-called crucial difference is that the piecewise $H(curl) \cap H(div)$ semi-norm, unlike the piecewise H^1 semi-norm, is too weak to control the jumps even if the mesh is refined. Hence the two terms involving the jumps must be included in the discretization so as to control the consistency error.

Through analyzing the error estimate of Maxwell's equations by Crouzeix-Raviart type nonconforming FEs, Shi et al. presented some work [23–25] where rectangular nonconforming FEs were constructed. In [23], the new nonconforming FEMs were proposed for approximating the Maxwell's equations on anisotropic meshes by mixed FE formulations in 2D, and 3D in [24]. In [25], the convergence analysis of Quasi-Wilson nonconforming FE approximation of Maxwell's equations was discussed under arbitrary quasi-uniform quadrilateral meshes. The subsequent work can also be found in [26], where a new mixed nonconforming FE space was constructed and applied to discuss the time-harmonic Maxwell's equations, and the analysis of extrapolation and superconvergence were also studied, respectively. Such these constructions have a same typical advantage, that is, the FEs' consistency errors are one order higher than their interpolation errors. Based on this crucial property, the traditional variational formula can be used without adding any 'stability' or 'penalty' term. However, the above researches are only carried out for PEC.

In practical computations, the electromagnetic fields are often truncated in a bounded domain with ABC, such as electromagnetic scattering problems. With the ABC, locally in space and time, there will arise two problems need to be considered: the well-posedness of the original problems in an artificial domain and reflection of waves at

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artificial boundaries. Once these two problems are solved, the continuous ABC can be approximated by discrete ABC, which is then coupled to a standard interior discretization of the differential equations [30]. In [30], the theoretical ABC of a half-space domain was developed for the Maxwell systems by considering the systems as a whole instead of each component of the electromagnetic fields individually. These boundary conditions allow any wave motion generated within the domain to pass through the boundary of the domain without generating any reflections back into the interior. By approximating the theoretical boundary conditions, a class of local ABC for the Maxwell systems could be constructed, and the well-posedness in the sense of Kreiss of the Maxwell systems with each of these local ABC could be established. The similar existence and uniqueness results with ABC were offered in [29].

It is known that in traditional finite difference methods, leapfrog scheme was usually used to discrete time-differential operator, which is simple in form with lower accuracy and CFL limits. In practice, the chosen time step has to be very small, which may result in the computing cost extremely expansive. Because ADI scheme is implicit and economic, Namiki [31] introduced ADI-FDTD to electromagnetic problems So as to overcome the difficulties originated from finite difference methods. From the investigation in [27,28,31–33], we find that Yee's scheme is a very important finite difference scheme for electric field E and magnetic field H. The key point leading to the success is that E and H have a discrete difference of half step.

The objective of this paper is to analyze and develop a nonconforming mixed FEM to approximate the Maxwell systems with ABC. We select the nonconforming FEMs with the degrees of freedom defined by the centroid of faces. By the traditional variational formula, without adding 'stability' or 'penalty' term, we prove that the discrete scheme is robust. The constructed nonconforming FE is in accord with Yee's scheme implicitly [33]. Then, the semi-discrete and backward Euler fully-discrete schemes are investigated. The implicit time-discrete scheme keeps the stability of long iteration, and the special construction of nonconforming mixed FE leads to the convergence. Numerical experiments show that the proposed element has a very good behavior as that of Nédélec's element.

The rest of this work is organized as follows. In Section 2, we describe Maxwell's equations with ABC. In Section 3, we construct the nonconforming mixed FE spaces in cubic meshes. In Section 4, we analyze the semi-discrete scheme and derive error estimates with ABC. This process can be regarded as a framework for approximating Maxwell's equations with nonconforming mixed FEM. In Section 5, we use backward Euler implicit scheme to discrete time differential operator in order to explore the effects of finite difference schemes with respect to time. In Section 6, we give a numerical example to verify our theoretical analysis. We find that the proposed element has a very good behavior as that of Nédélec's element if the exact solutions are smooth and a good quadrature rule is employed.

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2. The differential model

Let E and H denote the electric field and magnetic field, respectively. The time-dependent Maxwell's equations reads as

$$\begin{cases} \varepsilon E_t - \nabla \times H = -J & \text{in} \quad \Omega \times (0,T], \\ \mu H_t + \nabla \times E = 0 & \text{in} \quad \Omega \times (0,T], \end{cases} \tag{2.1}$$

for given J. The absorbing boundary condition of (2.1) is

$$aE + \nu \times H = 0, \quad \text{on} \quad \Gamma = \partial\Omega, \tag{2.2}$$

where ν denotes the unit outer normal to Γ. The initial conditions of (2.1) is

$$E(X,0) = E_0(X), \quad H(X,0) = H_0(X), \tag{2.3}$$

where $X = (x,y,z)$, E_0 and H_0 are given functions.

In this model, the medium parameters ε and μ are assumed by the positive constants. Also, assume that $a(X) \in W^{1,\infty}(\Omega)$ is a real-valued function on Γ such that

$$0 < a_{\min} \le a(X) \le a_{\max}, \quad \forall\, X \in \Gamma.$$

Let $(H^s(\Omega), \|\cdot\|_s)$ and $(H^s(\Gamma), |\cdot|_s)$ be the standard Sobolev spaces for any real number s, and the Hilbert space

$$H(curl;\Omega) = \left\{ \varphi \in [L^2(\Omega)]^3 : \nabla \times \varphi \in [L^2(\Omega)]^3 \right\}$$

be equipped with the natural norm and inner product

$$\begin{aligned} \|\varphi\|_{H(curl;\Omega)} &= \left(\|\varphi\|_0^2 + \|\nabla \times \varphi\|_0^2 \right)^{\frac{1}{2}}, \\ (\varphi,\psi)_{H(curl;\Omega)} &= (\varphi,\psi) + (\nabla \times \varphi, \nabla \times \psi), \end{aligned}$$

respectively. Define

$$H^*(curl;\Omega) = \left\{ \varphi \in H(curl;\Omega); \nu \times \varphi \in [L^2(\partial\Omega)]^3 \right\},$$

and denote $C^m[0,T;Y]$ by the space of m times continuously differentiable functions from $[0,T]$ into the Hilbert space Y.

The following generalized Green's formula [29] in $H(curl;\Omega)$ will be also useful. For each $\varphi, \psi \in H(curl;\Omega)$, there holds

$$(\nabla \times \varphi, \psi) - (\varphi, \nabla \times \psi) = \langle \nu \times \varphi, \psi \rangle_\Gamma = \int_\Gamma \nu \times \varphi \psi dS. \tag{2.4}$$

By (2.4), the weak formulation of (2.1) is: find $(E,H) \in C^1[0,T;H^*(curl;\Omega)] \times C^1[0,T;[L^2(\Omega)]^3]$, such that

$$\begin{cases} (\varepsilon E_t, \varphi) - (H, \nabla \times \varphi) + \langle aE, \varphi \rangle_\Gamma = (-J, \varphi) & \forall \varphi \in H^*(curl;\Omega), \\ (\mu H_t, \psi) + (\nabla \times E, \psi) = 0, & \forall \psi \in [L^2(\Omega)]^3, \end{cases} \tag{2.5}$$

for $0 < t \le T$ with the initial conditions (2.3).

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3. Nonconforming mixed FE spaces

For $0 < h < 1$, let T^h be a quasi-uniform partition of Ω into three-dimensional cuboid $\Omega_j \subset \Omega, j = i, \cdots, J$, with diameters bounded by h, $\overline{\Omega} = \cup_j \overline{\Omega}_j$, $\widehat{Q} = \widehat{Q}_{\widehat{x}} \times \widehat{Q}_{\widehat{y}} \times \widehat{Q}_{\widehat{z}}$ defined on $\widehat{K} = [-1, 1]^3$ by

$$\begin{cases} \widehat{Q}_{\widehat{x}} = span\{1, \widehat{y}, \widehat{z}, (\widehat{y}^2 - \frac{5}{3}\widehat{y}^4) - (\widehat{z}^2 - \frac{5}{3}\widehat{z}^4)\}, \\ \widehat{Q}_{\widehat{y}} = span\{1, \widehat{z}, \widehat{x}, (\widehat{z}^2 - \frac{5}{3}\widehat{z}^4) - (\widehat{x}^2 - \frac{5}{3}\widehat{x}^4)\}, \\ \widehat{Q}_{\widehat{z}} = span\{1, \widehat{x}, \widehat{y}, (\widehat{x}^2 - \frac{5}{3}\widehat{x}^4) - (\widehat{y}^2 - \frac{5}{3}\widehat{y}^4)\}. \end{cases} \tag{3.1}$$

Let $\widehat{\zeta}_i$, $(i = 1, \cdots, 6)$, be the centroid of the i^{th} face $\widehat{f}_i$ of $\widehat{K}$. For $\widehat{\varphi} \in \widehat{Q}$, define the interpolation $\widehat{\Pi} : [H^2(\widehat{K})]^3 \to \widehat{Q}$ as follows

$$(\widehat{\Pi}\widehat{\varphi} - \widehat{\varphi})(\widehat{\zeta}_i) = 0, \quad i = 1, \cdots, 6. \tag{3.2}$$

Next, let $\widehat{S} = \widehat{S}_{\widehat{x}} \times \widehat{S}_{\widehat{y}} \times \widehat{S}_{\widehat{z}}$ on $\widehat{K}$ with

$$\begin{cases} \widehat{S}_{\widehat{x}} = span\{1, \widehat{y} - \frac{10}{3}\widehat{y}^3, \widehat{z} - \frac{10}{3}\widehat{z}^3\}, \\ \widehat{S}_{\widehat{y}} = span\{1, \widehat{z} - \frac{10}{3}\widehat{z}^3, \widehat{x} - \frac{10}{3}\widehat{x}^3\}, \\ \widehat{S}_{\widehat{z}} = span\{1, \widehat{x} - \frac{10}{3}\widehat{x}^3, \widehat{y} - \frac{10}{3}\widehat{y}^3\}. \end{cases} \tag{3.3}$$

We further define a local interpolation operator $\widehat{P}$ from $[L^2(\widehat{K})]^3$ to $\to \widehat{S}$ as: for $\widehat{\psi} = (\widehat{\psi}_1, \widehat{\psi}_2, \widehat{\psi}_3)$,

$$\int_{\widehat{K}} (\widehat{P}\widehat{\psi} - \widehat{\psi})\widehat{q} = 0, \forall q \in \widehat{S}. \tag{3.4}$$

Obviously, $\widehat{P}$ also can be regarded as an L^2-projection.

Define $Q(\Omega_j)$ and $S(\Omega_j)$ by scaling and translating from $\widehat{Q}$ and $\widehat{S}$, respectively. $\mathcal{P}_l(D)$ denotes the space of polynomials of degree l on set D. Let

$$\Gamma_j = \partial\Omega_j \cap \Gamma, \quad \Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k = \Gamma_{kj},$$
$$\Lambda^h = \Big\{\lambda^h : \lambda^h|_{\Gamma_{jk}} = \lambda_{jk} \in \mathcal{P}_0 \times \mathcal{P}_0, \forall\, \Gamma_{jk} \in \Omega_j; \lambda_{jk} + \lambda_{kj} = 0\Big\}.$$

Let $\ll \cdot, \cdot \gg_{\Gamma_{jk}}$ be an approximation of $< \cdot, \cdot >_{\Gamma_{jk}}$ by mid-point rule on Γ_{jk}, that is, if ζ_{jk} is the centriod of face Γ_{jk}, then

$$\ll u, v \gg = |\Gamma_{jk}|(uv)(\zeta_{jk}).$$

Define the nonconforming mixed FE spaces as follows:

$$V^h = \Big\{\varphi \in [L^2(\Omega)]^3 : \varphi|_{\Omega_j} \in Q(\Omega_j), \sum_{jk} \ll \lambda_{jk}, \phi \gg_{\Gamma_{jk}} = 0, \qquad \forall \lambda_{jk} \in \Lambda^h\Big\},$$

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$$W^h = \left\{ \psi \in [L^2(\Omega)]^3 : \psi|_{\Omega_j} \in S(\Omega_j) \right\}.$$

Let $\Pi_h : [H^2(\Omega)]^3 \to V^h$ and $P_h : [L^2(\Omega)]^3 \to W^h$ be the associated interpolation operators defined, respectively. Then we have

$$(\psi - P_h\psi, \chi) = 0, \qquad \forall \chi \in W^h, \tag{3.5}$$

which implies

$$\sum_j (\psi - P_h\psi, \nabla \times \varphi)_j = 0, \qquad \forall \varphi \in V^h. \tag{3.6}$$

Let

$$\|u\|^2_{m,h} = \sum_j \|u\|^2_{m,\Omega_j}, \quad |u|^2_{m,h} = \sum_j |u|^2_{m,\Omega_j}, \quad |u|^2_{m,h,\Gamma} = \sum_j |u|^2_{m,\Gamma_j}.$$

From [2, 5], we have the following interpolation results.

Lemma 3.1. *Assume that the partition T^h $(0 < h < 1)$ is quasi-regular. Then there exists a constant C, independent of h, such that*

$$\|\varphi - \Pi_h\varphi\|_0 \le Ch\|\varphi\|_1, \quad \|\psi - P_h\psi\|_0 \le Ch\|\psi\|_1. \tag{3.7}$$

$$\sum_{\Omega_i} \|\nabla \times (\varphi - \Pi_h\varphi)\|_{0,\Omega_i} \le Ch\|\nabla \times \varphi\|_1. \tag{3.8}$$

Lemma 3.2. *For any $H \in [H^2(\Omega)]^3$, there exists*

$$\sum_j < \nu \times H, \phi >_{\partial\Omega_j \setminus \Gamma_j} \le Ch\|H\|_2\|\phi\|_0, \qquad \forall \phi \in V^h. \tag{3.9}$$

Proof. $\forall \phi = (\phi_1, \phi_2, \phi_3) \in V^h, F_l \subset \partial\Omega_j$, define

$$\bar{\phi}_{il} = \frac{1}{|F_l|} \int_{F_l} \phi_i ds, \quad 1 \le i \le 3, \quad 1 \le l \le 6.$$

By the property $\bar{\phi}_{il} = \frac{1}{|F_l|}\int_{F_l} \phi_i ds = \phi_i(m_l)$, where m_l is the center of face F_l and $F_1, \cdots, F_6$ is the left, right, back, front, low and up face of an element, respectively, we have

$$\sum_{\partial\Omega_j/\Gamma_j} \left[\left(\int_{F_3} - \int_{F_1} \right) (H_2\bar{\phi}_{33} - H_3\bar{\phi}_{23}) dydz + \left(\int_{F_2} - \int_{F_4} \right) (H_1\bar{\phi}|_{32} - H_3\bar{\phi}_{12}) dxdz \right.$$
$$\left. + \left(\left(\int_{F_6} - \int_{F_5} \right) \right) (H_1\bar{\phi}_{26} - H_2\bar{\phi}_{16}) dxdy \right] = 0.$$

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Therefore

$$\begin{aligned}
\sum_{\Omega_j/\Gamma_j}\int_{\partial\Omega_j} Hn\times\phi ds &= \sum_{\Omega_j/\Gamma_j}\Bigg[\Big(\int_{F_3}-\int_{F_1}\Big)(H_2\phi_3-H_3\phi_2)dydz \\
&\quad+\Big(\int_{F_2}-\int_{F_4}\Big)(H_1\phi_3-H_3\phi_1)dxdz+\Big(\int_{F_6}-\int_{F_5}\Big)(H_1\phi_2-H_2\phi_1)dxdy\Bigg], \\
&= \sum_{\Omega_j/\Gamma_j}\Bigg[\Big(\int_{F_3}-\int_{F_1}\Big)(H_2(\phi_3-\bar{\phi}_{33})-H_3(\phi_2-\bar{\phi}_{23}))dydz \\
&\quad+\Big(\int_{F_2}-\int_{F_4}\Big)(H_1(\phi_3-\bar{\phi}_{32})-H_3(\phi_1-\bar{\phi}_{12}))dxdz \\
&\quad+\Big(\int_{F_6}-\int_{F_5}\Big)(H_1(\phi_2-\bar{\phi}_{26})-H_2(\phi_1-\bar{\phi}_{16}))dxdy\Bigg] \\
&= \sum_{\Omega_j/\Gamma_j}(M_1+M_2+M_3)
\end{aligned}$$

With the same argument of [24], we have

$$|M_i|\le\frac{8}{3}h^2|H|_{2,\Omega_j}|\phi|_{1,\Omega_j},\quad i=1,2,3.$$

Then employing inverse inequality yields the desired result. □

The estimate in Lemma 3.2 is very crucial for all of the following error estimates by the traditional variational formula. However, it is not valid for the Crouzeix-Raviart nonconforming FE in [14–16]. Thus, the authors therein did not present a convergent scheme by the traditional variational formula and two 'stability' terms have to be added to make the scheme become robust.

4. Convergence of semi-discrete scheme

4.1. Error estimates

We first consider the semi-discrete scheme of (2.5): find $(E^h,H^h)\in V^h\times W^h$, such that

$$\begin{cases}(\varepsilon E_t^h,\varphi)-\sum\limits_j(H^h,\nabla\times\varphi)_j+\ll aE^h,\ \varphi\gg_\Gamma=(-J,\varphi) & \varphi\in V^h,\\ (\mu H_t^h,\psi)+(\nabla\times E^h,\psi)=0, & \psi\in W^h,\end{cases}\tag{4.1}$$

subject to the initial conditions

$$E^h(X,0)=\Pi_hE_0,\qquad H^h(X,0)=P_hH_0.\tag{4.2}$$

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Theorem 4.1. *Assume that* (E, H) *and* (E^h, H^h) *are the solutions of* (2.5) *and* (4.1)*, respectively. If* $E \in C^2[0,T;H^2(\Omega)], H \in C^2[0,T;H^{\frac{3}{2}}(\Omega)]$*, then there exists a constant* $C > 0$*, independent of* h *such that*

$$\|E - E^h\|_0 + \|H - H^h\|_0 + \|\nabla \times (E - E^h)\|_0 \le Ch^{1/2}. \tag{4.3}$$

Proof. Let $G(f,g) = \langle f, g\rangle_\Gamma - \ll f, g \gg_\Gamma$, we have

$$\begin{aligned}
&(\varepsilon E_t, \varphi) - \sum_j (H, \nabla \times \varphi)_j + \ll aE, \varphi \gg_\Gamma \\
&\qquad = (-J, \varphi) + \sum_j < \nu \times H, \phi >_{\partial\Omega_j \setminus \Gamma_j} - G(aE, \phi), \quad \varphi \in V^h, \\
&(\mu H_t, \psi) + (\nabla \times E, \psi) = 0, \quad \psi \in W^h.
\end{aligned}$$

Let $\xi = \Pi_h E - E^h,\ \eta = P_h H - H^h$. For $\forall \varphi \in V^h, \psi \in W^h$. We have $\nabla \times \varphi \in W^h$, and the error functions can be formulated by ξ and η satisfying

$$\begin{aligned}
&(\varepsilon \xi_t, \varphi) - \sum_j (\eta, \nabla \times \varphi)_j + \ll a\xi, \varphi \gg_\Gamma \\
&\quad = \big(\varepsilon(\Pi_h E - E)_t, \varphi\big) - \sum_j (P_h H - H, \nabla \times \varphi)_j + \ll a(\Pi_h E - E), \varphi \gg_\Gamma \\
&\qquad + \sum_j \langle \nu \times H, \varphi \rangle_{\partial\Omega_j \setminus \Gamma_j} - G(aE, \varphi)
\end{aligned} \tag{4.4}$$

$$(\mu \eta_t, \psi) + \sum_j (\nabla \times \xi, \psi)_j = (\mu(P_h H - H)_t, \psi) + \sum_j \big(\nabla \times (\Pi_h E - E), \psi\big)_j, \tag{4.5}$$

respectively. Taking $\psi = \nabla \times \xi$ in (4.5), we have

$$\begin{aligned}
\|\nabla \times \xi\|_{0,h}^2 &= -\sum_j (\mu \eta_t, \nabla \times \xi)_j + \sum_j \big(\mu(P_h H - H)_t, \nabla \times \xi\big)_j \\
&\quad + \sum_j (\nabla \times (\Pi_h E - E), \nabla \times \xi)_j \\
&\le C\Big(\|\eta_t\|_0 + h\|H_t\|_1 + h\|\nabla \times E\|_1\Big)\|\nabla \times \xi\|_{0,h},
\end{aligned}$$

which implies

$$\|\nabla \times \xi\|_{0,h} \le C\Big(\|\eta_t\|_0 + h\|H_t\|_1 + h\|\nabla \times E\|_1\Big). \tag{4.6}$$

Choosing $\varphi = \xi$ in (4.4) and $\psi = \eta$ in (4.5), then combining them, we have

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\|\xi\|_0^2 + \frac{\mu}{2}\frac{d}{dt}\|\eta\|_0^2 + \ll a\xi, \xi \gg_\Gamma = (\varepsilon(\Pi_h E - E)_t, \xi) + \ll a(\Pi_h E - E), \xi \gg_\Gamma \\
&\quad + \sum_j \langle \nu \times H, \xi \rangle_{\partial\Omega_j \setminus \Gamma_j} - G(aE, \xi) + (\mu(P_h H - H)_t, \eta) \\
&\quad + \sum_j (\nabla \times (\Pi_h E - E), \eta)_j - \sum_j (P_h H - H, \nabla \times \xi)_j =: \sum_{i=1}^{7} (Err)_i.
\end{aligned} \tag{4.7}$$

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Now we start to estimate the terms on the right-hand side of (4.7) individually. For the first term, we have

$$Err_1 \le \frac{\varepsilon}{4\epsilon_1}\|(\Pi_h E - E)_t\|_0^2 + \epsilon_1\|\xi\|_0^2 \le \frac{\varepsilon}{4\epsilon_1}h^2\|E_t\|_1^2 + \epsilon_1\|\xi\|_0^2. \tag{4.8}$$

By (3.2), the second term satisfies[-1ex]

$$Err_2 = \sum_j |\Gamma_j| a(\zeta_j)(\Pi_h E - E)(\zeta_j) = 0. \tag{4.9}$$

Since on Γ_{jk},

$$\nu \times H|_{\Gamma_{jk}} + \nu \times H|_{\Gamma_{kj}} = 0.$$

from the definition of V^h, we have

$$\begin{aligned} &< \nu \times H, \xi >_{\Gamma_{jk}} + < \nu \times H, \xi >_{\Gamma_{kj}} \\ =&< \nu \times H, [\xi] >_{\Gamma_{jk}} =< \nu \times H - \lambda_{jk}, [\xi] >_{\Gamma_{jk}}, \end{aligned}$$

where $[\xi]$ is the jump of ξ on face Γ_{jk}, λ_{jk} is the restriction to Γ_{jk} of an element $\lambda \in \Lambda^h$. Hence, if λ_{jk} is the L^2-projection of $\nu \times H \to (P_0 \times P_0)(\Gamma_{jk})$, there holds,

$$\begin{aligned} &\Big|\sum_j \langle \nu \times H, \xi\rangle_{\partial\Omega_j \setminus \Gamma_j}\Big| = \Big|\sum_j \langle \nu \times H - \lambda_{jk}, \xi\rangle_{\Gamma_{jk}}\Big| \\ \le &\sum_j |\nu \times H - \lambda_{jk}|_{1/2,\Gamma_{jk}} |\xi|_{-\frac{1}{2},\Gamma_{jk}}. \end{aligned}$$

For $0 < \delta < 1/2,\ \theta = 1/[2(1-\delta)]$, standard interpolation theorem gives

$$\begin{aligned} &\min_{\lambda_{jk}} \sum_j \Big|\nu \times H - \lambda_{jk}\Big|_{1/2,\Gamma_{jk}} \\ \le &\sum_j \Big|\nu \times H - \lambda_{jk}\Big|_{1,\Gamma_{jk}}^{1-\theta} \Big|\nu \times H - \lambda_{jk}\Big|_{\delta,\Gamma_{jk}}^{\theta} \\ \le &C\sum_j \Big|\nu \times H\Big|_{1,\Gamma_{jk}}^{1-\theta} \Big|\nu \times H\Big|_{1,\Gamma_{jk}}^{\theta} h^{(1-\delta)\theta} \le Ch^{1/2}\|H\|_{3/2}. \end{aligned}$$

From [6, 7], we know that

$$\|\xi\|_{-1/2,\Gamma_{jk}} \le C_1\Big(\|\xi\|_{0,\Omega_j} + \|\nabla \times \xi\|_{0,\Omega_j}\Big). \tag{4.10}$$

Thus, based on (4.6) and (4.10), the third term can be estimated by

$$\begin{aligned} Err_3 &\le Ch^{\frac{1}{2}}\|H\|_{3/2}\Big(\|\xi\|_0 + \|\nabla \times \xi\|_{0,h}\Big) \\ &\le Ch^{\frac{1}{2}}\|H\|_{3/2}\Big(\|\xi\|_0 + \|\eta_t\|_0 + h\|H_t\|_1 + h\|\nabla \times E\|_1\Big) \\ &\le \frac{Ch}{4\epsilon_2}\Big(\|H\|_{3/2}^2 + h^2\|H_t\|_1^2 + h^2\|\nabla \times E\|_1^2) + \epsilon_2(\|\xi\|_0^2 + \|\eta_t\|_0^2\Big). \end{aligned} \tag{4.11}$$

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For the fourth term, by mid-point rule, trace theorems and (3.2), we obtain

$$\begin{aligned}
|G(aE,\xi)| &= |<aE,\xi>_\Gamma - \ll aE,\xi\gg_\Gamma| \\
&\le \Big|\sum_j \int_{\Gamma_j} (a(s)E(s)\cdot\xi(s) - a(\zeta_j)E(s)\cdot\xi(s))ds\Big| \\
&\quad + \Big|\sum_j \int_{\Gamma_j} (a(\zeta_j)E(s)\cdot\xi(s) - a(\zeta_j)E(\zeta_j)\cdot\xi(s))ds\Big| \\
&\quad + \Big|\sum_j \int_{\Gamma_j} (a(\zeta_j)E(\zeta_j)\cdot\xi(s) - a(\zeta_j)E(\zeta_j)\cdot\xi(\zeta_j))ds\Big| \\
&\le Ch|a|_{1,\infty,\Gamma}|E|_{0,\Gamma}|\xi|_{0,\Gamma} + Ch|a|_{0,\infty,\Gamma}|E|_{1,\Gamma}|\xi|_{0,\Gamma} \\
&\le Ch|E|_2|\xi|_{0,\Gamma}. \qquad (4.12)
\end{aligned}$$

By quasi-uniform assumption on the subdivision [6, 7], it follows that

$$|\xi|_{0,\Gamma} \le C_2 h^{-\frac{1}{2}}\Big(\|\xi\|_0 + \|\nabla\times\xi\|_{0,h}\Big).$$

Therefore,

$$\begin{aligned}
Err_4 &\le Ch^{\frac{1}{2}}|E|_2\Big(\|\xi\|_0 + \|\nabla\times\xi\|_{0,h}\Big) \\
&\le \frac{Ch}{4\epsilon_3}\Big(\|E\|_2^2 + h^2\|H_t\|_1^2 + h^2\|\nabla\times E\|_1^2\Big) + \epsilon_3(\|\xi\|_0^2 + \|\eta_t\|_0^2\Big). \qquad (4.13)
\end{aligned}$$

For the last three terms, by (3.4), Lemma 3.1 and the fact $\nabla\times\xi \in W_h$, we have

$$Err_5 \le \frac{C\mu h^2}{\epsilon_4}\|H_t\|_1^2 + \epsilon_4\|\eta\|_0^2, \qquad (4.14)$$

$$Err_6 \le \frac{Ch^2}{4\epsilon_5}\|E\|_2^2 + \epsilon_5\|\eta\|_0^2, \qquad (4.15)$$

$$Err_7 = 0. \qquad (4.16)$$

By the estimates of (4.8)-(4.16), we have

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\|\xi\|_0^2 + \frac{\mu}{2}\frac{d}{dt}\|\eta\|_0^2 + \ll a\xi,\xi\gg_\Gamma \\
&\le (\epsilon_1+\epsilon_2+\epsilon_3)\|\xi\|_0^2 + (\epsilon_2+\epsilon_3)\|\eta_t\|_0^2 + (\epsilon_4+\epsilon_5)\|\eta\|_0^2 \\
&\quad + Ch\Big(\|E\|_2^2 + h\|E_t\|_2^2 + h\|E\|_2^2 + h^2\|\nabla\times E\|_1^2 + h\|H_t\|_1^2 + \|H\|_{3/2}^2\Big). \qquad (4.17)
\end{aligned}$$

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On the other hand, by taking the differential (4.4) and (4.5) with respect to time t, the following mixed scheme can be established

$$\begin{aligned}
&(\varepsilon\xi_{tt},\varphi)-\sum_j(\eta_t,\nabla\times\varphi)_j+\ll a\xi_t,P_\tau\varphi\gg_\Gamma\\
&=\Big(\varepsilon(\Pi_hE-E)_{tt},\varphi\Big)-\sum_j\Big((P_hH-H)_t,\nabla\times\varphi\Big)_j+\sum_j\langle\nu\times H_t,\varphi\rangle_{\partial\Omega_j\setminus\Gamma_j}\\
&\quad+\ll a(\Pi_hE-E)_t,\varphi\gg_\Gamma -G(aE_t,\varphi),
\end{aligned}\tag{4.18}$$

$$\begin{aligned}
&(\mu\eta_{tt},\psi)+\sum_j(\nabla\times\xi_t,\psi)_j\\
&=(\mu(P_hH-H)_{tt},\psi)+\sum_j(\nabla\times(\Pi_hE-E)_t,\psi)_j.
\end{aligned}\tag{4.19}$$

Choosing $\varphi=\xi_t$ in (4.18) and $\psi=\eta_t$ in (4.19), and summarizing them, another error formula can be set up

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\|\xi_t\|_0^2+\frac{\mu}{2}\frac{d}{dt}\|\eta_t\|_0^2+\ll a\xi_t,\xi_t\gg_\Gamma\\
&=(\varepsilon(\Pi_hE-E)_{tt},\xi_t)-\sum_j((P_hH-H)_t,\nabla\times\xi_t)_j+\sum_j\langle\nu\times H_t,\xi_t\rangle_{\partial\Omega_j\setminus\Gamma_j}\\
&\quad+\ll a(\Pi_hE-E)_t,\xi_t\gg_\Gamma -G(aE_t,\xi_t)+(\mu(P_hH-H)_{tt},\eta_t)\\
&\quad+\sum_j(\nabla\times(\Pi_hE-E)_t,\eta_t)_j=:\sum_{i=1}^7J_i.
\end{aligned}\tag{4.20}$$

Now we begin to estimate the terms on the right-hand side of (4.20) individually. Obviously, the first two terms can be estimated by

$$\begin{aligned}
J_1&=(\varepsilon(\Pi_hE-E)_{tt},\xi_t)\le\frac{\varepsilon}{4\tilde\epsilon_1}\|\Pi_hE_{tt}-E_{tt}\|_0^2+\tilde\epsilon_1\|\xi_t\|_0^2\\
&\le\frac{\varepsilon h^2}{4\tilde\epsilon_1}\|E_{tt}\|_2^2+\tilde\epsilon_1\|\xi_t\|_0^2,
\end{aligned}\tag{4.21}$$

$$J_2=-\sum_j((P_hH-H)_t,\nabla\times\xi_t)_j=0.\tag{4.22}$$

For the third term of the right-hand side of (4.20), with the same arguments of (4.11), we provide

$$\begin{aligned}
J_3&=\sum_j\langle\nu\times H_t,\xi_t\rangle_{\partial\Omega_j/\Gamma_j}\\
&=\frac{d}{dt}\sum_j\langle\nu\times H_t,\xi\rangle_{\partial\Omega_j/\Gamma_j}-\sum_j\langle\nu\times H_{tt},\xi\rangle_{\partial\Omega_j\setminus\Gamma_j}\\
&\le\Big|\frac{d}{dt}\sum_j\langle\nu\times H_t,\xi\rangle_{\partial\Omega_j\setminus\Gamma_j}\Big|+\frac{Ch}{4\tilde\epsilon_2}\Big(\|H_{tt}\|_{3/2}^2+h^2\|H_t\|_1^2
\end{aligned}$$

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$$+h^2\|\nabla\times E\|_1^2\Big)+\tilde{\epsilon}_2\Big(\|\xi\|_0^2+\|\eta_t\|_0^2\Big). \tag{4.23}$$

From (4.9), we know

$$J_4 =\ll a(\Pi_h E-E)_t,\xi_t\gg_\Gamma=0. \tag{4.24}$$

By the same technique in (4.13), we get

$$\begin{aligned} J_5 &= |-G(aE_t,\xi_t)| = \Big|\frac{d}{dt}G(aE_t,\xi)-G(aE_{tt},\xi)\Big| \\ &\le \frac{d}{dt}G(aE_t,\xi)+\frac{Ch}{4\tilde{\epsilon}_3}\Big(\|E_{tt}\|_2^2+h^2\|H_t\|_1^2+h^2\|\nabla\times E\|_1^2\Big) \\ &\quad +\tilde{\epsilon}_3\Big(\|\xi\|_0^2+\|\eta_t\|_0^2\Big). \end{aligned} \tag{4.25}$$

Similarly, by the same way in (4.14) and (4.15), we have

$$J_6=|(\mu(P_hH-H)_{tt},\eta_t)|\le\frac{C\mu h^2}{\tilde{\epsilon}_4}\|H_{tt}\|_1^2+\tilde{\epsilon}_4\|\eta_t\|_0^2 \tag{4.26}$$

$$J_7=\Big|\sum_j(\nabla\times(\Pi_hE-E)_t,\eta_t)_j\Big|\le\frac{Ch^2}{4\tilde{\epsilon}_5}\|E_t\|_2^2+\tilde{\epsilon}_5\|\eta_t\|_0^2. \tag{4.27}$$

Adding (4.21)-(4.27) together, (4.20) can be estimated as

$$\begin{aligned} &\frac{\varepsilon}{2}\frac{d}{dt}\|\xi_t\|_0^2+\frac{\mu}{2}\frac{d}{dt}\|\eta_t\|_0^2+\ll a\xi_t,\xi_t\gg_\Gamma \\ &\le\Big|\frac{d}{dt}\sum_j\langle\nu\times H_t,\xi\rangle_{\partial\Omega_j\setminus\Gamma_j}\Big|+\Big|\frac{d}{dt}G(aE_t,\xi)\Big| \\ &\quad +Ch\Big(\|E_{tt}\|_2^2+h\|E_t\|_2^2+h^2\|\nabla\times E\|_1^2+h^2\|H_t\|_1^2+\|H_{tt}\|_{3/2}^2\Big) \\ &\quad +\tilde{\epsilon}_3\|\xi\|_0^2+\tilde{\epsilon}_1\|\xi_t\|_0^2+(\tilde{\epsilon}_2+\tilde{\epsilon}_3+\tilde{\epsilon}_4+\tilde{\epsilon}_5)\|\eta_t\|_0^2. \end{aligned} \tag{4.28}$$

By choosing very small $\epsilon_1,\cdots,\epsilon_5>0,\tilde{\epsilon}_1,\cdots,\tilde{\epsilon}_5>0$, and by combining (4.17) with (4.28), we have

$$\begin{aligned} &\frac{\varepsilon}{2}\|\xi\|_0^2+\frac{\mu}{2}\|\eta\|_0^2+\frac{\varepsilon}{2}\|\xi_t\|_0^2+\frac{\mu}{2}\|\eta_t\|_0^2 \\ &\le C\int_T\Big(\|\xi\|_0^2+\|\eta\|_0^2+\|\xi_t\|_0^2+\|\eta_t\|_0^2\Big)dt+Ch. \end{aligned} \tag{4.29}$$

Finally, by using Gronwall inequality, triangle inequality, the initial conditions and (4.6), we obtain

$$\|E-E^h\|_0+\|H-H^h\|_0+\|\nabla\times(E-E^h)\|_0\le Ch^{\frac{1}{2}}, \tag{4.30}$$

which is the desired result. □

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4.2. An improved result

From the analysis of section 4.1, we can see that the error estimates of (4.11), (4.13), (4.23) and (4.25) are crucial. If the exact solutions are smooth, we can provide an improved result. In fact, (4.11) and (4.23) are of higher order from Lemma 3.2. Here we only need to give the further consideration of (4.13) and (4.25).

Assumed that $\hat{\Gamma} = [-1,1]^2$ is taken as the top face of the reference element $\hat{K}$. Define

$$Q_z = span\left\{1, x, y, \left(x^2 - \frac{5}{3}x^4\right) - \left(y^2 - \frac{5}{3}y^4\right)\right\}.$$

Let I denote bilinear interpolation over the four Gauss points as nodes when applying 2×2 Gauss quadrature on Γ. For $w \in Q_z$,

$$w = w_1 + w_2, \;\; w_1 \in span\{1, x, y\}, \;\; w_2 \in span\left\{\left(x^2 - \frac{5}{3}x^4\right) - \left(y^2 - \frac{5}{3}y^4\right)\right\}.$$

Then Ig is orthogonal to w_2 and w_2 vanishing at the Gauss points lead to

$$< Ig, w >_\Gamma - \ll Ig, w \gg_\Gamma = < Ig, w_2 >_\Gamma - \ll Ig, w_2 \gg_\Gamma = 0.$$

At the same time, (4.12) can be re-estimated by

$$|G(aE, \xi)| = | < aE - IE, \xi >_\Gamma | \le Ch^2 |E|_{2,\Gamma} |\xi|_{0,\Gamma} \tag{4.31}$$

after applying 2×2 Gauss rule by scaling Γ to mesh size h. Therefore, (4.13) and (4.25) are improved.

The summary of the above consideration can be generated by the following theorem which improves the results of Theorem 4.1 by half an order:

Theorem 4.2. *Assume that (E, H) and (E^h, H^h) are the solutions of* (2.5) *and* (4.1)*, respectively. If $E \in C^2[0,T; H^2(\Omega) \bigcap H^2(\partial\Omega)], H \in C^2[0,T; H^2(\Omega)]$, then there exists a constant $C > 0$, independent of h, such that*

$$\|E - E^h\|_0 + \|H - H^h\|_0 + \|\nabla \times (E - E^h)\|_0 \le Ch. \tag{4.32}$$

5. Convergence of backward euler fully-discrete scheme

Assume that the time interval $[0,T]$ is divided into M uniform subintervals by points $0 = t^0 < t^1 < t^2 < \cdots < t^M = T$, where $t^\kappa = \kappa\tau$, and we denote the kth subinterval by $I^\kappa = (t^{\kappa-1}, t^\kappa]$. Moreover, let $u^\kappa = u(\cdot, \kappa\tau)$ for $0 \le \kappa \le M$, and the backward Euler finite difference scheme be

$$\partial_\tau u^\kappa = (u^\kappa - u^{\kappa-1})/\tau, \;\; \bar{u}^\kappa = (u^\kappa + u^{k+1})/2.$$

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Then the fully discrete mixed FE scheme can be formulated as follows: For $\kappa = 1, 2, \cdots, M$, find $E_h^\kappa \in V^h,\ H_h^\kappa \in W^h$, such that

$$\begin{cases} (\varepsilon \partial_\tau E_h^\kappa, \varphi) - \sum\limits_j (\bar{H}_h^\kappa, \nabla \times \varphi)_j + \ll a \bar{E}_h^\kappa, \varphi \gg_\Gamma = (-J, \varphi), \quad \varphi \in V^h, \\ (\mu \partial_\tau H_h^\kappa, \psi) + \sum\limits_j (\nabla \times \bar{E}_h^\kappa, \psi)_j = 0, \quad \psi \in W^h, \end{cases} \tag{5.1}$$

subject to the initial conditions

$$E^h(X, 0) = \Pi_h E_0, \qquad H^h(X, 0) = P_h H_0.$$

The following inequalities will be useful in our error estimation and can be found in [18]:

Lemma 5.1. *For $B = [L^2(\Omega)]^3$, the following estimates can be offered*

$$(i)\ \ \|\partial_\tau u^\kappa\|_B^2 \le \frac{1}{\tau} \int_{t^{\kappa-1}}^{t^\kappa} \|u_t(s)\|_B^2 ds \qquad \forall u \in H^1(0, T; B), \tag{5.2}$$

$$(ii)\ \ \|\bar{u}^\kappa - \frac{1}{\tau} \int_{I^\kappa} u(s) ds\|_B^2 \le \tau \int_{I^\kappa} \|u_t(s)\|_B^2 ds \qquad \forall u \in H^1(0, T; B). \tag{5.3}$$

Now we are ready to state another main result of this paper:

Theorem 5.1. *Let (E^n, H^n) and (E_h^n, H_h^n) be the solutions of* (2.5) *and* (5.1) *at time $t = t^n$, respectively. If $E \in C^1[0, T; H^2(\Omega)], H \in C^1[0, T; H^{\frac{3}{2}}(\Omega)]$, then there exists a constant $C > 0$, independent of τ and h, such that*

$$\max_{1 \le n \le M} \left(\|E^n - E_h^n\|_0^2 + \|H^n - H_h^n\|_0^2 + \|\nabla \times (E^n - E_h^n)\|_0^2 \right) \le C(\tau^2 + h). \tag{5.4}$$

Proof. Denote the boundary truncation error by $G(f, g) =< f, g >_\Gamma - \ll f, g \gg_\Gamma$, then from the two equations of (2.1), we have

$$\begin{cases} (\varepsilon \partial_\tau E^\kappa, \varphi) - \sum\limits_j (\frac{1}{\tau} \int_{I^\kappa} H(s) ds, \nabla \times \varphi)_j + \ll a \frac{1}{\tau} \int_{I^\kappa} E(s) ds, \varphi \gg_\Gamma \\ \quad = (-J^\kappa, \varphi) + \sum\limits_j < \nu \times \frac{1}{\tau} \int_{I^\kappa} H(s) ds, \varphi >_{\partial\Omega_j \setminus \Gamma_j} - G(\frac{a}{\tau} \int_{I^\kappa} E(s) ds, \varphi), \quad \varphi \in V^h, \\ (\mu \partial_\tau H^\kappa, \psi) + \sum\limits_j (\nabla \times \frac{1}{\tau} \int_{I^\kappa} E(s) ds, \psi)_j = 0, \quad \psi \in W^h. \end{cases} \tag{5.5}$$

Let $\xi_h^\kappa = \Pi_h E^\kappa - E_h^\kappa$ and $\eta_h^\kappa = P_h H^\kappa - H_h^\kappa$. As to (4.4) and (4.5), we have

$$\begin{aligned} &(\varepsilon \partial_\tau \xi_h^\kappa, \varphi) - \sum_j (\bar{\eta}_h^\kappa, \nabla \times \varphi)_j + \ll a \bar{\xi}_h^\kappa, \varphi \gg_\Gamma \\ &= (\varepsilon \partial_\tau (\Pi_h E^\kappa - E^\kappa), \varphi) - \sum_j \left(P_h \bar{H}^\kappa - \frac{1}{\tau} \int_{I^\kappa} H(s) ds, \nabla \times \varphi \right)_j \end{aligned}$$

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$$+\frac{1}{\tau}\sum_j \left\langle \nu_j \times \int_{I^\kappa} H_j(s)ds, \varphi \right\rangle_{\partial\Omega_j\setminus\Gamma_j} - G\left(\frac{a}{\tau}\int_{I^\kappa} E(s)ds, \varphi\right)$$

$$+ \ll a\left(\Pi_h\bar{E}^\kappa - \frac{1}{\tau}\int_{I^\kappa} E(s)ds\right), \varphi \gg_\Gamma, \tag{5.6}$$

$$(\mu\partial_\tau\eta_h^\kappa, \psi) + \sum_j (\nabla\times\bar{\xi}_h^\kappa, \psi)_j$$

$$= (\mu\partial_\tau(P_hH^\kappa - H^\kappa), \psi) + \sum_j \left(\nabla\times(\Pi_h\bar{E}^\kappa - \frac{1}{\tau}\int_{I^\kappa} E(s)ds), \psi\right)_j. \tag{5.7}$$

Taking $\psi = \nabla\times\bar{\xi}_h^\kappa$ in (5.7), we obtain

$$\|\nabla\times\bar{\xi}_h^\kappa\|_0^2 = \sum_j (\nabla\times\bar{\xi}_h^\kappa, \nabla\times\bar{\xi}_h^\kappa)_j = -(\mu\partial_\tau\eta_h^\kappa, \nabla\times\bar{\xi}_h^\kappa)$$

$$+\left(\mu\partial_\tau(P_hH^\kappa - H^\kappa), \nabla\times\bar{\xi}_h^\kappa\right) + \sum_j \left(\nabla\times\left(\Pi_h\bar{E}^\kappa - \frac{1}{\tau}\int_{I^\kappa} E(s)ds\right), \nabla\times\bar{\xi}_h^\kappa\right)_j$$

$$\le C\left(\frac{(\|\eta_h^\kappa\|_0 + \|\eta_h^{\kappa-1}\|_0)}{\tau} + h\|H\|_1 + h\|\nabla\times\bar{E}^\kappa\|_1 + \tau\int_{I^\kappa}\|E_t(t)\|_0^2 dt\right)\|\nabla\times\bar{\xi}_h^\kappa\|_0,$$

which implies

$$\|\nabla\times\bar{\xi}_h^\kappa\|_0$$

$$\le C\left(\frac{(\|\eta_h^\kappa\|_0 + \|\eta_h^{\kappa-1}\|_0)}{\tau} + h\|H\|_1 + h\|\nabla\times E^\kappa\|_1 + \tau\int_{I^\kappa}\|E_t(t)\|_0^2 dt\right). \tag{5.8}$$

On the other hand, taking $\varphi = \bar{\xi}_h^\kappa$ and $\psi = \bar{\eta}_h^\kappa$ and substituting to (5.6) and (5.7), respectively, we have

$$(\varepsilon\partial_\tau\xi_h^\kappa, \bar{\xi}_h^\kappa) - \sum_j (\bar{\eta}_h^\kappa, \nabla\times\bar{\xi}_h^\kappa)_j + \ll a\bar{\xi}_h^\kappa, \bar{\xi}_h^\kappa \gg_\Gamma$$

$$= (\varepsilon\partial_\tau(\Pi_hE^\kappa - E^\kappa), \bar{\xi}_h^\kappa) - \sum_j \left(P_h\bar{H}^\kappa - \frac{1}{\tau}\int_{I^\kappa} H(s)ds, \nabla\times\bar{\xi}_h^\kappa\right)_j$$

$$+\frac{1}{\tau}\sum_j \left\langle \nu\times\int_{I^\kappa} H(s)ds, \bar{\xi}_h^\kappa\right\rangle_{\partial\Omega_j\setminus\Gamma_j} - G\left(\frac{a}{\tau}\int_{I^\kappa} E(s)ds, \bar{\xi}_h^\kappa\right)$$

$$+ \ll a\left(\Pi_h\bar{E}^\kappa - \frac{1}{\tau}\int_{I^\kappa} E(s)ds\right), \bar{\xi}_h^\kappa \gg_\Gamma, \tag{5.9}$$

$$(\mu\partial_\tau\eta_h^\kappa, \bar{\eta}_h^\kappa) + \sum_j (\nabla\times\bar{\xi}_h^\kappa, \bar{\eta}_h^\kappa)_j$$

$$= (\mu\partial_\tau(P_hH^\kappa - H^\kappa), \bar{\eta}_h^\kappa) + \sum_j \left(\nabla\times(\Pi_h\bar{E}^\kappa - \frac{1}{\tau}\int_{I^\kappa} E(s)ds), \bar{\eta}_h^\kappa\right)_j. \tag{5.10}$$

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Thus, combining (5.9) and (5.10), multiplying τ to the two sides, and applying the inequality, $p(p-q) \geq \frac{1}{2}(p^2-q^2), \forall p, q > 0$, we have

$$\begin{aligned}
&\frac{\varepsilon}{2}(\|\xi_h^\kappa\|_0^2 - \|\xi_h^{\kappa-1}\|_0^2) + \frac{\mu}{2}(\|\eta_h^\kappa\|_0^2 - \|\eta_h^{\kappa-1}\|_0^2) + \tau \ll a\bar{\xi}_h^\kappa, \bar{\xi}_h^\kappa \gg_\Gamma \\
&= \tau\Big(\varepsilon\partial_\tau(\Pi_h E^\kappa - E^\kappa), \bar{\xi}_h^\kappa\Big) - \tau\sum_j \left(P_h\bar{H}^\kappa - \frac{1}{\tau}\int_{I^\kappa} H(s)ds, \nabla\times\bar{\xi}_h^\kappa\right)_j \\
&\quad + \sum_j \left\langle \nu\times\int_{I^\kappa} H(s)ds, \bar{\xi}_h^\kappa\right\rangle_{\partial\Omega_j\setminus\Gamma_j} - \tau G\left(\frac{a}{\tau}\int_{I^\kappa} E(s)ds, \bar{\xi}_h^\kappa\right) \\
&\quad + \tau \ll a\left(\Pi_h\bar{E}^\kappa - \frac{1}{\tau}\int_{I^\kappa} E(t)dt\right), \bar{\xi}_h^\kappa \gg_\Gamma + \tau\Big(\mu\partial_\tau(P_h H^\kappa - H^\kappa), \bar{\eta}_h^\kappa\Big) \\
&\quad + \tau\sum_j \left(\nabla\times\left(\Pi_h\bar{E}^\kappa - \frac{1}{\tau}\int_{I^\kappa} E(s)ds\right), \bar{\eta}_h^\kappa\right)_j =: \sum_{i=1}^{7} I_i. \qquad (5.11)
\end{aligned}$$

With the similar argument of the semi-discrete scheme, the terms on the right-hand side of (5.11) can be estimated as:

$$I_1 = \tau\Big(\varepsilon\partial_\tau(\Pi_h E^\kappa - E^\kappa), \bar{\xi}_h^\kappa\Big) \leq \frac{1}{4\bar{\epsilon}_1}\tau^2 h^2\|E_t\|_2^2 + \bar{\epsilon}_1\|\bar{\xi}_h^\kappa\|_0^2; \qquad (5.12)$$

$$\begin{aligned}
I_2 &= \tau\sum_j \left(P_h\bar{H}^\kappa - \frac{1}{\tau}\int_{I^\kappa} H(s)ds, \nabla\times\bar{\xi}_h^\kappa\right)_j \\
&= \tau\sum_j \left(P_h\bar{H}^\kappa - \bar{H}^\kappa, \nabla\times\bar{\xi}_h^\kappa\right)_j + \tau\sum_j \left(\bar{H}^\kappa - \frac{1}{\tau}\int_{I^\kappa} H(s)ds, \nabla\times\xi_h^\kappa\right)_j \\
&\leq C\tau^2\int_{I^\kappa}\|H_t\|_0^2 dt\left(\frac{(\|\eta_h^\kappa\|_0 + \|\eta_h^{\kappa-1}\|_0)}{\tau} + h\|H\|_1 + h\|\nabla\times E\|_0^2 + \tau\int_{I^\kappa}\|E_t\|_0^2 dt\right) \\
&\leq \frac{\tau^2}{4\bar{\epsilon}_2}\left(\int_{I^\kappa}\|H_t\|_0^2 dt + \tau^2 h^2\|H\|_1^2 + \tau^2 h^2\|\nabla\times E\|_0^2 + \tau^2\left(\int_{I^\kappa}\|E_t\|_0^2 dt\right)^2\right) \\
&\quad + \bar{\epsilon}_2\Big(\|\eta_h^\kappa\|_0^2 + \|\eta_h^{\kappa-1}\|_0^2\Big); \qquad (5.13)
\end{aligned}$$

$$\begin{aligned}
I_3 &= \sum_j \left\langle \nu\times\int_{I^\kappa} H(t)dt, \bar{\xi}_h^\kappa\right\rangle_{\partial\Omega_j\setminus\Gamma_j} \leq C\tau\sum_j \left\langle \nu\times H^\kappa, \bar{\xi}_h^\kappa\right\rangle_{\partial\Omega_j\setminus\Gamma_j} \\
&\leq Ch^{1/2}\tau\|H\|_{3/2}\Big(\|\bar{\xi}_h^\kappa\|_0 + \|\nabla\times\bar{\xi}_h^\kappa\|_0\Big) \\
&\leq Ch^{1/2}\tau\|H\|_{3/2}\left(\|\bar{\xi}_h^\kappa\|_0 + \frac{(\|\eta_h^\kappa\|_0 + \|\eta_h^{\kappa-1}\|_0)}{\tau} + h\|H\|_1\right. \\
&\quad \left. + h\|\nabla\times E\|_0^2 + \tau\int_{I^\kappa}\|E_t\|_0^2 dt\right)
\end{aligned}$$

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$$\leq \bar{\epsilon}_3\Big(\|\bar{\xi}_h^\kappa\|_0^2+\|\eta_h^\kappa\|_0^2+\|\eta_h^{\kappa-1}\|_0^2\Big)+\frac{Ch}{4\bar{\epsilon}_3}\Bigg((\tau h)^2\|H\|_{3/2}^2$$

$$+(\tau h)^2\|H\|_1^2+(\tau h)^2\|\nabla\times E\|_2^2+\tau^4\bigg(\int_{I^\kappa}\|E_t\|_0^2dt\bigg)^2\Bigg); \tag{5.14}$$

$$I_4=\tau G\bigg(\frac{a}{\tau}\int_{I^\kappa}E(t)dt,\bar{\xi}_h^\kappa\bigg)$$

$$\leq Ch^{\frac{1}{2}}\int_{I^\kappa}\|E\|_2dt(\|\bar{\xi}_h^\kappa\|_0+\|\nabla\times\bar{\xi}_h^\kappa\|_0)$$

$$\leq Ch^{\frac{1}{2}}\tau\|E^k\|_2\bigg(\|\bar{\xi}_h^\kappa\|_0+\frac{(\|\eta_h^\kappa\|_0+\|\eta_h^{\kappa-1}\|_0)}{\tau}+h\|H\|_1+h\|\nabla\times E\|_0^2+\tau\int_{I^\kappa}\|E_t\|_0^2dt\bigg)$$

$$\leq \bar{\epsilon}_4(\|\bar{\xi}_h^\kappa\|_0^2+\|\eta_h^\kappa\|_0^2+\|\eta_h^{\kappa-1}\|_0^2)+\frac{Ch}{4\bar{\epsilon}_4}\Bigg((\tau h)^2\|E\|_2^2+(\tau h)^2\|H\|_1^2$$

$$+(\tau h)^2\|\nabla\times E\|_2^2+\tau^4\bigg(\int_{I^\kappa}\|E_t\|_0^2dt\bigg)^2\Bigg); \tag{5.15}$$

$$I_5=\tau\ll a\bigg(\Pi_h\bar{E}^\kappa-\frac{1}{\tau}\int_{I^\kappa}E(t)dt\bigg),\bar{\xi}_h^\kappa\gg_\Gamma$$

$$\leq \tau\ll a(\Pi_h\bar{E}^\kappa-\bar{E}^\kappa,\bar{\xi}_h^\kappa\gg_\Gamma+\tau\ll a\bigg(\bar{E}^\kappa-\frac{1}{\tau}\int_{I^\kappa}E(t)dt\bigg),\bar{\xi}_h^\kappa\gg_\Gamma$$

$$\leq C\tau^2\int_{I^\kappa}\|E_t\|_{0,\Gamma}^2dt(\|\bar{\xi}_h^\kappa\|_0+\|\nabla\times\bar{\xi}_h^\kappa\|_0)$$

$$\leq C\tau^2\int_{I^\kappa}\|E_t\|_{0,\Gamma}^2dt\bigg(\|\bar{\xi}_h^\kappa\|_0+\frac{(\|\eta_h^\kappa\|_0+\|\eta_h^{\kappa-1}\|_0)}{\tau} \tag{5.16}$$

$$+h\|H\|_1+h\|\nabla\times E\|_0^2+\tau\int_{I^\kappa}\|E_t\|_0^2dt\bigg)$$

$$\leq \bar{\epsilon}_5\Big(\|\bar{\xi}_h^\kappa\|_0^2+\|\eta_h^\kappa\|_0^2+\|\eta_h^{\kappa-1}\|_0^2\Big)+\frac{C\tau^2}{4\bar{\epsilon}_5}\Bigg(\tau^2\int_{I^\kappa}\|E_t\|_{0,\Gamma}^2dt$$

$$+(\tau h)^2\|H\|_1^2+(\tau h)^2\|\nabla\times E\|_2^2+\tau^4(\int_{I^\kappa}\|E_t\|_0^2dt)^2\Bigg); \tag{5.17}$$

$$I_6=\tau\Big(\mu\partial_\tau(P_hH^\kappa-H^\kappa),\bar{\eta}_h^\kappa\Big)\leq\frac{C\mu}{4\bar{\epsilon}_6}h^2\|H\|_1^2+\bar{\epsilon}_6\|\bar{\eta}_h^\kappa\|_0^2, \tag{5.18}$$

$$I_7=\tau\sum_j\bigg(\nabla\times(\Pi_h\bar{E}^\kappa-\frac{1}{\tau}\int_{I^\kappa}E(s)ds)_j,\bar{\eta}_h^\kappa\bigg)$$

$$=\tau\sum_j\bigg(\nabla\times(\Pi_h\bar{E}^\kappa-\bar{E}^\kappa)_j,\bar{\eta}_h^\kappa,\bigg)+\tau\sum_j\bigg(\nabla\times\bigg(\bar{E}^\kappa-\frac{1}{\tau}\int_{I^\kappa}E(s)ds\bigg)_j,\bar{\eta}_h^\kappa\bigg)$$

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$$\leq Ch\|\nabla\times E\|_0\|\bar{\eta}_h^\kappa\|_0 + C\tau^2\int_{I^\kappa}\|E_t\|_0^2dt\|\bar{\eta}_h^\kappa\|_0$$

$$\leq \frac{C}{4\bar{\epsilon}_7}\left(h^2\|\nabla\times E\|_0^2 + \tau^4\left(\int_{I^\kappa}\|E_t\|_0^2dt\right)^2\right) + \bar{\epsilon}_7\|\bar{\eta}_h^\kappa\|_0^2. \tag{5.19}$$

Adding (5.12)-(5.19) together, we can choose proper $\bar{\epsilon}_1,\cdots,\bar{\epsilon}_7 > 0$ such that

$$\frac{\varepsilon}{2}\left(\|\xi_h^k\|_0^2 - \|\xi_h^{k-1}\|_0^2\right) + \frac{\omega\mu}{2}\left(\|\eta_h^k\|_0^2 - \|\eta_h^{k-1}\|_0^2\right)$$

$$\leq \epsilon\left(\|\xi_h^k\|_0^2 + \|\eta_h^k\|_0^2\right) + C(h+\tau^2). \tag{5.20}$$

Then, summing k from 1 to $n, 1 < n \leq M$ on both sides of (5.20), we have

$$\frac{\varepsilon}{2}\|\xi_h^n\|_0^2 + \frac{\mu}{2}\|\eta_h^n\|_0^2$$

$$\leq \left(\frac{\varepsilon}{2}\|\xi_h^0\|_0^2 + \frac{\mu}{2}\|\eta_h^0\|_0^2\right) + \sum_{k=1}^{n}\epsilon'\left(\|\xi_h^k\|_0^2 + \|\eta_h^k\|_0^2\right) + C(h+\tau^2).$$

According to the discrete Gronwall inequality and the initial values,

$$\|\xi_h^n\|_0^2 + \|\eta_h^\kappa\|_0^2 \leq C(\tau^2+h). \tag{5.21}$$

Substituting (5.21) to (5.8), we derive

$$\|\nabla\times\xi_h^k\|_0^2 \leq C(\tau^2+h).$$

Finally, the desired result comes from a combination of the triangle inequality and Lemma 3.1. □

Theorem 5.2. *Let (E^n, H^n) and (E_h^n, H_h^n) be the solutions of* (2.5) *and* (5.1) *at time $t = t^n$, respectively.If $E \in C^1[0,T;H^2(\Omega)\bigcap H^2(\partial\Omega)], H \in C^1[0,T;H^2(\Omega)]$, then there exists a constant C, independent of τ and h, such that*

$$\max_{1\leq n\leq M}\left(\|E^n - E_h^n\|_0^2 + \|H^n - H_h^n\|_0^2 + \|\nabla\times(E^n - E_h^n)\|_0^2\right) \leq C(\tau^2+h^2). \tag{5.22}$$

Proof. The proof is similar to that of **Theorem 5.1**. We can use the result in (4.31) to improve the convergence order of (5.14) and (5.15). □

6. Numerical experiments

We consider the time-harmonic Maxwell's equations with ABC

$$E_t - \nabla\times H = -J, \qquad \text{in} \quad (0,T]\times\Omega,$$

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Table 1: Errors at t=0.5 with time step $\tau = 0.01$.

$N \times N \times N$	NonFE error$_h$	order	Né-FE error$_h$	order
$2 \times 2 \times 2$	0.2729	-	0.3253	-
$4 \times 4 \times 4$	0.1489	0.8743	0.1668	0.9634
$8 \times 8 \times 8$	0.0757	0.9760	0.0828	1.0106
$16 \times 16 \times 16$	0.0380	0.9943	0.0411	1.0110

Table 2: Errors at t=0.7 with time step $\tau = 0.01$.

$N \times N \times N$	NonFE error$_h$	order	Né-FE error$_h$	order
$2 \times 2 \times 2$	0.3425	-	0.3978	-
$4 \times 4 \times 4$	0.1855	0.8848	0.2037	0.9653
$8 \times 8 \times 8$	0.0942	0.9779	0.1014	1.0062
$16 \times 16 \times 16$	0.0473	0.9948	0.0501	1.0168

Table 3: Errors at t=1 with time step $\tau = 0.01$.

$N \times N \times N$	NonFE error$_h$	order	Né-FE error$_h$	order
$2 \times 2 \times 2$	0.4572	-	0.5382	-
$4 \times 4 \times 4$	0.2436	0.8848	0.2752	0.9675
$8 \times 8 \times 8$	0.1234	0.9819	0.1368	1.0081
$16 \times 16 \times 16$	0.0473	0.9957	0.0677	1.0157

$$
\begin{aligned}
&H_t + \nabla \times E = 0, && \text{in} \quad (0,T] \times \Omega,\\
&E + \nu \times H = g, && \text{on} \quad (0,T] \times \partial\Omega,\\
&E(x,y,z,0) = E_0(x,y,z), \quad H(x,y,z,0) = H_0(x,y,z), && \text{in} \quad \Omega,
\end{aligned}
$$

where $\Omega = [0,1]^3, T = 1$. Let $\epsilon = \mu = a = 1$, and the electric field E and magnetic field H are

$$
\begin{aligned}
E &= \Big(\sin y_1 \sin z_1,\ \sin z_1 \sin x_1,\ \sin x_1 \sin y_1\Big) e^t,\\
H &= \frac{\pi}{4}\Big(\sin x_1(\cos z_1 - \cos y_1),\ \sin y_1(\cos x_1 - \cos z_1),\ \sin z_1(\cos y_1 - \cos x_1)\Big) e^t,
\end{aligned}
$$

respectively, with $x_1 = \pi x/4$, $y_1 = \pi y/4$, $z_1 = \pi z/4$.

The errors of our nonconforming FE and Nédélec's FE at $t = 0.5, 0.7$ and 1 are listed in the Tables 1-3, respectively, where N is the number of partition in $x, y, z-$direction. We denote by τ the time step and let

$$
error_h = \max_{1 \le n \le M} (\|E^n - E_h^n\|_0 + \|H^n - H_h^n\|_0 + \|\nabla \times (E^n - E_h^n)\|_0),
$$

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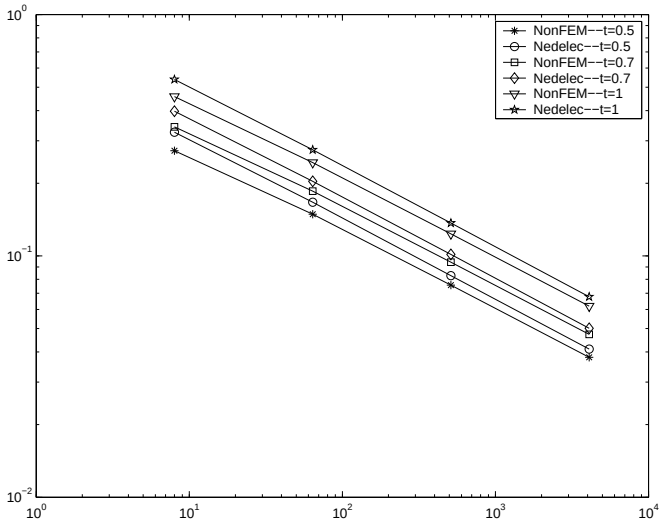


Figure 1: Error curves at t=0.5, 0.7 and 1 for Nédélec's element and our nonconforming finite element.

$$order = \log\left(\frac{error_h}{error_{\frac{h}{2}}}\right)/\log 2,$$

be the error and convergence order, respectively. For simplicity, NonFE error$_h$ and Né-FE error$_h$ denote the errors of our proposed nonconforming FE and Nédélec's FE with respect to h, respectively. From Tables 1-3, we can see that the convergence order is $O(h)$ for the provided nonconforming mixed finite element, which is consistency with the result in Theorem 4.2 and Theorem 5.2.

At the same time, we also plot the error curves in Fig. 1. One may find that the proposed element indeed has a very good behavior as that of Nédélec's element if the exact solutions are smooth and a good quadrature rule is employed.

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