Teaching Notes

Learning from a mistake

Seldom do we encounter a situation where we find a *new problem* in a *mistake*, but, when we do, it counts as something special and merits attention of others. While stating the sine rule a student wrote that in any triangle ABC,

$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C} = \frac{1}{2R}$$

instead of

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R},$$

where the angles A, B and C are measured in radians and $BC = a$, $CA = b$ and $AB = c$. Evidently, what the student wrote is incorrect but interestingly there exists a triangle for which the incorrect statement holds true. It is the equilateral triangle. But is it the only such triangle?

Observe that $A + B + C = \pi$ and if we assume that $A \leqslant B \leqslant C$ then for a non-degenerate triangle to exist we must have

$$A + B = \pi - C > C \tag{1}$$

implying $C < \frac{1}{2}\pi$. Thus such a triangle must necessarily be acute-angled. Now, observe that the statement that the student wrote gives away another piece of information, namely that the function

$$f(x) = \frac{\sin x}{x}$$

agrees at three points A, B and C, where each of them is in the interval $\left(0, \frac{1}{2}\pi\right)$. We claim that

$$f(A) = f(B) = f(C) \Rightarrow A = B = C,$$

is true. To prove this we study the nature of the derivative $f'(x)$ on $\left(0, \frac{1}{2}\pi\right)$. Note that

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x\,(x - \tan x)}{x^2},$$

and if $g(x) = x - \tan x$ then $g'(x) = 1 - \sec^2 x = -\tan^2 x < 0$ on $\left(0, \frac{1}{2}\pi\right)$ which shows that $f'(x) < 0$ on $\left(0, \frac{1}{2}\pi\right)$, implying that $f(x)$ is decreasing and hence injective. Thus,

$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C} = \frac{1}{2R} \tag{$*$}$$

holds if, and only if, $A = B = C = \frac{1}{3}\pi$. That is $(*)$ is true if, and only if, the triangle is equilateral.

The same proof, with slight modification, would also work if the angles A, B and C were measured in degrees.

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