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Pricing Model for Convertible Bonds: A Mixed Fractional Brownian Motion with Jumps

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Abstract. A mathematical model to price convertible bonds involving mixed fractional Brownian motion with jumps is presented. We obtain a general pricing formula using the risk neutral pricing principle and quasi-conditional expectation. The sensitivity of the price to changing various parameters is discussed. Theoretical prices from our jump mixed fractional Brownian motion model are compared with the prices predicted by traditional models. An empirical study shows that our new model is more acceptable.

AMS subject classifications: 60J75, 60G22, 91G80

Key words: Mixed fractional Brownian motion, Poisson jump, convertible bond, empirical study.

1. Introduction

A convertible bond is a complex financial product that enables the holder to exchange the bond for the issuer's underlying stock in some specified circumstance. The characteristics of both bond and equity make the valuation of convertible bonds quite difficult. Indeed, the convertible bond trade is an emerging market, and it is important to consider more factors that influence convertible bond pricing than allowed for by existing pricing methods.

Theoretical research on convertible bond pricing was initiated by Ingersoll [1], who applied the well known Black-Scholes-Merton options pricing model. Following his work, Brennan & Schwartz [2] used corporate value as the basic variable to price convertible bonds, and then took into account the uncertainty inherent in interest rates and also the possibility of senior debt in the firm's capital structure [3]. Nyborg [4] considered the more complicated call and put features in convertible bond pricing under stochastic interest rates. In these ways, researchers have gradually added various factors to increase the accuracy of convertible bond pricing.

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The above articles all regard the price movement as a geometric Brownian motion. However, many empirical studies demonstrate that the distributions of the logarithmic returns on the financial asset usually exhibit self-similarity properties, heavy tails and long-range dependence in both auto-correlations and cross-correlations, and volatility clustering [5-10]. Indeed, the most common stochastic process that exhibits long-range dependence is a fractional Brownian motion (FBM). Furthermore, an FBM produces a burstiness in the sample path behaviour, an important aspect of financial time series. Consequently, it is natural to replace the Brownian motion with an FBM to make stochastic models more realistic [11-15].

Classical Itô theory cannot be applied to an FBM, and defining an associated proper stochastic integral is difficult [16]. A major difficulty is that an FBM is not semi-martingale, so to take the long memory property into account it is reasonable to prefer a mixed fractional Brownian motion (MFBM) in order to capture the price fluctuations of a financial asset [17-18]. An MFBM is essentially a family of Gaussian processes in a linear combination of Brownian motion and FBM — a class of long memory processes with the Hurst parameter $H \in (1/2, 1)$. The first work in economics using an MFBM is in Ref. [19], where for $H \in (3/4, 1)$ it was proven that an MFBM is equivalent in law to a Brownian motion, and hence financial markets driven by the MFBM is arbitrage-free. Recent additional applications have also been documented [20]. However, all of the above-mentioned earlier research considered that the logarithmic returns of the underlying stock are independent identically distributed normal random variables, whereas the empirical study of asset return indicates that discontinuities or jumps are an essential component of financial asset price series [21-23]. Merton [24] proposed a jump-diffusion process involving a Poisson jump, given the observed abnormal fluctuations in stock prices. Based on his theory, several authors have modelled the price of a convertible bond as a Brownian motion with Poisson jumps [25-27].

This article provides a theoretical, numerical and empirical contribution to the study of convertible bonds. To capture jumps or discontinuities and account for the long memory property, a combination of Poisson jumps and mixed fractional Brownian motion is used. As in Ref. [19], we assume $H \in (3/4, 1)$ throughout, an assumption validated by many previous empirical studies [20, 28]. Our jump mixed fractional Brownian motion (JMFBM) model produces empirically observed distributions of stock price changes that are skewed, leptokurtic, long memory and possess fatter tails than comparable normal distributions. A JMFBM model to price convertible bonds has not been investigated before, and our new model allows us to explore the sensitivities of the convertible bond price to changes in various relevant parameters. Numerical experiments supplemented by an empirical study indicate that our JMFBM model more closely predicts the actual market in convertible bonds than any purely Brownian motion model.

The rest of the article is organized as follows. Some MFBM results are recalled in Section 2, and we present the JMFBM pricing model for convertible bonds in Section 3. Section 4 contains the sensitivity analysis for the convertible bond price. We numerically compare our JMFBM model with traditional models in Section 5, and discuss the relationship between the convertible bond price and the parameters of the Hurst and jump. In

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Section 6, an empirical study shows that our model is more acceptable. Our concluding remarks are in Section 7.

2. Preliminaries

We first recall some essential results — cf. also Refs. [22,29,30].

Definition 2.1. An MFBM with parameters a, b and H is a linear combination of Brownian motion and FBM with Hurst parameter H, defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ for any $t \in \mathbb{R}_+$ by $M_t^H = aB_t + bB_t^H$, where a and b are two real constants such that $(a,b) \neq (0,0)$, B_t is a Brownian motion, B_t^H is an FBM with Hurst parameter $H \in (0,1)$, which is independent of B_t.

Proposition 2.1. *The MFBM M_t^H satisfies the following properties:*

(i) M_t^H *is a centred Gaussian process and not Markovian;*
(ii) $M_t^H = 0$, $\mathbb{P}$*-almost surely;*
(iii) the covariance of $M_t^H(a,b)$ *and* $M_s^H(a,b)$ *for any* $t,s \in \mathbb{R}_+$ *is given by*

$$\operatorname{cov}\left(M_t^H, M_t^H\right) = a^2\,(t \wedge s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{2.1}$$

where $\wedge$ *denotes* $min\{s,t\}$*;*
(iv) the increments of $M_t^H(a,b)$ *are stationary and mixed-self-similar for any* $h > 0$*:*

$$M_{ht}^H(a,b) \triangleq M_t^H\left(ah^{\frac{1}{2}}, bh^H\right), \tag{2.2}$$

where $\triangleq$ *means "to have the same law";*
(v) the increments of M_t^H *are positively correlated if* $1/2 < H < 1$*, uncorrelated if* $H = 1/2$ *and negatively correlated if* $0 < H < 1/2$*;*
(vi) the increments of M_t^H *are long-range dependent if and only if* $H > 1/2$*;*
(vii) for all $t \in R_+$*,*

$$E\left[(M_t^H(a,b))^n\right] = \begin{cases} 0\,, & n = 2l+1\,, \\ \dfrac{(2l)!}{2^l l!}(a^2 t + b^2 t^{2H})^l, & n = 2l\,. \end{cases} \tag{2.3}$$

Lemma 2.1. *For every* $t \in [0,T]$*, if* $\sigma, \varepsilon \in C$ *then*

$$\hat{\mathbb{E}}_{\mathbb{P}}\left[e^{\sigma(B_T + \varepsilon B_T^H)} | \mathscr{F}_t\right] = e^{\sigma(B_t + \varepsilon B_t^H) + \frac{\sigma^2}{2}(T-t) + \frac{\sigma^2\varepsilon^2}{2}(T^{2H} - t^{2H})}, \tag{2.4}$$

where $\mathbb{P}$ *is a risk-neutral probability measure and* $\hat{\mathbb{E}}_{\mathbb{P}}[\cdot|\mathscr{F}_t]$ *denotes the quasi-conditional expectation with respect to* $\mathscr{F}_t$ *under the probability measure* $\mathbb{P}$*.*

Proof. The proof is similar to that of Lemma A.1. in Ref. [29] and so omitted here. □

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3. Pricing Model for Convertible Bonds in the JMFBM Environment

3.1. The JMFBM market

Consider a market consisting of one bond (a riskless asset) and one stock (a risky asset). The price A_t of the bond evolves (for $t \in [0, T]$) according to the differential equation and initial price

$$dA_t = rA_t dt\,, \qquad A_0 = 1\,, \tag{3.1}$$

where the interest rate r is assumed constant.

The price S_t of the stock is assumed to satisfy

$$dS_t = S_t\big[(\mu - \lambda\rho)dt + \sigma(dB_t + \varepsilon dB_t^H) + UdN_t\big]\,, \quad S_0 = S\,, \tag{3.2}$$

where the drift μ and volatility σ are assumed constant, B_t is a Brownian motion, B_t^H is a fractional Brownian motion with Hurst parameter $H \in (3/4, 1)$, N_t is a Poisson process with intensity λ, $U_1, \cdots, U_n$ are jump sizes of the stock price (a sequence of independent identically distributed random variables), $\rho = E(U)$ and $\ln(1+U) \sim N(\mu_U, \sigma_U)$. All four sources of randomness (the Brownian motion B_t, the fractional Brownian motion B_t^H, the Poisson process N_t and the jump size U) are assumed to be independent.

We consider a probability space

$$\begin{aligned}&\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P}\big)\,,\\ &\mathscr{F}_t = \sigma[B_s; s \le t] \vee \sigma[B_s^H; s \le t] \vee \sigma[N((0,s]; s \le t] \vee \sigma[U_i, i \ge 1]\,.\end{aligned}$$

Since S_t satisfies Eq. (3.2), let $y_t = \ln S_t$, using the $Itô$ formula we get

$$\begin{aligned}S_T = S_t \exp\Big[&(\mu - \lambda\rho - \frac{1}{2}\sigma^2)(T-t) - \frac{1}{2}\sigma^2\varepsilon^2(T^{2H} - t^{2H})\\ &+ \sigma(B_T - B_t) + \sigma\varepsilon(B_T^H - B_t^H) + \sum_{i=1}^{N_{T-t}} \ln(1+U_i)\Big].\end{aligned} \tag{3.3}$$

Theorem 3.1. *For $T \ge t$, the quasi-conditional expectation of the stock price is*

$$\hat{\mathbb{E}}_{\mathbb{P}}\big[S(T)|\mathscr{F}_t\big] = S_t e^{\mu(T-t)}\,. \tag{3.4}$$

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Proof. From Eq. (3.3) and Lemma 2.1,

$$\begin{aligned}
\hat{\mathbb{E}}_{\mathbb{P}}\big[S(T)|\mathscr{F}_t\big] &= \hat{\mathbb{E}}_{\mathbb{P}}\bigg[S_t \exp\bigg\{\Big(\mu-\lambda\rho-\frac{1}{2}\sigma^2\Big)(T-t)-\frac{1}{2}\sigma^2\varepsilon^2(T^{2H}-t^{2H}) \\
&\quad +\sigma(B_T-B_t)+\sigma\varepsilon(B_T^H-B_t^H)+\sum_{i=1}^{N_{T-t}}\ln(1+U_i)\bigg\}\bigg|\mathscr{F}_t\bigg] \\
&= \hat{\mathbb{E}}_{\mathbb{P}}\bigg[S_t\prod_{i=1}^{N_{T-t}}(1+U_i)\exp\bigg[\Big(\mu-\lambda\rho-\frac{1}{2}\sigma^2\Big)(T-t) \\
&\quad -\frac{1}{2}\sigma^2\varepsilon^2(T^{2H}-t^{2H})+\sigma(B_T-B_t)+\sigma\varepsilon(B_T^H-B_t^H)\bigg]\bigg|\mathscr{F}_t\bigg] \\
&= S_t\exp\bigg[\Big(\mu-\lambda\rho-\frac{1}{2}\sigma^2\Big)(T-t)-\frac{1}{2}\sigma^2\varepsilon^2(T^{2H}-t^{2H})\bigg] \\
&\quad \times\hat{\mathbb{E}}_{\mathbb{P}}\bigg[\prod_{i=1}^{N_{T-t}}(1+U_i)\bigg|\mathscr{F}_t\bigg]\hat{\mathbb{E}}_{\mathbb{P}}\bigg[e^{\sigma\left(B_T-B_t+\varepsilon(B_T^H-B_t^H)\right)}\bigg|\mathscr{F}_t\bigg] \\
&= S_t e^{(\mu-\lambda\rho)(T-t)}\hat{\mathbb{E}}_{\mathbb{P}}\bigg[\prod_{i=1}^{N_{T-t}}(1+U_i)\bigg|\mathscr{F}_t\bigg], \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathbb{E}}_{\mathbb{P}}\bigg[\prod_{i=1}^{N_{T-t}}(1+U_i)\bigg|\mathscr{F}_t\bigg] &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n e^{-\lambda(T-t)}}{n!}\hat{\mathbb{E}}_{\mathbb{P}}\bigg[\prod_{i=1}^{n}(1+U_i)|\mathscr{F}_t\bigg] \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n e^{-\lambda(T-t)}}{n!}\prod_{i=1}^{n}\hat{\mathbb{E}}_{\mathbb{P}}\big[(1+U_i)|\mathscr{F}_t\big] \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n e^{-\lambda(T-t)}}{n!}(1+\rho)^n \\
&= e^{\lambda\rho(T-t)}\sum_{n=0}^{\infty}\frac{\lambda^n(1+\rho)^n(T-t)^n e^{-\lambda(1+\rho)(T-t)}}{n!} \\
&= e^{\lambda\rho(T-t)}. \qquad (3.6)
\end{aligned}$$

From Eqs. (3.5) and (3.6),

$$\hat{\mathbb{E}}_{\mathbb{P}}\big[S(T)|\mathscr{F}_t\big] = S_t e^{\mu(T-t)},$$

completing the proof. □

Now let $\mathbb{Q}$ be a risk-neutral probability, $\hat{\mathbb{E}}_{\mathbb{Q}}[\cdot|\mathscr{F}_t]$ denote the quasi-conditional expectation with respect to $\mathscr{F}_t$ under the probability measure $\mathbb{Q}$, and r a risk-free interest rate. Then under the probability measure $\mathbb{Q}$, the quasi-conditional expectation of the stock price can be represented as

$$\hat{\mathbb{E}}_{\mathbb{Q}}\big[S_T|\mathscr{F}_t\big] = S_t e^{r(T-t)}. \qquad (3.7)$$

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From Eqs. (3.4) and (3.7), under the risk-neutral probability $\mathbb{Q}$ the price S_t of the stock satisfies the following jump diffusion model:

$$dS_t = S_t\Big[(r-\lambda\rho)dt + \sigma(dB_t + \varepsilon dB_t^H) + UdN_t\Big], \tag{3.8}$$

such that

$$\begin{aligned} S_T = \ & S_t \exp\Bigg[\Big(r-\lambda\rho-\frac{1}{2}\sigma^2\Big)(T-t) - \frac{1}{2}\sigma^2\varepsilon^2(T^{2H}-t^{2H}) \\ & + \sigma(B_T - B_t) + \sigma\varepsilon(B_T^H - B_t^H) + \sum_{i=1}^{N_{T-t}} \ln(1+U_i)\Bigg]. \end{aligned} \tag{3.9}$$

Let us consider a convertible bond with maturity T and face value F, with a constant coupon rate $\alpha < r$. To derive the convertible bonds pricing formula in a JMFBM market, we make the following assumptions:

(1) the capital markets are perfect with no transaction costs or taxes, and equal access to information for all investors;
(2) security trading is continuous;
(3) there are no riskless arbitrage opportunities;
(4) the convertible bond cannot be called or putted, and the issuer does not default;
(5) the convertible bond can be converted to the underlying stock at maturity date T, and the conversion price C_v is a constant.

3.2. Pricing model

Let $V(T,S_T)$ be the value of the convertible bond with maturity date T, and $P_b = Fe^{\alpha T}$ the value of a straight bond with coupon rate α. At the maturity date T:

- if the stock value FS_T/C_v on converting the bond into stock is smaller than the face value F (if the stock price S_T is less than the conversion price C_v), then the bond holder will not convert the bond into stock and receive payments including principle and coupons, and the convertible bond is just a straight bond;
- if the stock price S_T is more than conversion price C_v but FS_T/C_v is less than straight bond value P_b, the holder will not convert the bond into stock and the convertible bond value equals the straight bond value P_b;
- if the converted stock value FS_T/C_v is more than the straight bond value P_b, the bond holder must exercise the option, converting the bond into stock with a value equal to the convertible bond value.

Thus we have

$$V(T,S_T) = \begin{cases} P_b\,, & S_T < C_v\,, \\ P_b\,, & C_v \le S_T \le C_v P_b/F\,, \\ FS_T/C_v\,, & S_T > C_v P_b/F\,, \end{cases} \tag{3.10}$$

such that the convertible bond value $V(T,S_T) = \max\{P_b, FS_T/C_v\}$ at time T.

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Lemma 3.1. *For any $t \in [0,T]$, the price of a bounded $\mathscr{F}_T$-measurable claim $F(T) \in L^2(\Omega,\mathscr{F},\mathbb{Q})$ is given by*

$$F(t) = e^{-r(T-t)}\hat{\mathbb{E}}_{\mathbb{Q}}\Big[F(T)|\mathscr{F}_t\Big], \tag{3.11}$$

where r is a constant risk-free interest rate.

Proof. The proof is similar to that of Theorem A.1. in [29] and is therefore omitted. □

From Eq. (3.9), we know that $B_T - B_t \sim N(0,T-t)$ and $B_T^H - B_t^H \sim N(0,T^{2H}-t^{2H})$.

If $Z_1 \sim N(0,1)$ and $Z_2 \sim N(0,1)$, then $B_T - B_t = \sqrt{T-t}Z_1$ and $B_T^H - B_t^H = \sqrt{T^{2H}-t^{2H}}Z_2$.

Given that $\ln(1+U_i) \sim N(\mu_U,\sigma_U^2)$ and $U_1,\cdots,U_n$ is a sequence of independent identically distributed random variables, we have $\sum_{i=1}^n \ln(1+U_i) \sim N(n\mu_U, n\sigma_U^2)$; and for $Z_3 \sim N(0,1)$ we have $\sum_{i=1}^n \ln(1+U_i) = n\mu_U + \sqrt{n}\sigma_U Z_3$. Then for $Z_n \sim N(0,1)$, it follows that

$$\sigma_n\sqrt{T-t}Z_n = \sigma\sqrt{T-t}Z_1 + \sigma\varepsilon\sqrt{T^{2H}-t^{2H}}Z_2 + \sqrt{n}\sigma_U Z_3\,, \tag{3.12}$$

where $\sigma_n^2 = \sigma^2 + \sigma^2\varepsilon^2(T^{2H}-t^{2H})/(T-t) + n\sigma_U^2/(T-t)$. Consequently, letting $N_{T-t} = n$ from Eqs. (3.9) and (3.12) we obtain

$$\begin{aligned} S_T^n &= S_t \exp\left[\Big(r - \lambda\rho - \frac{1}{2}\sigma^2\Big)(T-t) - \frac{1}{2}\sigma^2\varepsilon^2(T^{2H}-t^{2H}) + n\mu_U + \sigma_n\sqrt{T-t}Z_n\right] \\ &= S_t \exp\left[\Big(r_n - \frac{1}{2}\sigma_n^2\Big)(T-t) + \sigma_n\sqrt{T-t}Z_n\right], \end{aligned} \tag{3.13}$$

where $r_n = r - \lambda\rho + (\mu_U + \frac{1}{2}\sigma_U^2)\,n/(T-t)$.

Now we give the pricing model for convertible bonds in the JMFBM environment.

Theorem 3.2. *Suppose that the stock price S_t satisfies Eq. (3.8). Then for any $t \in [0,T]$, the valuation $V(t,S_t)$ of a convertible bond with face value F and conversion price C_v is*

$$V(t,S_t) = \sum_{n=0}^{\infty} \frac{\lambda'^n (T-t)^n e^{-\lambda'(T-t)}}{n!}\left[\frac{FS_t}{C_v}N(d_{n1}) + P_b e^{-r_n(T-t)}N(-d_{n2})\right], \tag{3.14}$$

where

$$\begin{aligned} d_{n1} &= \frac{\ln\frac{FS_t}{P_bC_v} + (r_n + \frac{1}{2}\sigma_n^2)(T-t)}{\sigma_n\sqrt{T-t}}\,, & d_{n2} &= \frac{\ln\frac{FS_t}{P_bC_v} + (r_n - \frac{1}{2}\sigma_n^2)(T-t)}{\sigma_n\sqrt{T-t}}\,, \\ \sigma_n^2 &= \sigma^2 + \frac{\sigma^2\varepsilon^2(T^{2H}-t^{2H})}{T-t} + \frac{n\sigma_U^2}{T-t}\,, & r_n &= r - \lambda\rho + \frac{n}{T-t}\left(\mu_U + \frac{1}{2}\sigma_U^2\right), \\ \lambda' &= (1+\rho)\lambda\,, \qquad \rho = e^{\mu_U + \frac{1}{2}\sigma_U^2} - 1\,, & P_b &= Fe^{\alpha T}\,. \end{aligned}$$

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Proof. In a risk-neutral world, from Lemma 3.1 it follows that the valuation at $t \in [0, T]$ of a convertible bond is

$$\begin{aligned} V(t,S_t) &= e^{-r(T-t)}\hat{\mathbb{E}}_{\mathbb{Q}}\left[\max\left\{P_b, \frac{F}{C_v}S_T\right\}\middle|\mathscr{F}_t\right] \\ &= e^{-r(T-t)}\frac{F}{C_v}\hat{\mathbb{E}}_{\mathbb{Q}}\left[S_T 1_{\{S_T>\frac{P_bC_v}{F}\}}\middle|\mathscr{F}_t\right] + e^{-r(T-t)}P_b\hat{\mathbb{E}}_{\mathbb{Q}}\left[1_{\{S_T\le\frac{P_bC_v}{F}\}}\middle|\mathscr{F}_t\right] \\ &= e^{-r(T-t)}\frac{F}{C_v}I_1 + e^{-r(T-t)}P_bI_2\,, \end{aligned} \tag{3.15}$$

where $I_1 = \hat{\mathbb{E}}_{\mathbb{Q}}[S_T 1_{\{S_T>\frac{P_bC_v}{F}\}}|\mathscr{F}_t]$, and $I_2 = \hat{\mathbb{E}}_{\mathbb{Q}}[1_{\{S_T\le\frac{P_bC_v}{F}\}}|\mathscr{F}_t]$. We first compute I_2.

Setting $d^*_{n2} = [\ln\frac{P_bC_v}{FS_t} - (r_n - \frac{1}{2}\sigma_n^2)(T-t)]/(\sigma_n\sqrt{T-t})$, from Eqs. (3.9) and (3.13) we obtain

$$\begin{aligned} I_2 &= \sum_{n=0}^{\infty}\hat{\mathbb{E}}_{\mathbb{Q}}\left[1_{\{S_T^n\le\frac{P_bC_v}{F}\}}\middle|\mathscr{F}_t\right]P(N_{T-t}=n) \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^ne^{-\lambda(T-t)}}{n!}\hat{\mathbb{E}}_{\mathbb{Q}}\left[1_{\{S_T^n\le\frac{P_bC_v}{F}\}}\middle|\mathscr{F}_t\right] \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^ne^{-\lambda(T-t)}}{n!}\int_{-\infty}^{d^*_{n2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^ne^{-\lambda(T-t)}}{n!}N(d^*_{n2}) \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^ne^{-\lambda(T-t)}}{n!}N(-d_{n2})\,, \end{aligned} \tag{3.16}$$

where $d_{n2} = [\ln\frac{FS_t}{P_bC_v} + (r_n - \frac{1}{2}\sigma_n^2)(T-t)]/(\sigma_n\sqrt{T-t})$. Now we compute I_1.

From Eqs. (3.9), (3.13) and (3.15) we obtain

$$\begin{aligned} I_1 &= \sum_{n=0}^{\infty}\hat{\mathbb{E}}_{\mathbb{Q}}\left[S_T^n 1_{\{S_T^n>\frac{P_bC_v}{F}\}}\middle|\mathscr{F}_t\right]P(N_{T-t}=n) \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^ne^{-\lambda(T-t)}}{n!}\hat{\mathbb{E}}_{\mathbb{Q}}\left[S_t e^{(r_n-\frac{1}{2}\sigma_n^2)(T-t)+\sigma_n\sqrt{T-t}Z_n}1_{\{S_T^n>\frac{P_bC_v}{F}\}}\middle|\mathscr{F}_t\right] \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^ne^{-\lambda(T-t)}}{n!}S_t e^{r_n(T-t)}\int_{d^*_{n2}}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sigma_n\sqrt{T-t})^2}{2}}dx \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^ne^{-\lambda(T-t)}}{n!}S_t e^{r_n(T-t)}N(d_{n1})\,, \end{aligned} \tag{3.17}$$

where $d_{n1} = [\ln\frac{FS_t}{P_bC_v} + (r_n + \frac{1}{2}\sigma_n^2)(T-t)](\sigma_n\sqrt{T-t})$.

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The valuation of a convertible bond can therefore be represented as

$$V(t,S_t)=\sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n e^{-\lambda(T-t)}}{n!}\left[\frac{FS_t}{C_v}e^{-r(T-t)}e^{r_n(T-t)}N(d_{n1})+P_b e^{-r(T-t)}N(-d_{n2})\right]. \tag{3.18}$$

Since $\ln(1+U)\sim N(\mu_U,\sigma_U)$, we get $1+\rho=e^{\mu_U+\frac{1}{2}\sigma_U^2}$, we have

$$\begin{aligned} e^{-r(T-t)} &= (1+\rho)^n e^{-\lambda\rho(T-t)}e^{-[r-\lambda\rho+\frac{n}{T-t}(\mu_U+\frac{1}{2}\sigma_U^2)](T-t)} \\ &= (1+\rho)^n e^{-\lambda\rho(T-t)}e^{-r_n(T-t)}, \end{aligned} \tag{3.19}$$

hence the valuation of a convertible bond can be written as

$$V(t,S_t)=\sum_{n=0}^{\infty}\frac{\lambda'^n(T-t)^n e^{-\lambda'(T-t)}}{n!}\left[\frac{FS_t}{C_v}N(d_{n1})+P_b e^{-r_n(T-t)}N(-d_{n2})\right]$$

where $\lambda'=(1+\rho)\lambda$. □

4. Sensitivity Analysis of Convertible Bond Pricing

We now discuss the sensitivity of the convertible bond price with respect to various changing parameters. Measuring these sensitivities is a basic tools in risk management, as trading convertible bonds without that knowledge can result in high losses.

Theorem 4.1. *Let $V=V(t,S_t)$ be the price of a convertible bond at time $t\in[0,T]$. With reference to Theorem* 3.2*, the influence of the common parameters can be represented as*

$$\Delta=\frac{\partial V}{\partial S}=\sum_{n=0}^{\infty}\frac{\lambda'^n(T-t)^n e^{-\lambda'(T-t)}}{n!}\frac{F}{C_v}N(d_{n1}),$$

$$\Gamma=\frac{\partial^2 V}{\partial S^2}=\sum_{n=0}^{\infty}\frac{\lambda'^n(T-t)^n e^{-\lambda'(T-t)}}{n!}\frac{F}{C_v S_t\sigma_n\sqrt{T-t}}N'(d_{n1}),$$

$$\nabla_F=\frac{\partial V}{\partial F}=\sum_{n=0}^{\infty}\frac{\lambda'^n(T-t)^n e^{-\lambda'(T-t)}}{n!}\left[\frac{S_t}{C_v}N(d_{n1})+e^{\alpha T}e^{-r_n(T-t)}N(-d_{n2})\right],$$

$$\nabla_{C_v}=\frac{\partial V}{\partial C_v}=\sum_{n=0}^{\infty}\frac{\lambda'^n(T-t)^n e^{-\lambda'(T-t)}}{n!}\left(-\frac{FS_t}{C_v^2}N(d_{n1})\right),$$

$$\nu_\sigma=\frac{\partial V}{\partial\sigma}=\sum_{n=0}^{\infty}\frac{\lambda'^n(T-t)^n e^{-\lambda'(T-t)}}{n!}P_b e^{-r_n(T-t)}N'(-d_{n2})\frac{\sigma}{\sigma_n}\left(1+\varepsilon^2\frac{T^{2H}-t^{2H}}{T-t}\right)\sqrt{T-t},$$

$$\rho_r=\frac{\partial V}{\partial r}=\sum_{n=0}^{\infty}\frac{\lambda'^n(T-t)^n e^{-\lambda'(T-t)}}{n!}\left[-P_b e^{-r_n(T-t)}(T-t)N(-d_{n2})\right].$$

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Proof. The proof is immediate from the chain rule of a compound function. □

Remark 4.1. From Theorem 4.1, we can easily deduce that $\Delta \geq 0$, $\nabla_F \geq 0$, $\nu_\sigma \geq 0$, $\Gamma \geq 0$, $\nabla_{C_V} \leq 0$ and $\rho_r \leq 0$, so the valuation of a convertible bond is an increasing function of S_t, F and σ, and a decreasing function of C_v and ρ_r.

Theorem 4.2. *Suppose $V = V(t, S_t)$ is the price of a convertible bond at time $t \in [0, T]$. From Theorem* 3.2*, the influence of the Hurst parameter can be expressed as*

$$\frac{\partial V}{\partial H} = \sum_{n=0}^{\infty} \frac{\lambda'^n (T-t)^n e^{-\lambda'(T-t)}}{n!} P_b e^{-r_n(T-t)} N'(-d_{n2}) \frac{\sigma^2 \varepsilon^2}{\sigma_n \sqrt{T-t}} (T^{2H} \ln T - t^{2H} \ln t) . \quad (4.1)$$

Remark 4.2. From Theorem 4.2, we can easily deduce that $\partial V / \partial H \geq 0$, so the valuation of a convertible bond will increase with increasing Hurst parameter.

Theorem 4.3. *Let $V = V(t, S_t)$ be the price of a convertible bond at time $t \in [0, T]$. From Theorem* 3.2*, the influence of the jump parameters can be represented as*

$$\begin{aligned}
\frac{\partial V}{\partial \lambda} &= \sum_{n=0}^{\infty} \frac{\lambda'^n (T-t)^n e^{-\lambda'(T-t)}}{n!} (1+\rho) \left(\frac{n}{\lambda'} - (T-t) \right) \left[\frac{FS_t}{C_v} N(d_{n1}) + P_b e^{-r_n(T-t)} N(-d_{n2}) \right] \\
&\quad + \sum_{n=0}^{\infty} \frac{\lambda'^n (T-t)^n e^{-\lambda'(T-t)}}{n!} P_b e^{-r_n(T-t)} \rho (T-t) N(-d_{n2}) , \\
\frac{\partial V}{\partial \mu_U} &= \sum_{n=0}^{\infty} \frac{\lambda'^n (T-t)^n e^{-\lambda'(T-t)}}{n!} \lambda e^{\mu_U + \frac{1}{2}\sigma_U^2} \left(\frac{n}{\lambda'} - (T-t) \right) \left[\frac{FS_t}{C_v} N(d_{n1}) + P_b e^{-r_n(T-t)} N(-d_{n2}) \right] \\
&\quad + \sum_{n=0}^{\infty} \frac{\lambda'^n (T-t)^n e^{-\lambda'(T-t)}}{n!} P_b e^{-r_n(T-t)} \left[\left(\lambda e^{\mu_U + \frac{1}{2}\sigma_U^2} - \frac{n}{T-t} \right) (T-t) N(-d_{n2}) \right] , \\
\frac{\partial V}{\partial \sigma_U} &= \sum_{n=0}^{\infty} \frac{\lambda'^n (T-t)^n e^{-\lambda'(T-t)}}{n!} \lambda \sigma_U e^{\sigma_U + \frac{1}{2}\sigma_U^2} \left(\frac{n}{\lambda'} - (T-t) \right) \left[\frac{FS_t}{C_v} N(d_{n1}) + P_b e^{-r_n(T-t)} N(-d_{n2}) \right] \\
&\quad + \sum_{n=0}^{\infty} \frac{\lambda'^n (T-t)^n e^{-\lambda'(T-t)}}{n!} P_b e^{-r_n(T-t)} \left[\left(\lambda \sigma_U e^{\mu_U + \frac{1}{2}\sigma_U^2} - \frac{n\sigma_U}{T-t} \right) (T-t) N(-d_{n2}) \right. \\
&\quad \left. + \frac{n\sigma_U}{\sigma_n \sqrt{T-t}} N'(-d_{n2}) \right] ,
\end{aligned}$$

Proof. The proof is again immediate. □

Fig. 1 intuitively displays the influence of the jump parameters on the valuation of the convertible bond.

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Table 1: Chosen parameter values.

Model	α	r	σ	F	C_v	T	t	H	ε	λ	μ_U	σ_U
BM	0.021	0.0387	0.125	100	8.22	5	0					
FBM	0.021	0.0387	0.125	100	8.22	5	0	0.762				
MFBM	0.021	0.0387	0.125	100	8.22	5	0	0.762	1			
$JMBM^a$	0.021	0.0387	0.125	100	8.22	5	0	0.762	1	1.5	0.00067	0.0012
$JMBM^b$	0.021	0.0387	0.125	100	8.22	5	0	0.762	1	7.5	-0.00067	0.0018

Table 2: Pricing results from the different models.

S	V_{B-M}	V_{F-M}	V_{M-M}	$V^{*\,a}_{J-M}$	$V^{*\,b}_{J-M}$
5	92.1919	92.4439	95.9998	96.0058	96.0128
6	94.2145	95.5998	100.3687	100.4107	100.4225
7	98.4865	100.8985	106.2863	106.2901	106.2963
8	105.1724	108.0915	113.5531	113.5629	113.5832
9	113.8785	116.7764	121.9306	121.9526	122.0285
10	124.0144	126.5580	131.1913	131.2691	131.2957

*The maximum number of iterations is 100.

[a]The parameters for this calculations are in the fourth line of Table 1.

[b]The parameters for calculations are in the fifth line of Table 1.

5. Numerical Experiment

We now discuss the implementation of our jump mixed fractional Brownian motion pricing model, and show the effects of the Hurst and jump parameters. Let us first present simulation results from different pricing models with appropriately chosen parameters, to compare the theoretical convertible bond prices rendered by: (1) the pure Brownian motion (BM) model, (2) the pure fractional Brownian motion (FBM) model, (3) the pure mixed fractional Brownian motion (MFBM) model, and (4) our jump mixed fractional Brownian motion (JMBFM) model. The codes are written in Matlab.

Table 1 presents the parameters adopted. The first row displays the parameters for the BM model, the second row for the FBM model, he third row for the MFBM model, and the fourth row (which has low jump parameters) provides the parameters for calculating the prices by our JMFBM model. The fifth row has high jump parameters, for a second calculation using our JMFBM model.

The computed prices from the respective models are presented in Table 2. Here S denotes the underlying stock price, V_{B-M} the prices computed by the BM model, V_{F-M} the prices from the FBM model, V_{M-M} the prices from the MFBM model, and V_{J-M} the prices computed from the JMFBM model. Comparing the columns V_{B-M}, V_{F-M}, V_{M-M} and V^{*a}_{J-M}, we see that the prices increase as the stock price increases, and column V^{*a}_{J-M} shows higher prices than those obtained from the BM, FBM and MFBM models. Table 2 also shows that the convertible bond prices obtained by the MFBM and JMFBM models are close to each

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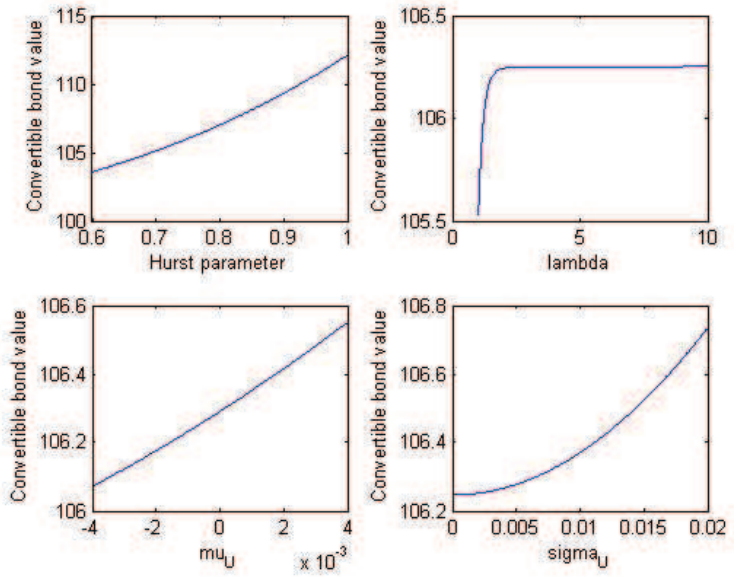


Figure 1: Value of the convertible bond.

Table 3: BCCB information.

Convertible bond code	113001	Issue date	2/6/2010	Listing date	18/6/2010
Underlying stock code	601988	Expiration date	2/6/2016	Face value	100RMB
First conversion price	4.02	Duration	6 years	Coupon rate	0.005

other, which is mainly because the jump parameters are very low. From columns V_{J-M}^{*a} and V_{J-M}^{*b} in Table 2, we find that the convertible bond prices for the large jump parameters model are notably higher.

We next present convertible bond values obtained from our JMFBM model for different parameters. Fig. 1 displays the values of a convertible bond versus its parameters H, λ, μ_U and σ_U. The default parameters were $S = 7$, $F = 100$, $\alpha = 0.021$, $r = 0.0387$, $\sigma = 0.125$, $H = 0.762$, $C_v = 8.22$, $\lambda = 7.5$, $\mu_U = -0.00067$, $\sigma_U = 0.0018$, $\varepsilon = 1$, $T = 5$, and $t = 0$. We see that the convertible bond value is an increasing function with respect to H, λ, μ_U and σ_U.

6. Empirical Analysis

To test the performance of our model, we consider one of the largest convertible bonds issued in China, the Bank of China's convertible bond (BCCB). For our empirical study, we adopt the underlying stock's closing prices and the convertible bond's market values from 2 June 2010 to 31 December 2012, obtained from CSMAR (China Stock Market & Accounting Research Database). Information on the BCCB is shown in Table 3. We first need to estimate the required thirteen parameters for our JMFBM model. Fortunately, from Table 3 we have

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$F = 100$, $\alpha = 0.005$, $C_v = 4.02$, and $T = 6$. According to historical data, we have the riskless interest rate $r = 0.035$ in 2012, and the underlying stock's closing prices from 2 June 2010 to 31 December 2012. We chose $t \in [0, 2.5]$ and $\varepsilon = 1$, and estimated the remaining parameters as follows.

(1) To estimate the volatility of stock price empirically, let $S(i)$ = closing price of the stock at the end of i-th interval, such that with $W(i) = \ln(S(i)/S(i-1))$ we have

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} W^2(i) - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} W(i)\right)^2} \Big/ \sqrt{\frac{1}{250}}\,.$$

Using this method, we obtain $\sigma = 0.1469$.

(2) Using R/S analysis methodology, the estimated Hurst parameter is $H = 0.758$ — cf. Ref. [31].

(3) Obtain the jump parameters by calculating the underlying stock returns over the estimation period.

The algorithm involves three steps:

- Obtain the first six sample moments k_s, $s = 1$ to 6 — viz. $k_s = (1/T)\sum_{t=1}^{T}(\Delta G_t)^s$, where ΔG_t is the change in the natural log of the stock price during time t and T is the number of days in the estimation period.
- Using the sample moments, determine the required sample cumulants, where relationships between the sample moments and cumulants are [32]

$$\begin{aligned} y_1 &= k_1, \quad y_2 = k_2 - k_1^2, \quad y_4 = k_4 - 4k_3k_1 - 3k_2^2 + 12k_2k_1^2 - 6k_1^4, \\ y_6 &= k_6 - 6k_5k_1 - 15k_4k_2 + 30k_4k_1^2 - 10k_3^2 + 12k_3k_2k_1 - 120k_3k_1^3 \\ &\quad + 30k_2^3 - 270k_2^2k_1^2 + 360k_2k_1^4 - 120k_1^6. \end{aligned} \tag{6.1}$$

- Using the sample cumulates, obtain the estimating parameters

$$\lambda = 25y_4^3/3y_1^2, \qquad \mu_U = y_2 - 5y_4^2/3y_6, \qquad \sigma_U^2 = \log(y_6/5y_4 + 1). \tag{6.2}$$

Using this method, we get $\lambda = 1.0204$, $\mu_U = 3.9819 \times 10^{-5}$, $\sigma_U = 0.0159$.

Putting the above parameters into the BM model and our JMFBM model, we obtain the convertible bond values from 18 June 2010 to 31 December 2012, and then compare these values with actual market values. Fig. 2 shows the results from the BM model and our JMFBM model. The main features are that our model values are mostly higher than the BM model values, and the actual market values are much closer to the values computed via our JMFBM model than from the BM model, indicating that our model is more acceptable.

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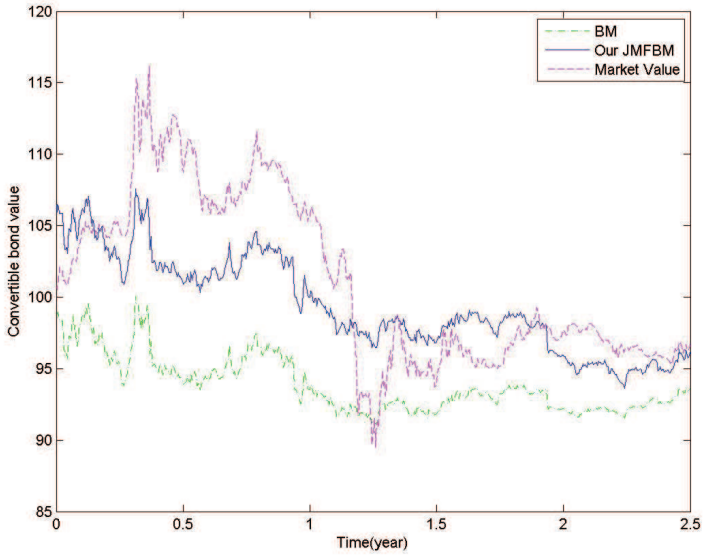


Figure 2: Comparison of the market values with our JMFBM model values and the BM model values.

7. Conclusion

Convertible bonds are popular financial derivatives, with an essential role in the Chinese financial market. Pricing them efficiently and accurately is very important, in both theory and practice. To capture the long memory and discontinuous property, this article focuses on the problem of pricing convertible bonds in a jump mixed fractional Brownian environment. After formulating our convertible bond pricing model, we discussed the pricing formula for the mixed fractional Brownian motion with jumps involved. We compared theoretical values obtained from other available valuation models with the values from our model in a numerical simulation. Finally, we presented an empirical study using actual data from the Chinese convertible bond market, which showed that our JMFBM model is more acceptable.

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