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Theory for two-dimensional electron and ion Bernstein–Greene–Kruskal modes in a magnetized plasma

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Abstract. A theory for two-dimensional electron and ion Bernstein–Greene–Kruskal (BGK) modes in a magnetized space plasma is presented. The BGK modes are constructed using the energy and the canonical angular momentum of the particles, which are conserved in a cylindrically symmetric potential. The typical length scale of the BGK modes is of the same order or larger than the thermal gyroradius of the particles. The results are relevant for understanding the properties of observed localized structures in the Earth's magnetosphere and auroral zone, as well as in laboratory magnetoplasmas.

1. Introduction

Localized electrostatic pulses are frequently observed by various spacecrafts in the Earth's magnetosphere and auroral acceleration region. These pulses were first recorded in the frequency domain and were denoted broad electrostatic noise (BEN). When the waveforms of the pulses were resolved in the time domain, it was recognized that they were most probably coherent nonlinear structures [1, 2]. These pulses have been interpreted as electron and ion Bernstein–Greene–Kruskal (BGK) modes/holes. Pulses that are associated with a negative potential usually propagate along the geomagnetic field lines with a speed comparable to the ion thermal speed and are interpreted as ion holes, while the pulses with a positive potential propagate with a much larger speed, in the range of the electron thermal speed, and are associated with trapped electron islands/electron holes. The amplitudes of the electric potential of such localized holes are usually somewhat smaller than the thermal energy of the particles. Some measurements indicate that the electrostatic pulses are localized not only along the geomagnetic field lines, but also perpendicular to the geomagnetic field. While the electric field signal along the geomagnetic field lines usually exhibit a bipolar structure in time, the electric field components perpendicular to the geomagnetic field line sometimes show a monopolar electric signal with an amplitude comparable with the field signal along the geomagnetic field lines. To explain these observations, one is led to the conclusion that the electrostatic pulses have an extension in multiple space dimensions. In the laboratory, electrostatic double layers and electron holes have been observed in magnetized plasmas [3, 4], as well as ion holes behind ion-acoustic

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shocks [5]. Kinetic effects have been considered previously in theoretical treatments of nonlinear structures. Sagdeev [6] discussed the reflection of ions by the positive potential of ion acoustic shocks, while Gurevich [7] considered solitary ion-acoustic waves with trapped electrons. Undamped periodic electron waves involving trapped electrons were found by Holloway and Dorning [8,9] and by Schamel [10], who also constructed solitary electron and ion holes [11–13] in one-dimensional space.

On the other hand, Ng and Bhattacharjee showed that it is not possible to construct localized structures that depend only on the energy in two or three space dimensions [14]. Instead, they used a combination of the energy integral and the conserved angular momentum to construct three-dimensional spherically symmetric electron BGK modes in an unmagnetized plasma [14], and two-dimensional cylindrically symmetric electron BGK modes in a magnetized plasma [15]. Eliasson and Shukla [16] used the conserved canonical momentum to construct one-dimensional electron and ion BGK modes perpendicular to the external magnetic field. The existence of large-amplitude solitary ion-acoustic waves in a magnetized plasma was reported by Shukla and Yu [17]. Krasovsky et al. [18] discussed the trapping of particles in an external three-dimensional potential well in the presence of an external magnetic field, and predicted existence criteria for trapped particle BGK modes in a magnetized plasma. In strong magnetic fields, where the electrons move strictly along the magnetic field lines, Chen et al. [19] constructed three-dimensional electron holes while, by using the gyro-kinetic and drift-kinetic approximations, Jovanović et al. [20] developed a theory for electron phase-space holes in a magnetized plasma in the small-amplitude limit. A review of recent theories of electron and ion holes is contained in Eliasson and Shukla [21].

In this paper, we present a theory for finite-amplitude electron and ion BGK modes that are associated with a localized bell-shaped potential in two-dimensional space. They are constructed using the conserved particle energy and canonical angular momentum of the particles, from which exact solutions of the Vlasov equation are found. By integrating the Vlasov equation over velocity space, the particle densities are obtained as functions of the radial coordinate and the self-consistent potential, which is found by solving the Poisson equation numerically.

2. Theory

The time-stationary Vlasov equation in a magnetized plasma reads

$$\mathbf{v}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(-\nabla\phi + \mathbf{v}\times\mathbf{B}_0/c)\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} = 0, \tag{1}$$

for particle species α (α equals e for electrons and i for ions) where ϕ and B_0 is the electrostatic potential and the strength of the geomagnetic field, respectively, c is the speed of light in vacuum, m_α is the particle mass, and $q_\mathrm{e} = -e$ ($q_\mathrm{i} = e$) is the electron (ion) charge where e is the magnitude of the electron charge. Equation (1) is governed by the equations for the particle trajectories

$$\frac{d\mathbf{v}}{dt} = \frac{q_\alpha}{m_\alpha}(-\nabla\phi + \mathbf{v}\times\mathbf{B}_0/c), \tag{2}$$

and

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}. \tag{3}$$

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The equations of motion (2) and (3) admit the first constant of motion (energy integral)

$$\mathscr{E}_\alpha = \frac{m_\alpha v^2}{2} + q_\alpha \phi, \tag{4}$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$. Equation (4) expresses the invariance of (2) and (3) under translation in time.

In a magnetized plasma, we have a natural symmetry axis along the direction of the magnetic field lines. Letting the geomagnetic field to be directed along the z-axis, i.e. $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, we can find constant of motions in some simple geometries. For a cylindrically symmetric potential, $\phi = \phi(z, r_\perp)$, we have the constant of motion

$$L_z = [\mathbf{r} \times (m_\alpha \mathbf{v} - q_\alpha \mathbf{r} \times \hat{\mathbf{z}} B_0/2c)] \cdot \hat{\mathbf{z}}, \tag{5}$$

which expresses the conservation of the z component of the canonical angular momentum. Writing out the components of (5), we have

$$\begin{aligned} L_{z\alpha} &= m_\alpha [v_y x - v_x y + \omega_{c\alpha}(x^2 + y^2)/2] \\ &= -\frac{m_\alpha}{2\omega_{c\alpha}} [v_x^2 + v_y^2 - (v_x - \omega_{c\alpha} y)^2 - (v_y + \omega_{c\alpha} x)^2], \end{aligned} \tag{6}$$

where we have denoted the electron and ion gyrofrequencies (including signs) as $\omega_{ce} = -eB_0/m_e c$ and $\omega_{ci} = eB_0/m_i c$, respectively. A new conserved quantity can be formed as a linear combination of (4) and (6) as

$$\begin{aligned} \mathscr{E}_{L\alpha} &= \mathscr{E}_\alpha + \omega_c L_z \\ &= \frac{m_\alpha}{2} [(v_x - \omega_{c\alpha} y)^2 + (v_y + \omega_{c\alpha} x)^2 + v_z^2] + q_\alpha \phi, \end{aligned} \tag{7}$$

which has a similar form as $\mathscr{E}_\alpha$. The conserved quantities $\mathscr{E}_\alpha$ and $\mathscr{E}_{L\alpha}$ will be used to construct multi-dimensional BGK modes in a plasma.

We will here concentrate on the cylindrically symmetric case governed by (4) and (7). If we introduce cylindrical coordinates in $\mathbf{r}$ space, namely $x = r_\perp \cos(\varphi)$, $y = r_\perp \sin(\varphi)$, $v_x = v_\perp \cos(\varphi) - v_\varphi \sin(\varphi)$, $v_y = v_\perp \sin(\varphi) + v_\varphi \cos(\varphi)$, and make the change of variables $v_\perp = u\cos(\theta)$ and $v_z = u\sin(\theta)$, then the conserved quantities (4) and (7) are expressed as

$$\mathscr{E}_\alpha = \frac{m_\alpha}{2}(u^2 + v_\varphi^2) + q_\alpha \phi, \tag{8}$$

and

$$\mathscr{E}_{L\alpha} = \frac{m_\alpha}{2}[u^2 + (v_\varphi - \omega_c r_\perp)^2] + q_\alpha \phi. \tag{9}$$

Integrating $f_\alpha(\mathscr{E}_\alpha, \mathscr{E}_{L\alpha})$ over velocity space gives the particle number density n_α as a function of $r_\perp$ and ϕ,

$$n_\alpha(r_\perp, \phi) = 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} f_\alpha u \, du \, dv_\varphi. \tag{10}$$

The potential ϕ is found by solving the Poisson equation

$$\nabla^2 \phi = \frac{1}{r_\perp} \frac{\partial}{\partial r_\perp} \left(r_\perp \frac{\partial \phi}{\partial r_\perp} \right) + \frac{\partial^2 \phi}{\partial z^2} = 4\pi e(n_e - n_0), \tag{11}$$

with the boundary condition $\partial\phi/\partial r_\perp = 0$ at $r_\perp = 0$ and $\phi = 0$ at $r_\perp = \infty$.

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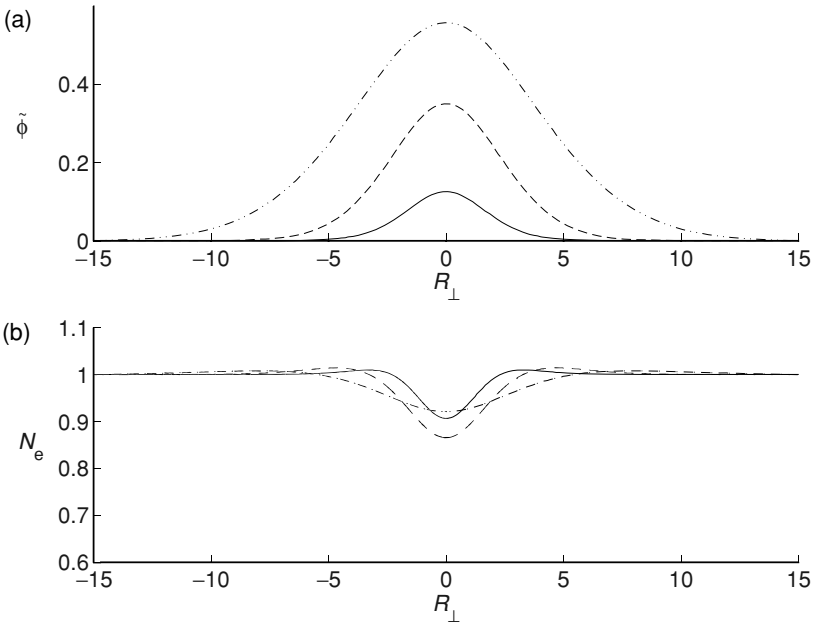


Figure 1. The profiles of (a) the normalized electrostatic potential $\widetilde{\phi}$ and (b) the electron density N_e associated with a cylindrically symmetric kinetic structure/electron hole as a function of the radial coordinate $R_\perp$, with immobile ions ($N_i = 1$). The amplitude parameter was chosen as $a_e = 0.5$, the frequency ratio $\omega_{pe}/|\omega_{ce}| = 4$, and the length scale parameter $D_e = 1$ (solid lines), $D_e = 2$ (dashed lines), and $D_e = 4$ (dash-dotted lines).

In a similar manner as in [14–16], we will construct localized kinetic structures of the form

$$f_\alpha = \frac{n_0}{(2\pi)^{3/2}V_{T\alpha}^3}\exp\left(-\frac{\mathscr{E}_\alpha}{m_\alpha V_{T\alpha}^2}\right)\left[1 - a_\alpha \exp\left(-\frac{\mathscr{E}_{L\alpha}}{D_\alpha^2 m_\alpha V_{T\alpha}^2}\right)\right], \tag{12}$$

where a_α is the amplitude and D_α is the scale size of the localized structure. Integrating the particle distribution function (12) over velocity space, we obtain from (10) the number density

$$n_\alpha = n_0 \exp\left(-\frac{q_\alpha \phi}{T_\alpha}\right)\left[1 - \mathscr{A}_\alpha \exp\left(-\frac{r_\perp^2}{2\lambda_{D\alpha}^2 \mathscr{R}_\alpha^2} - \frac{1}{D_\alpha^2}\frac{q_\alpha \phi}{T_\alpha}\right)\right], \tag{13}$$

where $\lambda_{D\alpha} = V_{T\alpha}/\omega_{p\alpha}$ is the electron/ion Debye radius, $\mathscr{R}_\alpha = (1+D_\alpha^2)^{1/2}\omega_{p\alpha}/|\omega_{c\alpha}|$, and $\mathscr{A}_\alpha = a_\alpha D_\alpha^3/(1+D_\alpha^2)^{3/2}$. We see that $\mathscr{R}_\alpha$ is larger than $\omega_{p\alpha}/|\omega_{c\alpha}|$ for all choices of D_α, which means that the perpendicular scale length of the kinetic structures must be of the same order or larger than the particle's thermal gyroradius. Inserting the particle densities (13) into the Poisson equation (11) and normalizing the variables, we finally obtain

$$\frac{1}{R_\perp}\frac{\partial}{\partial R_\perp}\left(R_\perp \frac{\partial \widetilde{\phi}}{\partial R_\perp}\right) + \frac{\partial^2 \widetilde{\phi}}{\partial Z^2} = N_e - N_i, \tag{14}$$

with

$$N_e = \exp(\widetilde{\phi})\left[1 - \mathscr{A}_e \exp\left(-\frac{R_\perp^2}{2\mathscr{R}_e^2} + \frac{\widetilde{\phi}}{D_e^2}\right)\right], \tag{15}$$

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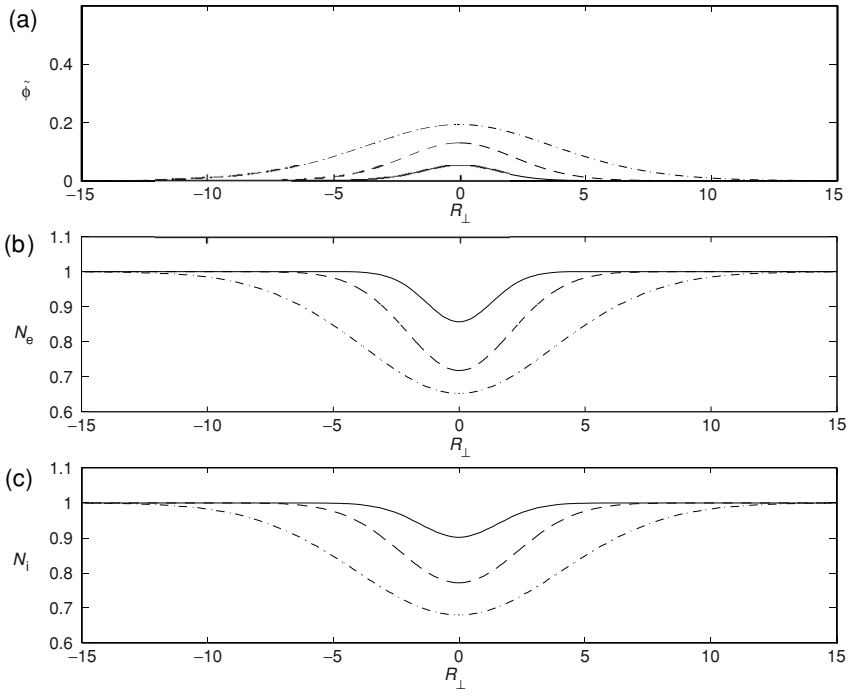


Figure 2. The profiles of (a) the normalized electrostatic potential $\widetilde{\phi}$, (b) the electron density N_e, and (c) the ion density N_i associated with a cylindrically symmetric kinetic structure/electron hole as a function of the radial coordinate $R_\perp$, with Boltzmann distributed ions ($a_i = 0$). The amplitude parameter for the electrons was chosen as $a_e = 0.5$, the frequency ratio $\omega_{pe}/|\omega_{ce}| = 4$, and the length scale parameter $D_e = 1$ (solid lines), $D_e = 2$ (dashed lines), and $D_e = 4$ (dash-dotted lines). We used the electron to ion temperature ratio $\tau = 2$.

and

$$N_i = \exp(-\tau\widetilde{\phi})\left[1 - \mathscr{A}_i \exp\left(-\frac{\tau R_\perp^2}{2\mathscr{R}_i^2} - \frac{\tau\widetilde{\phi}}{D_i^2}\right)\right], \tag{16}$$

where $N_e = n_e/n_0$, $N_i = n_i/n_0$, $R_\perp = r_\perp/\lambda_{De}$, $Z = z/\lambda_{De}$, $\widetilde{\phi} = e\phi/T_e$, and $\tau = T_e/T_i$.

3. Numerical results

The fully three-dimensional problem is problematic both theoretically and numerically, as it involves the integration of the Poisson equation in $R_\perp$ and Z space. Here we will therefore concentrate on the purely cylindrical problem ($\partial/\partial Z = 0$) and postpone the fully three-dimensional problem for a future study. We have solved (16) numerically with $\partial/\partial Z = 0$ for different sets of parameters and have presented the results in Figs 1–3. In Fig. 1, we assume that the ions are immobile and uniformly distributed so that $N_i = 1$. We have taken a positive value $a_e = 0.5$ for the amplitude and different values for the width parameter D_e. We see that the structure is characterized by a localized positive electrostatic potential. The positive potential attracts the electrons, which are hindered in neutralizing the plasma by the external magnetic field. In Fig. 2, we assume that the ions are Boltzmann

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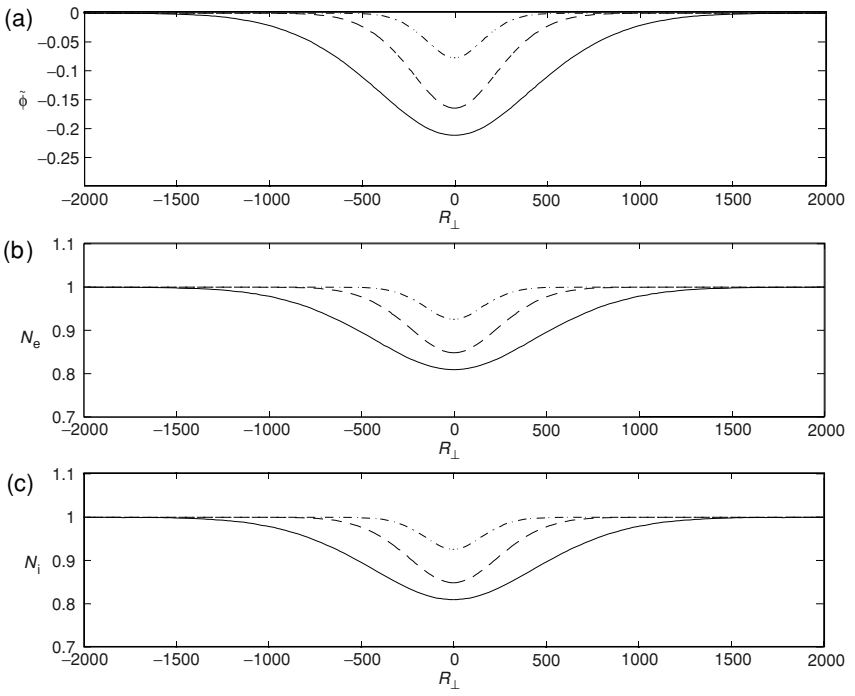


Figure 3. The profiles of (a) the normalized electrostatic potential $\widetilde{\phi}$, (b) the electron density N_e, and (c) the ion density N_i associated with a cylindrically symmetric kinetic structure/ion hole as a function of the radial coordinate $R_\perp$, with Boltzmann distributed electrons ($a_e = 0$). The amplitude parameter for the ions was chosen as $a_i = 0.5$, the frequency ratio $\omega_{pe}/|\omega_{ce}| = 4$, and the length scale parameter $D_i = 1$ (solid lines), $D_i = 2$ (dashed lines), and $D_i = 4$ (dash-dotted lines). We used the electron to ion temperature ratio $\tau = 2$ and the ion (proton) to electron mass ratio $m_i/m_e = 1836$.

distributed, where we have taken the electron–ion temperature ratio $\tau = 2$ and $a_i = 0$. We have used different length scales D_e for the electrons while keeping $a_e = 0.5$ constant. We observe that the localized positive potential is associated with a depletion of both the electrons and ions, so that the ions are partially neutralizing the plasma. Therefore, the amplitude of the potential becomes smaller than for the case of immobile ions in Fig. 1. In Fig. 3, we display a case for Boltzmann distributed electrons ($A_e = 0$), while the parameters for the ions were chosen as $a_i = 0.5$ and $\tau = 2$. We used the mass ratio for protons $m_i/m_e = 1836$ and a magnetic field such that $\omega_{pe}/|\omega_{ce}| = 4$. Here, we see a localized negative potential which is on the length scale of the ion thermal gyroradius, and which is associated with a depletion of both ion and electron densities. Here, the electrons almost completely neutralize the plasma.

4. Conclusions

In conclusion, we have presented a theory for two-dimensional electron and ion BGK modes in a magnetized plasma. The BGK modes are constructed with the help of the conserved energy and canonical angular momentum of the particles. These nonlinear structures have a transverse length scale comparable to, or larger than,

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the particle's thermal gyroradius. The ion BGK modes are large-scale structures, larger than the ion thermal gyroradius, and are thus almost neutralized by the electrons. The present results should be helpful in understanding the properties of kinetic structures in space and laboratory plasmas.

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