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Pure Projective Approximations

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Abstract

A notion of good behavior is introduced for a definable subcategory $\mathcal{X}$ of left R-modules. It is proved that every finitely presented left R-module has a pure projective left $\mathcal{X}$-approximation if and only if the associated torsion class $\mathcal{T}_{\mathcal{X}}$ of finite type in the functor category (mod-R, Ab) is coherent, i.e., the torsion subobject of every finitely presented object is finitely presented. This yields a bijective correspondence between such well-behaved definable subcategories and preenveloping subcategories $\mathcal{C}$ of the category Add(R-mod) of pure projective left R-modules. An example is given of a preenveloping subcategory $\mathcal{C} \subseteq$ Add(R-mod) that does not arise from a covariantly finite subcategory of finitely presented left R-modules. As a general example of this good behavior, it is shown that if R is a ring over which every left cotorsion R-module is pure injective, then the smallest definable subcategory $\mathcal{X}(R\text{-proj})$ containing every finitely generated projective module is well-behaved.

0. *Introduction*

Let R be an associative ring with identity and consider a subcategory $\mathcal{B} \subseteq R$-mod, closed under finite direct sums and summands, of the category of finitely presented left R-modules. Lenzing [**14**] proved that a left R-module M belongs to $\varinjlim \mathcal{B}$, the category of direct limits of objects from $\mathcal{B}$, if and only if there is a pure epimorphism $\varinjlim \oplus_i B_i \to M$ with $B_i \in \mathcal{B}$. Crawley–Boevey [**8**, section 4] and H. Krause [**13**, proposition 3·11] used this characterization to determine when the subcategory $\varinjlim \mathcal{B} \subseteq R$-Mod is definable, i.e., closed under direct limits, direct products and pure submodules. They showed that $\varinjlim \mathcal{B} \subseteq R$-Mod is definable if and only if the subcategory $\mathcal{B} \subseteq R$-mod is covariantly finite [**6**], i.e., preenveloping [**10**, p. 129] in R-mod. Among the definable categories, such categories $\mathcal{X} = \varinjlim \mathcal{B} \subseteq R$-Mod may be characterized [**13**, theorem 3·12] by the property that every finitely presented module has a finitely presented left $\mathcal{X}$-approximation. The results of this paper suggest that the natural setting for these ideas is not within the category of finitely presented modules but, rather, in the wider context of the category Add(R-mod) of pure projective left R-modules.

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In this broader context, one abandons the characterization of the objects of $\lim \mathcal{B}$ as direct limits in favor of the characterization in terms of pure epimorphisms: if $\mathcal{C} \subseteq \vec{\mathrm{Add}}(R\text{-mod})$ is a subcategory closed under summands and arbitrary direct sums, then, as in [**1**], we let $\tilde{\mathcal{C}} \subseteq R$-Mod be the category of those modules M for which there is a pure epimorphism $g : C \to M$, with $C \in \mathcal{C}$. The first result decribes the situation when $\tilde{\mathcal{C}}$ is definable.

THEOREM 4. *The following are equivalent for a subcategory* $\mathcal{C} \subseteq \mathrm{Add}(R\text{-mod})$ *closed under arbitrary direct sums and summands*:

(i) $\mathcal{C}$ *is a preenveloping subcategory of* $\mathrm{Add}(R\text{-mod})$;
(ii) *the associated subcategory* $\tilde{\mathcal{C}} \subseteq R$-Mod *is definable*;
(iii) $\tilde{\mathcal{C}}$ *is a preenveloping subcategory of* R-Mod.

Conversely, the next result describes the situation when a definable subcategory $\mathcal{X}$ is of the form $\tilde{\mathcal{C}}$. Together with Theorem 4, it shows that the association $\mathcal{C} \mapsto \tilde{\mathcal{C}}$ is a bijective correspondence between preenveloping subcategories of $\mathrm{Add}(R\text{-mod})$, closed under arbitrary direct sums and sums, and such definable subcategories, and that the inverse of this association is given by $\mathcal{X} \mapsto \mathcal{X} \cap \mathrm{Add}(R\text{-mod})$.

THEOREM 8. *The following are equivalent for a definable subcategory* $\mathcal{X}$ *of* R-Mod:

(i) *every finitely presented module has a pure projective* $\mathcal{X}$*-approximation*;
(ii) $\mathcal{X} = \tilde{\mathcal{C}}$, *where* $\mathcal{C} = \mathcal{X} \cap \mathrm{Add}(R\text{-mod})$ *is a preenveloping subcategory of* $\mathrm{Add}(R\text{-mod})$;
(iii) $\mathcal{X} = \tilde{\mathcal{C}}$, *for some subcategory* $\mathcal{C} \subseteq \mathrm{Add}(R\text{-mod})$ *closed under arbitrary direct sums and direct summands*.

If $\mathcal{B} \subseteq R$-mod is closed under finite direct sums and summands, then $\mathcal{C} = \mathrm{Add}(\mathcal{B})$ is the closure of $\mathcal{B}$ in $\mathrm{Add}(R\text{-mod})$ under arbitrary direct sums and summands. Lenzing's result implies that $\tilde{\mathcal{C}} = \lim \mathcal{B}$. If $\mathcal{B} \subseteq R$-mod is covariantly finite, then $\mathcal{C}$ is a preenveloping subcategory of $\mathrm{Add}(R\text{-}\vec{\mathrm{mod}})$ so that our theory subsumes that of the covariantly finite subcategories of R-mod. On the other hand, Dubrovin and Puninski [**9**] give an example (Example 6) of a ring R for which there exists a nontrivial preenveloping subcategory $\mathcal{C} \subseteq \mathrm{Add}(R\text{-mod})$ that contains no nonzero finitely presented modules.

There is a bijective correspondence [**13**, corollary 2·4] between definable subcategories $\mathcal{X} \subseteq R$-Mod and hereditary torsion classes $\mathcal{T}$ of finite type in the category $(\text{mod-}R, \mathrm{Ab})$ of additive functors $F : \text{mod-}R \to \mathrm{Ab}$. If $\mathcal{B} = \mathrm{add}(\mathcal{B}) \subseteq R$-mod is covariantly finite, then the correspondence associates to the definable subcategory $\mathcal{X} = \lim \mathcal{B}$ a hereditary torsion class $\mathcal{T}_{\mathcal{X}}$ that has a property we call *coherence*: the torsion subobject of every finitely presented object in $(\text{mod-}R, \mathrm{Ab})$ is finitely presented. It is easy to verify that if $\mathcal{C} \subseteq \mathrm{Add}(R\text{-mod})$ is preenveloping, then the associated hereditary torsion class $\mathcal{T}_{\tilde{\mathcal{C}}}$ of finite type is also coherent (Proposition 13). Our conviction that $\mathrm{Add}(R\text{-mod})$ is the natural setting in which to put forth the present theory stems from Theorem 14, which asserts the converse. This yields a fourth characterization of those definable subcategories $\mathcal{X}$ that are of the form $\tilde{\mathcal{C}}$.

PROPOSITION 13 and THEOREM 14. *Let* $\mathcal{X} \subseteq R$-Mod *be a definable subcategory. The associated hereditary torsion class* $\mathcal{T}_{\mathcal{X}}$ *of finite type is coherent if and only if every finitely presented left* R*-module has a pure projective left* $\mathcal{X}$*-approximation*.

An interesting example of this phenomenon occurs in the module theory of those rings whose class of pure injective left modules is closed under extension. Such rings have been studied by Xu [**22**, section 3·5], who characterized them by the property that every left cotorsion module is pure injective. Equivalently, the pure injective envelope $m : M \to \mathrm{PE}(M)$

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of every left module M is its cotorsion envelope, and therefore has a flat cokernel. We prove (Corollary 17) that for such rings R, the hereditary torsion theory of finite type associated to the smallest definable category $\mathcal{X}(R\text{-proj}) \subseteq R\text{-Mod}$ containing the finitely generated projective left R-modules is coherent. This is a weakening of the coherence condition, for a ring R is right coherent if and only if the category of left flat modules $R\text{-Flat} = \varinjlim (R\text{-proj}) \subseteq R\text{-Mod}$ is definable. More generally, we prove.

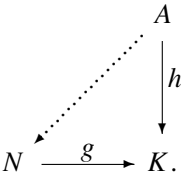
THEOREM 16. *Given* $\mathcal{B} = \text{add}(\mathcal{B}) \subseteq R\text{-mod}$, *let* $\mathcal{X}(\mathcal{B}) \subseteq R\text{-Mod}$ *be the smallest definable subcategory containing* $\mathcal{B}$. *If every left* R*-module* M *has the property that the first pure cosyzygy* $\text{PE}(M)/M$ *belongs to* $\varinjlim \mathcal{B}$, *then the hereditary torsion class* $\mathcal{T}_{\mathcal{X}(\mathcal{B})}$ *of finite type is coherent and locally noetherian.*

The results and techniques used in this paper may be expressed, perhaps even more naturally, in the model-theoretic idiom. The reader is invited to consult the companion paper [**12**], which describes the content of this paper as a study in the model theory of modules [**15**].

Throughout this paper, R will denote an associative ring with identity. The unqualified term *module* will refer to a unital left R-module and the unadorned symbol $\otimes$ will always mean $\otimes_R$. The category of left (resp., right) R-modules is denoted by R-Mod (resp., Mod-R); the subcategory of finitely presented left (resp., right) R-modules by R-mod (resp., mod-R); and the category of abelian groups by Ab. Subcategories will be always assumed to be *full*. Given a subcategory $\mathcal{C} \subseteq R\text{-Mod}$, we denote by $\text{Add}(\mathcal{C})$ (resp., $\text{add}(\mathcal{C})$) the smallest subcategory of R-Mod that contains $\mathcal{C}$ and is closed under arbitrary (resp., finite) direct sums and direct summands. If $\mathcal{C} = \{C\}$ consists of one object, we will use the notation $\text{Add}(C)$ (resp., $\text{add}(C)$) to denote $\text{Add}(\mathcal{C})$ (resp., $\text{add}(\mathcal{C})$).

1. *Categories of pure projective modules*

A morphism $g: N \to K$ of modules is a *pure epimorphism* if every morphism $h: A \to K$ from a finitely presented module A, lifts to N along g,



The module K is *pure projective* if every pure epimorphism $g: N \to K$ has a retraction, $r: K \to N$, $gr = 1_K$. Warfield [**20**] characterized the pure projective modules as direct summands of direct sums of finitely presented modules. For that reason, we denote by $\text{Add}(R\text{-mod})$ the category of pure projective modules. *Throughout the paper, we will denote by* $\mathcal{C} \subseteq \text{Add}(R\text{-mod})$ *a subcategory closed under arbitrary direct sums and direct summands*. Define $\tilde{\mathcal{C}} \subseteq R\text{-Mod}$ to be the subcategory of those modules K for which there exists a pure epimorphism $g: C \to K$. It is readily verified that this subcategory $\tilde{\mathcal{C}} \subseteq R\text{-Mod}$ is closed under arbitrary direct sums and pure epimorphic images.

LEMMA 1. *A module* K *belongs to* $\tilde{\mathcal{C}}$ *if and only if every morphism* $h: A \to K$ *from a finitely presented module* A *factors through an object in* $\mathcal{C}$.

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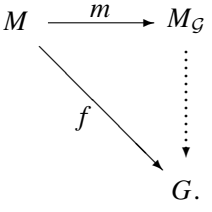
Proof. If $g: C \to K$ is a pure epimorphism, then h factors through g, by definition. Conversely, every module K admits [**20**] a pure epimorphism $k: \oplus_i A_i \to K$, with each A_i finitely presented. If each component morphism $k_i: A_i \to K$ factors through $g_i: C_i \to K$, then the coproduct $\oplus g_i: \oplus_i C_i \to K$ is a pure epimorphism.

A submodule $i: M \subseteq N$ is *pure* if the natural quotient morphism $g: N \to N/M$ is a pure epimorphism. Equivalently, for every finitely presented right R-module D_R, the induced morphism $D \otimes i: D \otimes M \to D \otimes N$ of abelian groups is a monomorphism.

PROPOSITION 2. *The subcategory $\tilde{\mathcal{C}} \subseteq R$-*Mod *is closed under pure submodules.*

Proof. The proof is entirely analogous to the proof of [**13**, lemma 3·10]. Just keep in mind [**21**, 34·1(2)] that if C is a pure projective module and $M \subseteq C$ is finitely generated, then C/M is also pure projective.

Suppose that $\mathcal{G} \subseteq \mathcal{A}$ is an inclusion of categories and that M is an object of $\mathcal{A}$. A (*left*) $\mathcal{G}$*-approximation* of M is a morphism $m: M \to M_{\mathcal{G}}$ with $M_{\mathcal{G}} \in \mathcal{G}$, such that every morphism $f: M \to G$ with $G \in \mathcal{G}$ factors through m, as indicated in the commutative diagram



A $\mathcal{G}$-approximation of M is also called a $\mathcal{G}$*-preenvelope* of M. The subcategory $\mathcal{G} \subseteq \mathcal{A}$ is *preenveloping* if every object $M \in \mathcal{A}$ has a $\mathcal{G}$-approximation.

In order to verify that a subcategory $\mathcal{C} \subseteq$ Add(R-mod) is preenveloping, it suffices to check that every finitely presented module A has a $\mathcal{C}$-approximation $a: A \to A_{\mathcal{C}}$. For then, the direct sum $\oplus_i A_i$ has the $\mathcal{C}$-approximation

$$\oplus a_i: \oplus_i A_i \longrightarrow \oplus_i (A_i)_{\mathcal{C}},$$

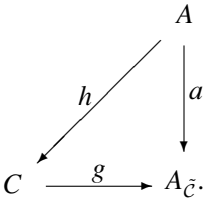
which also serves as a $\mathcal{C}$-approximation of any direct summand of $\oplus_i A_i$.

LEMMA 3. *A pure projective module A has a $\mathcal{C}$-approximation if and only if it has a $\tilde{\mathcal{C}}$-approximation. Moreover, if $a: A \to A_{\mathcal{C}}$ is a $\mathcal{C}$-approximation, then it is a $\tilde{\mathcal{C}}$-approximation.*

Proof. Let $a: A \to A_{\mathcal{C}}$ be a $\mathcal{C}$-approximation of A. If K is an object of $\tilde{\mathcal{C}}$, there is a pure epimorphism $g: C \to K$ from an object in $\mathcal{C}$. As A is pure projective, every morphism $h: A \to K$ factors through $g: C \to K$ as $h = gf: A \to C \to K$. Since $f: A \to C$ factors through the $\mathcal{C}$-approximation $a: A \to A_{\mathcal{C}}$, so does h.

Suppose, on the other hand, that $a: A \to A_{\tilde{\mathcal{C}}}$ is a $\tilde{\mathcal{C}}$-approximation of A. There is a pure epimorphism $g: C \to A_{\tilde{\mathcal{C}}}$, with $C \in \mathcal{C}$, so that there exists a factorization

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But then $h\colon A \to C$ is a $\tilde{\mathcal{C}}$-approximation, and therefore a $\mathcal{C}$-approximation.

A subcategory $\mathcal{X} \subseteq R$-Mod is *definable* if it is closed under direct limits, pure submodules and direct products. Because $\tilde{\mathcal{C}}$ is closed under direct sums and pure epimorphic images, it is closed under direct limits, and we have already noted (Proposition 2) that it is closed under pure submodules. By the result [**17**, corollary 3·5(c)] of Rada and Saorin, every definable subcategory of R-Mod is a preenveloping subcategory of R-Mod, which implies (ii) $\Rightarrow$ (iii) of the following theorem.

THEOREM 4. *The following are equivalent for a subcategory* $\mathcal{C} = \mathrm{Add}(\mathcal{C}) \subseteq \mathrm{Add}(R\text{-mod})$:

(i) $\mathcal{C}$ *is a preenveloping subcategory of* Add(R-mod);
(ii) *the associated subcategory* $\tilde{\mathcal{C}} \subseteq R$-Mod *is definable*;
(iii) $\tilde{\mathcal{C}}$ *is a preenveloping subcategory of* R-Mod.

Proof. (i) $\Rightarrow$ (ii). If $\mathcal{C} \subseteq \mathrm{Add}(R\text{-mod})$ is preenveloping and $K_i \in \tilde{\mathcal{C}}$, for some index set $i \in I$, then the argument given in [**8**, theorem 4·2] or [**13**, lemma 3·8] shows that every morphism $f\colon A \to \prod_i K_i$ from a finitely presented module A factors through its $\mathcal{C}$-approximation $a\colon A \to A_{\mathcal{C}}$. Thus $\prod_i K_i \in \tilde{\mathcal{C}}$. (iii) $\Rightarrow$ (i). Immediate from Lemma 3.

Given a subcategory $\mathcal{B} \subseteq R$-mod, of finitely presented modules, denote by $\mathcal{C} = \mathrm{Add}(\mathcal{B})$ the associated subcategory of pure projective modules. Lenzing [**14**] proved that $\tilde{\mathcal{C}}$ consists of those modules that may be represented as a direct limit of objects from $\mathrm{add}(\mathcal{B}) \subseteq R$-mod. A subcategory $\mathcal{B} \subseteq R$-mod is called *covariantly finite* if it is a preenveloping subcategory of R-mod.

PROPOSITION 5. *If* $\mathcal{B} = \mathrm{add}(\mathcal{B})$ *is a covariantly finite subcategory of* R-mod, *then* $\mathcal{C} = \mathrm{Add}(\mathcal{B})$ *is a preenveloping subcategory of* Add(R-mod) *with the property that every finitely presented module has a finitely presented* $\mathcal{C}$*-approximation. Conversely, if* $\mathcal{C} \subseteq \mathrm{Add}(R\text{-mod})$ *is a preenveloping subcategory with the property that every finitely presented module has a finitely presented* $\mathcal{C}$*-approximation, then* $\mathcal{B} = \mathcal{C} \cap R$-mod *is covariantly finite and* $\mathcal{C} = \mathrm{Add}(\mathcal{B})$.

Proof. Suppose that $\mathcal{B} = \mathrm{add}(\mathcal{B})$ is covariantly finite and let A be a finitely presented module. A $\mathcal{B}$-approximation $a\colon A \to A_{\mathcal{B}}$ is an Add($\mathcal{B}$)-approximation. For, if $C \in \mathrm{Add}(\mathcal{B})$ and a morphism $f\colon A \to C$ is given, we have another module C' in Add($\mathcal{B}$) such that $C \oplus C' = \oplus_i B_i$, with $B_i \in \mathcal{B}$. The image of f is finitely generated so that, after possibly combining finitely many of the B_i, it lies in one of the finitely presented summands B_0. The lifting of f, $f'\colon A \to B_0 \oplus B'$ therefore factors through a. But f is the composition of this lifting with projection onto C, so it also factors through a.

The only nontrivial aspect of the second statement is the claim that $\mathcal{C} \subseteq \mathrm{Add}(\mathcal{B})$. Well, if every finitely presented module has a $\mathcal{C}$-approximation that belongs to $\mathcal{B}$, then the argument just before Lemma 3 indicates that every pure projective module has a $\mathcal{C}$-approximation that

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belongs to Add($\mathcal{B}$). If $C \in \mathcal{C}$, then the identity morphism $1_C: C \to C$ must factor through an object of Add($\mathcal{B}$), and therefore $C \in$ Add($\mathcal{B}$).

Example 6. Let S be a countable nearly simple uniserial domain [**16**] and take a nonzero element s from the maximal ideal of S. Consider the finitely presented uniserial right S-module S/sS as a left module over its endomorphism ring

$$R = \mathrm{End}_S(S/sS).$$

A deep study of the ideal structure of R has been undertaken by Dubrovin and Puninski [**9**]. They prove that the ideal

$$I = \{r : S/sS \to S/sS \mid r \text{ is not a monomorphism}\}$$

considered as a left R-module is an example of an indecomposable projective module that is not finitely generated.

PROPOSITION 7. *With the ring R and ideal I as given in Example* 6, Add(I) $\subseteq$ Add(R-mod) *is a preenveloping subcategory that contains no nonzero finitely presented modules.*

Proof. Because the ring R is right coherent [**9**, lemma 4·4], the category R-Flat of flat R-modules is a definable subcategory. It is of the form $\tilde{\mathcal{C}}$, where $\mathcal{C} = R$-Proj is the category of projective modules, so by Theorem 4, R-Proj is preenveloping in Add(R-mod). Thus to prove that Add(I) is preenveloping in Add(R-mod), it suffices to verify that it is preenveloping in R-Proj.

By [**9**, theorem 7·3], every object of R-Proj is a direct sum of a free module and a direct sum of copies of I, so it is enough to produce an Add(I)-approximation of R. This is done using the fact [**9**, corollary 3·6] that $I = rR$ is cyclic as a right ideal of R. For then, the left R-module I is endocyclic, and an Add(I)-approximation of R is given [**2**, proposition 1·2] by the morphism $f: R \to I$ determined by the value $f(1) = r$. Finally, [**9**, corollary 7·4] implies that Add(I) $\subseteq R$-Proj contains no nonzero finitely presented modules.

The following is a characterization of the definable categories that are of the form $\tilde{\mathcal{C}}$. It shows that the rule $\mathcal{X} \mapsto \mathcal{X} \cap$ Add(R-mod) is a bijective correspondence from the collection of such categories to the collection of preenveloping subcategories of Add(R-mod). This rule is the inverse of the association $\mathcal{C} \mapsto \tilde{\mathcal{C}}$.

THEOREM 8. *The following are equivalent for a definable subcategory $\mathcal{X}$ of* R-Mod:

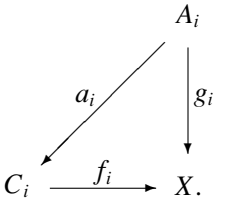
(i) *every finitely presented module has a pure projective $\mathcal{X}$-approximation*;

(ii) $\mathcal{X} = \tilde{\mathcal{C}}$, *where* $\mathcal{C} = \mathcal{X} \cap$ Add(R-mod) *is a preenveloping subcategory of* Add(R-mod);

(iii) $\mathcal{X} = \tilde{\mathcal{C}}$, *for some subcategory* $\mathcal{C} =$ Add($\mathcal{C}$) $\subseteq$ Add(R-mod).

Proof. (i) $\Rightarrow$ (ii). If a finitely presented module A has a pure projective $\mathcal{X}$-approximation, then that approximation is in fact a $\mathcal{C}$-approximation of A. Since $\mathcal{C} \subseteq \mathcal{X}$, and definable subcategories are closed under pure epimorphisms (this follows from [**13**, theorem 2·1(3)]), $\mathcal{X} \supseteq \tilde{\mathcal{C}}$. To see that $\mathcal{X} \subseteq \tilde{\mathcal{C}}$, let $X \in \mathcal{X}$ and consider a pure epimorphism $g: \oplus_i A_i \to X$, with each A_i finitely presented. Each component function $g_i: A_i \to X$ factors through a pure projective $\mathcal{X}$-approximation $a_i: A_i \to C_i$,

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But then the morphism $\oplus f_i : \oplus_i C_i \to X$ is a pure epimorphism from an object in $\mathcal{C}$.

(iii) $\Rightarrow$ (i). Let A be finitely presented. Since $\mathcal{X}$ is a definable subcategory, there is an $\mathcal{X}$-approximation $a : A \to A_{\mathcal{X}}$. If $\mathcal{X} = \tilde{\mathcal{C}}$, then Lemma 3 implies that A has a pure projective $\tilde{\mathcal{C}}$-approximation.

2. *Coherent hereditary torsion theories of finite type*

In this section, we give a torsion-theoretic characterization of those definable subcategories $\mathcal{X} \subseteq R$-Mod that have the form $\tilde{\mathcal{C}}$, where $\mathcal{C} \subseteq \mathrm{Add}(R\text{-mod})$ is preenveloping. In order to express this characterization, we recall the general correspondence [**13**, corollary 2·4] between definable subcategories of R-Mod and hereditary torsion classes of finite type in the functor category (mod-R, Ab) (cf. [**4**]) of additive functors $F : \text{mod-}R \to \mathrm{Ab}$. The category (mod-$R$, Ab) is a Grothendieck category whose notion of exactness is inherited from exactness in the category Ab.

The *finitely generated projective* objects of (mod-R, Ab) are the *representable* functors (cf. [**19**, corollary IV·7·5]), i.e., the functors that, up to isomorphism, have the form $(D, -) := \mathrm{Hom}_R(D, -)$, with $D \in \text{mod-}R$. A functor F in (mod-R, Ab) is *finitely presented* if there is a presentation of F by finitely generated projective objects

$$(D_2, -) \xrightarrow{\eta} (D_1, -) \longrightarrow F \longrightarrow 0.$$

A finitely presented object $F \in$ (mod-R, Ab) is *coherent* if every finitely generated subobject is finitely presented. In the sequel, we invoke the following fact without comment.

Fact 9 [3]. Every finitely presented object of (R-mod, Ab) is coherent. Equivalently, the subcategory fp(mod-R, Ab) $\subseteq$ (mod-R, Ab) of finitely presented functors is abelian.

There is a full and faithful embedding of R-Mod into (mod-R, Ab), given by $M \mapsto - \otimes_R M$. This implies that every morphism $\eta : - \otimes M \to - \otimes N$ is of the form $\eta = - \otimes f$ where $f : M \to N$ is a morphism of R-modules. If A is a finitely presented module, with free presentation

$$R^m \xrightarrow{g} R^n \longrightarrow A \longrightarrow 0,$$

then the induced exact sequence in (mod-R, Ab),

$$- \otimes R^m \xrightarrow{-\otimes g} - \otimes R^n \longrightarrow - \otimes A \longrightarrow 0,$$

is a presentation of $- \otimes A$ by finitely generated projective objects of (mod-R, Ab). This is because $- \otimes R^n \cong (R^n, -)$, and it implies that $- \otimes A$ is a finitely presented functor, whenever A is a finitely presented module. The objects of (mod-R, Ab), having the form $- \otimes M$ may be recovered using the following characterization.

Fact 10 [**7**, lemma 1·4]. An object $G \in$ (mod-R, Ab) is isomorphic to a functor of the form $- \otimes M$, with $M \in R$-Mod, if and only if it is *fp-injective,* that is, $\mathrm{Ext}^1(F, G) = 0$ for every finitely presented functor F in (mod-R, Ab).

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If $\mathcal{X} \subseteq R$-Mod is a definable subcategory, then [**13**, corollary 2·4] the torsion class

$$\mathcal{T}_{\mathcal{X}} = \{G \mid (G, - \otimes X) = 0 \text{ for all } X \in \mathcal{X}\}$$

of (mod-R, Ab) is a hereditary torsion class of finite type. This means that $\mathcal{T}_{\mathcal{X}}$ is closed under subobjects and every finitely generated object of $\mathcal{T}_{\mathcal{X}}$ is the quotient object of a finitely presented object from $\mathcal{T}_{\mathcal{X}}$. Conversely, if $\mathcal{T} \subseteq$ (mod-R, Ab) is a hereditary torsion class of finite type, then the subcategory

$$\mathcal{X}_{\mathcal{T}} = \{X \in R\text{-Mod} \mid - \otimes X \text{ is } \mathcal{T}\text{-torsion-free}\}$$

is definable. We will need the following property of this bijective correspondence between definable subcategories of R-Mod and hereditary torsion classes of finite type in (mod-R, Ab).

PROPOSITION 11. *Let M be a module and $\mathcal{X} \subseteq R$-Mod a definable subcategory. If $m\colon M \to M_{\mathcal{X}}$ is an $\mathcal{X}$-approximation of M, then*

$$t(- \otimes M) = \operatorname{Ker}(- \otimes m),$$

where t is the torsion radical associated to $\mathcal{T}_{\mathcal{X}}$.

Proof. As $- \otimes M_{\mathcal{X}}$ is torsion-free, it is clear that $t(- \otimes M) \subseteq \operatorname{Ker}(- \otimes m)$. On the other hand, consider the injective envelope $\eta\colon - \otimes M/t(- \otimes M) \to E$. It is a monomorphism into a functor of the form $E = - \otimes N$, because it is fp-injective. This functor is also torsion-free, so that N belongs to $\mathcal{X}$. The following commutative diagram arises,

$$\begin{array}{ccc} - \otimes M & \xrightarrow{- \otimes m} & - \otimes M_{\mathcal{X}} \\ \downarrow & \searrow^{- \otimes f} & \downarrow \\ - \otimes M/t(- \otimes M) & \xrightarrow{\eta} & - \otimes N. \end{array}$$

Since η is a monomorphism, $\operatorname{Ker}(- \otimes m) \subseteq t(- \otimes M)$.

Let $\mathcal{T} \subseteq$ (mod-R, Ab) be a hereditary torsion class of finite type, with associated torsion radical t. We will call $\mathcal{T}$ *coherent* if the torsion subobject $t(F) \leqslant F$ of every finitely presented object F of (mod-R, Ab) is itself finitely presented. Since the finitely presented objects of (mod-R, Ab) are coherent, this is equivalent to the condition that $t(F)$ be finitely generated.

LEMMA 12. *A hereditary torsion class $\mathcal{T} \subseteq$ (mod-R, Ab) of finite type is coherent if for every finitely presented left R-module A, the torsion subobject $t(- \otimes A)$ of the finitely presented functor $- \otimes A$ is finitely generated.*

Proof. By [**5**, lemma 7·1], every finitely presented object F of fp(mod-R, Ab) admits a monomorphism into an object of the form $- \otimes A$, where $A \in R$-mod. As the torsion class $\mathcal{T}$ is hereditary, one obtains the equality

$$t(F) = F \cap t(- \otimes A).$$

But the functor $- \otimes A$ is coherent, so that the intersection of two finitely generated subfunctors is itself finitely generated.

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An object G of (mod-R, Ab) is *locally coherent* if every finitely generated subobject $F \leqslant G$ is coherent. For example, if C is a pure projective module, then the functor $- \otimes C$ is locally coherent. For, C is a direct summand of some direct sum $\oplus_j A_j$, of finitely presented modules A_j, so that $- \otimes C$ is a coproduct factor of the coproduct of the coherent objects $- \otimes A_j$.

PROPOSITION 13. *If $\mathcal{C} \subseteq$ Add(R-mod) is preenveloping, then the hereditary torsion class $\mathcal{T}_{\tilde{\mathcal{C}}}$ of finite type is coherent.*

Proof. By Lemma 12, it suffices to verify for every finitely presented module A that the torsion subobject $t(- \otimes A)$ is finitely generated. By Theorem 8, there is a pure projective $\tilde{\mathcal{C}}$-approximation of A given by $a: A \to A_{\tilde{\mathcal{C}}}$. By Proposition 11, the torsion subobject of $- \otimes A$ is given by the kernel of $- \otimes a$. But the image of $- \otimes a$ is a finitely generated subobject of the locally coherent object $- \otimes A_{\tilde{\mathcal{C}}}$. It is therefore finitely presented, which implies that the kernel of $- \otimes a$ is finitely generated.

Gruson and Jensen [**11**, proposition 5·8] showed that the modules M for which the functor $- \otimes M$ is locally coherent are precisely the Mittag-Leffler modules introduced by Raynaud and Gruson, who proved [**18**, theorem 2·2·1] that every countably generated Mittag–Leffler module is pure projective.

THEOREM 14. *Let $\mathcal{T} \subseteq$ (mod-R, Ab) be a hereditary torsion class of finite type and $\mathcal{X} = \mathcal{X}_{\mathcal{T}}$ the corresponding definable category. If $\mathcal{T}$ is coherent, then every finitely presented module has a pure projective $\mathcal{X}$-approximation.*

Proof. For every finitely presented module A, we will construct an $\mathcal{X}$-approximation $a: A \to A_{\mathcal{X}}$ where $A_{\mathcal{X}}$ is a countably generated Mittag–Leffler module. Since $t(- \otimes A) \subseteq - \otimes A$ is finitely generated, the quotient object $- \otimes A/t(- \otimes A)$ is finitely presented, so by [**5**, lemma 7·1], there is a monomorphism

$$- \otimes A/t(- \otimes A) \longrightarrow - \otimes A_1$$

of functors, where $A_1 \in R$-mod. Applying this observation successively yields a sequence of morphisms

$$A = A_0 \xrightarrow{f_{01}} A_1 \xrightarrow{f_{12}} A_2 \xrightarrow{f_{23}} \cdots$$

with the property that each A_n is finitely presented, and the epi-mono factorization of $- \otimes f_{n,n+1}$ is given by

$$- \otimes f_{n,n+1}: - \otimes A_n \longrightarrow - \otimes A_n/t(- \otimes A_n) \longrightarrow - \otimes A_{n+1}.$$

Consider the direct limit $f_n: A_n \to A_{\mathcal{X}} = \varinjlim A_n$. Because the image $\mathrm{Im}(- \otimes f_{n,n+1})$ is torsion-free and the remaining morphisms $- \otimes f_{k,k+1}$, $k > n$, annihilate the torsion part, the epi-mono factorization of $- \otimes f_n$ is given by

$$- \otimes f_n: - \otimes A_n \longrightarrow - \otimes A_n/t(- \otimes A_n) \longrightarrow - \otimes A_{\mathcal{X}}.$$

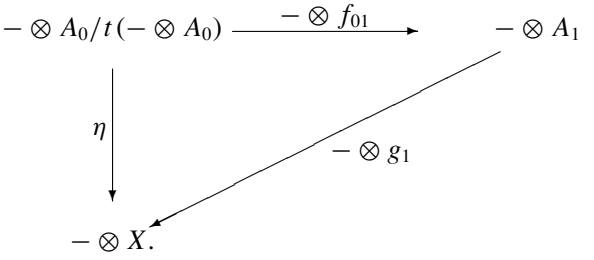
This implies that

$$- \otimes A_{\mathcal{X}} = \sum_n - \otimes A_n/t(- \otimes A_n)$$

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is an increasing union of coherent objects, which is therefore locally coherent. Hence the module $A_{\mathcal{X}}$ is Mittag–Leffler. As the union is countable, the module $A_{\mathcal{X}}$ is countably generated. Furthermore, each quotient $- \otimes A_n/t(- \otimes A_n)$ is torsion-free. The ordered union $- \otimes A_{\mathcal{X}}$ is therefore also torsion-free and so $A_{\mathcal{X}}$ belongs to $\mathcal{X}$.

To see that the morphism $f_0\colon A_0 \to A_{\mathcal{X}}$ is an $\mathcal{X}$-approximation of $A = A_0$, suppose that a morphism $g\colon A_0 \to X$ with $X \in \mathcal{X}$ is given. As $- \otimes X$ is torsion free, the corresponding morphism of functors $- \otimes g\colon - \otimes A_0 \to - \otimes X$ induces a morphism η on the torsion-free quotient $-\otimes A_0/t(-\otimes A_0)$. Because $-\otimes X$ is fp-injective, this morphism extends to $-\otimes A_1$,



Applying this argument successively results in a sequence of morphisms $g_n\colon A_n \to X$ from the ordered system $\{f_{n,n+1}\colon A_n \to A_{n+1}\}$. The morphism $g\colon A \to X$ therefore factors through the direct limit $f_0\colon A \to A_{\mathcal{X}}$.

The following corollary is immediate from Proposition 13 and Theorem 14.

COROLLARY 15. *There is a bijective correspondence between coherent hereditary torsion theories of finite type* $\mathcal{T} \subseteq (\text{mod-}R, \text{Ab})$ *and preenveloping subcategories* $\mathcal{C} \subseteq \text{Add}(R\text{-mod})$. *The correspondence is given by*

$$\mathcal{T} \;\mapsto\; \mathcal{X}_{\mathcal{T}} \cap \text{Add}(R\text{-mod}),$$

whose inverse is given by the rule $\mathcal{C} \mapsto \mathcal{T}_{\tilde{\mathcal{C}}}$.

3. *The first pure cosyzygy*

If M is a module, then the injective envelope of the functor $- \otimes M$ in $(\text{mod-}R, \text{Ab})$ is given by the morphism

$$- \otimes m\colon - \otimes M \longrightarrow - \otimes \text{PE}(M),$$

where $m\colon M \to \text{PE}(M)$ is the pure injective envelope of M (cf. [**20**]).

THEOREM 16. *Let* $\mathcal{B} = \text{add}(\mathcal{B}) \subseteq R\text{-mod}$ *and suppose that for every module* M, *the first pure cosyzygy* $\text{PE}(M)/M$ *belongs to* $\varinjlim \mathcal{B}$. *If* $\mathcal{X}(\mathcal{B}) \subseteq R\text{-Mod}$ *denotes the smallest definable subcategory containing* $\mathcal{B}$, *then the hereditary torsion class* $\mathcal{T}_{\mathcal{X}(\mathcal{B})}$ *of finite type is coherent and locally noetherian.*

Proof. We will prove that if F is a finitely presented functor in $(\text{mod-}R, \text{Ab})$, then every ascending chain

$$0 = F_0 \leqslant F_1 \leqslant F_2 \leqslant \cdots \leqslant F_n \leqslant \cdots \leqslant F,$$

with the F_i finitely generated and torsion, is stationary. An immediate consequence will be that $\mathcal{T}_{\mathcal{X}(\mathcal{B})}$ is coherent, and that every finitely presented torsion object is noetherian.

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So let $E_i = E(F/F_i)$ be the injective envelope of F/F_i in (mod-R, Ab). Consider the morphism $\alpha\colon \Sigma_i F_i \to \amalg_i E_i$, whose restriction to F_j maps to $F_j/F_i \subseteq E_i$ for all $i \leqslant j$ (cf. [**19**, proposition V·4·3]). It is part of a commutative diagram

$$\begin{array}{ccccccccc}
 & & \Sigma_i F_i & \xrightarrow{\iota} & F & \xrightarrow{\eta} & - \otimes B & & \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\
0 & \longrightarrow & \amalg_i E_i & \longrightarrow & E(\amalg_i E_i) & \xrightarrow{\pi} & - \otimes N & \longrightarrow & 0,
\end{array}$$

where $\iota\colon \Sigma_i F_i \to F$ is the natural inclusion and the bottom row is exact, and begins with the injective envelope of $\amalg_i E_i$. Fact 10 implies that for each i, there is a module M_i such that $E_i \cong - \otimes M_i$. Thus $\amalg_i E_i \cong - \otimes (\oplus_i M_i)$ and $E(\amalg_i E_i)$ is isomorphic to $- \otimes M$, where $M = \mathrm{PE}(\oplus_i M_i)$. The cokernel N of the pure injective envelope is therefore the first pure cosyzygy of $\oplus_i M_i$. By hypothesis, it is a direct limit $N = \varinjlim B_i$ of objects from $B_i \in \mathcal{B}$. As F is finitely presented, and $- \otimes N = \varinjlim - \otimes B_i$, the commutative diagram indicates how the morphism $\pi\beta\colon F \to - \otimes N$ factors through some $- \otimes B = - \otimes B_i$. Since $B \in \mathcal{B}$, the functor $- \otimes B$ is torsion-free, so the composition $\eta\iota$ in the top row is 0.

Consider the subobject $\mathrm{Ker}\,\eta \subseteq F$. It contains $\Sigma_i F_i$, but it is contained in $\mathrm{Ker}\,\pi\beta$, so the image $\beta(\mathrm{Ker}\,\eta)$ of the restriction of β to $\mathrm{Ker}\,\eta$ is in $\amalg_i E_i$. Now F and $- \otimes B$ are both coherent objects, so that $\mathrm{Ker}\,\eta$ is finitely generated. Therefore $\beta(\mathrm{Ker}\,\eta)$ is contained in a finite coproduct of the E_i, from which it follows that the image of α is contained in a finite coproduct of the E_i, and hence that the chain is stationary.

The most interesting special case of the theorem is when $\mathcal{B} = R$-proj is the category of finitely generated projective modules. In that case $\varinjlim \mathcal{B} = R$-Flat, the category of flat R-modules, and the hypothesis of the theorem supposes that the first pure cosyzygy of every module is flat. The rings R over which every left R-module has a flat first pure cosyzygy have been studied by Xu [**22**, section 3·5], who characterized them by the property that the class of pure injective left R-modules is closed under extension. Equivalently, every left cotorsion R-module is pure injective; recall that a left R-module M is *cotorsion* if $\mathrm{Ext}^1(Z, M) = 0$ for every flat module Z, and that cotorsion modules represent a homologically defined generalization of pure injective modules.

COROLLARY 17. *If every left cotorsion module is pure injective, then the hereditary torsion class $\mathcal{T}_{\mathcal{X}(R-\mathrm{proj})}$ of finite type is coherent and locally noetherian.*

Examples of rings over which every cotorsion module is pure injective include von Neumann regular rings and left pure semisimple rings. Unfortunately, we know of no example of such a ring that is not coherent.

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