Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2015.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

99.E (Michael Fox)

The tangents to the parabola $(t^2,2t)$ at points with integer parameters a, b and c intersect to form a triangle whose circumdiameter is d. Prove that for any given a there is an infinity of pairs b, c for which d is an integer, and show how we can obtain as many of these pairs as we please.

99.F (Michel Bataille)

Let ABC be a triangle with sides $BC = \sqrt{10 - 2\sqrt{2}}$, $CA = \sqrt{5 - \sqrt{2}}$, $AB = 1$ and let P interior to the triangle be such that $\dfrac{BP}{\sin\beta}\cdot\dfrac{CP}{\sin\gamma} = 4\cdot\dfrac{AP^2}{\sin^2\alpha}$ where $\alpha = \angle BPC$, $\beta = \angle CPA$, $\gamma = \angle APB$. Show that the minimal value of AP is $\sqrt{5\sqrt{2} - 7}$.

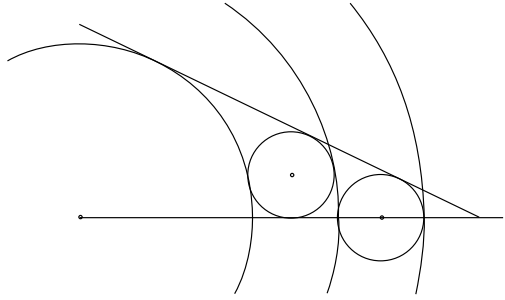
99.G (John D. Mahony)

In the diagram below, the three large arcs are portions of evenly spaced concentric circles: the inner circle (C_1) has radius R. The two small circles have the same radius (r) and satisfy the following conditions:

- They each touch two of the large arcs;
- They share a common tangent with C_1;
- The line joining the centre of C_1 to the centre of the outer small circle is a tangent to the inner small circle.

(a) Find a polynomial of lowest degree satisfied by the ratio $\frac{r}{R}$.

(b) How, in practice, would you draw the diagram by hand?



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99.H (Isaac Sofair)

Prove that the cosines c_1, c_2, c_3, c_4 of the four angles of a plane quadrilateral satisfy the relation

$$S_1^4 - 4S_1^2S_2 + 8S_1S_3 - 4S_1^2S_4 + 4S_3^2 = 0$$

where S_k is the sum of the products of the four cosines taken k at a time; for example, $S_3 = c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + c_2c_3c_4$.

Solutions and comments on **98.I**, **98.J**, **98.K**, **98.L** (November 2014).

98.I (Michael Fox)

The points C, D, E lie in the plane of a variable triangle ABC. Triangle BCD is described anti-clockwise with $\angle B = 63°$, $\angle C = 31°$, triangle CAE is described clockwise with $\angle C = 60°$, $\angle A = 63°$, and triangle ABF is described clockwise with $\angle A = 31°$, $\angle B = 120°$.

Establish the truth or falsity of each of the following statements:

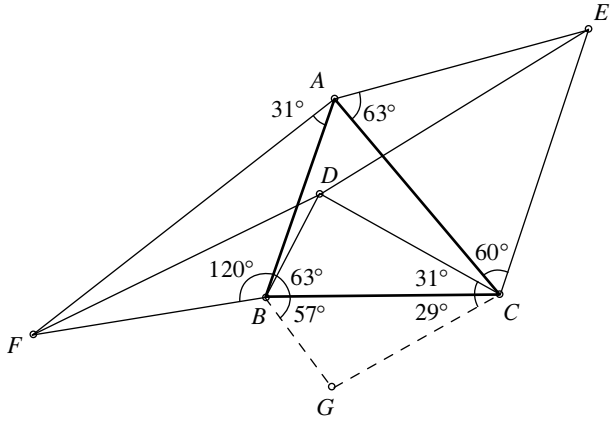
(i) The points D, E, F are collinear.

(ii) The lengths DE and DF are equal.

(iii) The ratio $DE : EF$ is constant.

Answer: (i) and (iii) are true; (ii) is false.

We give the proposer's solution below: Michel Bataille and Stan Dolan also gave very neat solutions by using complex number representations for the points involved.



Write $P(\theta, k)$ for the similarity with centre P that combines a rotation through an angle θ and an enlargement with factor k. If this is followed by $Q(\phi, l)$ the result is a similarity with angle $\theta + \phi$ and factor kl.

As in the Figure, construct triangle BCG clockwise with $\angle B = 57°$, $\angle C = 29°$. Note that $\angle GBD = 120°$, $\angle DCG = 60°$. Now consider the

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similarity $C\left(60°, \frac{AC}{EC}\right)$, which maps E onto A, and find the image of D. Since CD is rotated through 60°, the image of D lies on the line CG. The factor $\frac{AC}{EC} = \frac{\sin 57°}{\sin 63°}$, and $CD = BC\frac{\sin 63°}{\sin 86°}$. In the enlargement, therefore, the length CD becomes $BC\frac{\sin 63°}{\sin 86°}\cdot\frac{\sin 57°}{\sin 63°} = BC\frac{\sin 57°}{\sin 86°} = CG$. Thus the image of D is G, and ED maps onto AG.

The similarity with centre B that maps A onto F is $B\left(120°, \frac{FB}{AB}\right)$. The line BG rotates to lie along BD. The scale factor is equal to $\frac{\sin 31°}{\sin 29°}$, and we find that the length of BG becomes $BC\frac{\sin 31°}{\sin 86°}$, which equals BD; hence G maps back onto D, and AG maps onto FD.

The combined similarity therefore maps ED onto FD. The angle is 60° + 120°, i.e. 180°, hence E, D, F are collinear, whatever the shape of triangle ABC. Thus (i) is always true.

The scale factor of the combined similarity is $\frac{\sin 57°}{\sin 63°}\cdot\frac{\sin 31°}{\sin 29°}$. This evaluates to 0.99995… , so $DF \approx 0.99995DE$, and D is never the midpoint of DF. Thus (ii) is always false. (The error is small: if $EF = 100$ m then D is just over 1 mm from the true midpoint.)

As Michel Bataille observed, we can also show that $\frac{\sin 57°}{\sin 63°}\cdot\frac{\sin 31°}{\sin 29°} \neq 1$ by contradiction; otherwise $\sin 57° \sin 31° = \sin 63° \sin 29°$ leads to $\cos 88° - \cos 26° = \cos 92° - \cos 34°$ or $\cos 34° - \cos 26° = -2 \sin 2°$ which easily yields the contradiction that $\cos 2° = 1$.

Finally, since DEF is a straight line, $DE : EF = DE : DE + DF = 1 : 1.99995$ and is constant. Thus (iii) is true.

Correct solutions were received from: M. Bataille, S. Dolan (2 solutions), M. G. Elliott, J. A. Mundie and the proposer, M. Fox.

98.J (Stan Dolan)

Let A and B be positive odd integers with no common factor. If each of the fractions $\frac{A}{B}, \frac{2A}{B}, \frac{3A}{B}, \ldots, \frac{(B-1)A}{2B}$ is rounded to the nearest integer, show that the sum of these integers is even.

First, my apology that some respondents were thrown by the form of the last fraction: it was intended to be read as the '$\frac{1}{2}(B-1)$th' term in the sequence, $\frac{\frac{1}{2}(B-1)A}{B}$.

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The proposer's solution which follows was notably succinct.

For $1 \leqslant i \leqslant B-1$, let $iA = \lambda_i B + \delta_i$ where λ_i, δ_i are integers with $1 \leqslant \delta_i \leqslant B-1$. Then $\{\delta_1, \delta_2, \ldots, \delta_{B-1}\} = \{1, 2, \ldots, B-1\}$ and $\delta_i + \delta_{B-i} = B$.

For $1 \leqslant i \leqslant \frac{1}{2}(B-1)$, let $x_i = \delta_i$ if $\delta_i < \frac{1}{2}B$ and otherwise let $x_i = B - \delta_i$. Suppose there are n of these latter integers corresponding to the fractions that round up: we need to show that $\sum_{i=1}^{\frac{1}{2}(B-1)} \lambda_i + n \equiv 0 \pmod 2$.

Summing for $1 \leqslant i \leqslant \frac{1}{2}(B-1)$, we have

$$\sum iA = \sum \lambda_i B + \sum \delta_i$$

so that

$$\sum i \equiv \sum \lambda_i + \sum x_i + nB \pmod 2.$$

But $\sum i = \sum x_i$ because $\{x_1, x_2, \ldots, x_{\frac{1}{2}(B-1)}\} = \{1, 2, \ldots, \frac{1}{2}(B-1)\}$. Thus $\sum \lambda_i + n \equiv 0 \pmod 2$ as required.

Chris Starr showed that if the sequence of function is continued on as far as $\frac{(B-1)A}{B}$, the sum of the rounded fraction integers is $\frac{1}{2}A(B-1)$ which is only even if $B \equiv 1 \pmod 4$. This full range of fractions also featured in Student Problems 2013.6 with solution in the March 2014 *Gazette*, pp. 144-145.

Correct solutions were received from: C. Starr and the proposer S. Dolan.

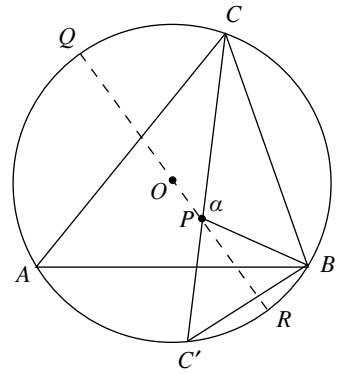
98.K (Isaac Sofair)

Let sides BC, CA, AB of triangle ABC subtend angles α, β, γ at an internal point P. If Δ is the area of triangle ABC, R its circumradius and d the distance of P from the circumcentre, express $\dfrac{R^2 - d^2}{\Delta}$ in terms of the angles α, β, γ, A, B, C.

The solution which follows is based on those submitted by Michel Bataille and M. G. Elliott; the former shortened his by reference to previous results from the solutions of **96.J** (July 2013) and **95.L** (July 2012).

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The Figure shows triangle ABC with its circumcircle, centre O, chord CC' through P and diameter QR through P. In triangle BPC', $\angle PC'B = A$ (same segment) and $\angle PBC' = \alpha - A$ (angle sum of a triangle) so, from the sine rule, $PB = \dfrac{PC' \sin A}{\sin(\alpha - A)}$.



Also $CP.PC' = RP.PQ = (R + d)(R - d) = R^2 - d^2$ so that

$$PB.PC = \frac{(R^2 - d^2)\sin A}{\sin(\alpha - A)}. \qquad (*)$$

Splitting triangle ABC into triangles BPC, CPA, APB gives

$$\Delta = \tfrac{1}{2}PB.PC \sin\alpha + \tfrac{1}{2}PC.PA \sin\beta + \tfrac{1}{2}PA.PB \sin\gamma$$

and hence from (*) and the analogous formulae for $PC.PA$ and $PA.PB$ we obtain

$$\Delta = \tfrac{1}{2}(R^2 - d^2)\left[\frac{\sin A \sin\alpha}{\sin(\alpha - A)} + \frac{\sin B \sin\beta}{\sin(\beta - B)} + \frac{\sin C \sin\gamma}{\sin(\gamma - C)}\right]$$

or

$$\frac{R^2 - d^2}{\Delta} = 2\left[\frac{\sin A \sin\alpha}{\sin(\alpha - A)} + \frac{\sin B \sin\beta}{\sin(\beta - B)} + \frac{\sin C \sin\gamma}{\sin(\gamma - C)}\right]^{-1}.$$

Simplifying the latter fraction gives a numerator of $2\sin(\alpha - A)\sin(\beta - B)\sin(\gamma - C)$ and a denominator for which the proposer derived the following equivalent cyclic sums/products

$$\sum \sin A \sin\alpha \sin(\beta - B)\sin(\gamma - C)$$

$$= \sum \sin B \sin C \sin\alpha \sin(\alpha - A)$$

$$= \sum \sin A \sin\beta \sin\gamma \sin(\alpha - A)$$

$$= 2\prod \sin A \sin\alpha + \sum \sin^2\alpha \cos A \sin B \sin C$$

$$= 2\prod \sin A \sin\alpha - \sum \sin^2 A \cos\alpha \sin\beta \sin\gamma$$

with the neat corollary that

$$\sum \sin^2\alpha \cos A \sin B \sin C + \sum \sin^2 A \cos\alpha \sin\beta \sin\gamma = 0.$$

Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, J. A. Mundie and the proposer I. Sofair.

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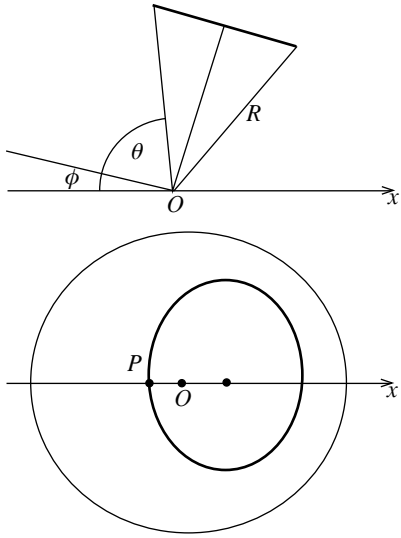
98.L (M. G. Elliott)

Find the length of the shortest day from sunrise to sunset for a point on the Earth's surface with latitude 51°30′ N using the following model of the Earth's motion around the Sun.

Let the centre of the Earth move in a plane and orbit the Sun once every $365\frac{1}{4}$ days (1 year) in a circular path of radius 93 000 000 miles. Let the Earth's axis, joining the north and south poles, be fixed in direction and be inclined at 23° 26′ 15″ to the normal vector of the plane. In 1 year, let the Earth make $366\frac{1}{4}$ revolutions from west to east about its axis. Let the Earth and Sun be spheres with diameters of 7927 miles and 864 950 miles respectively.

Two impressively detailed solutions were received. The proposer's solution used coordinate transformations to obtain the key equation required; Stan Dolan's (which follows) made greater use of geometry.

Let R be the radius of the Earth and O its centre. The relevant circle of latitude and its elliptical projection onto the plane of the ecliptic are as shown in Figure 1, where the x-axis is the projection of the Earth's axis.



(i) Here $\theta = 51.5°$, $\phi = 23.4375°$ and the circle of latitude has radius $R\cos\theta$.

(ii) The distance from O to the projection of the centre of the circle of latitude is $R\sin\theta\sin\phi$.

FIGURE 1

Let the shortest day be $2T$ hours at the point P where the winter solstice occurs at noon. (The shortest day for other points at latitude θ will be marginally longer.) At T hours before or after noon the coordinates of the projection of P are then

$$(X, Y) = \left(R\left(\sin\theta\sin\phi - \cos\theta\cos 15T^\circ\cos\phi\right), \pm R\cos\theta\sin 15T^\circ\right)$$

Slightly more than half of the Earth's surface is in direct daylight at any one

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time because the sun is not a point source of light: see Figure 2 where

$$\sin\delta = \frac{864950 - 7927}{2 \times 93 \times 10^6} = 0.00460765.$$

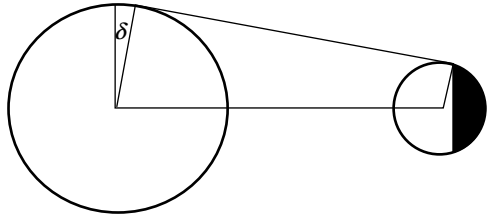
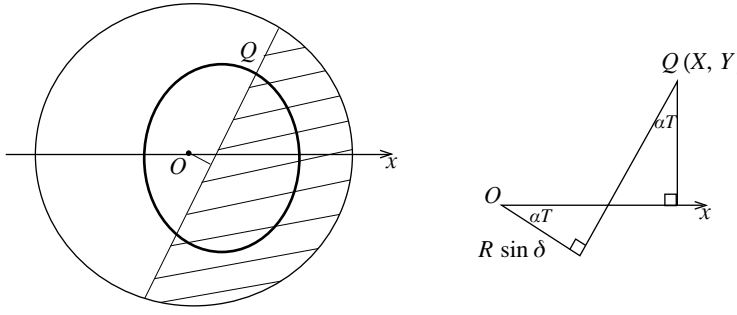


FIGURE 2

In T hours the region of shadow rotates $\alpha T°$, in the same sense as the Earth's rotation, where $\alpha = \frac{15}{365.25}$. This is shown in Figure 3, where the distance from O to the region of shadow is $R \sin\delta$.



(i) (ii)

FIGURE 3

A precise equation for T is then, from Figure 3(ii):

$$\tan\alpha T = \frac{X - R\sin\delta\cos\alpha T}{Y + R\sin\delta\sin\alpha T} = \frac{\sin\theta\sin\phi - \cos\theta\cos 15T°\cos\phi - \sin\delta\cos\alpha T}{\cos\theta\ \sin 15T° + \sin\delta\sin\alpha T}$$

and hence

$$\cos 15T° = \tan\theta\tan\phi - \frac{\cos\theta\sin 15T°\ \tan\alpha T + \sin\delta\sin\alpha T\tan\alpha T + \sin\delta\cos\alpha T}{\cos\theta\cos\phi}. \quad (*)$$

Considering the magnitudes of the various terms, it can be seen that an excellent and very simple approximation is given by

$$\cos 15T° \approx \tan\theta\ \tan\phi \approx 0.545004.$$

Then $2T \approx 7.596673$, i.e. approximately 7 hours 36 minutes.

A numerical solution of (*) gives the actual duration as 7 hours 41 minutes 35 seconds to the nearest second.

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This answer is based upon the assumptions made in the statement of this problem. Various factors have not been considered, such as the eccentricity of the Earth's orbit and the Earth's atmosphere affecting indirect daylight. One very important factor which should be mentioned is that the Earth is not a perfect sphere.

The references below are to earlier *Gazette* articles which discuss variation in the length of daylight but model the sun as a point source of light.

References

1. J. F. Ling, The variation in the length of daylight, *Math. Gaz.*, **73** (December 1989) pp. 302-305.
2. P. K. Aravind, On the variation in the length of daylight in a simple model, *Math. Gaz.*, **77** (March 1993) pp. 92-94.

Correct solutions were received from: S. Dolan and the proposer M. G. Elliott.
doi:10.1017/mag.2015.58 N.J.L.

In the Pipeline

The following articles are expected to appear in the November issue.

The Presidential Address	*Lynne McClure*
The hitchhiker triangle and the problem of perimeter = area	*Tony Crilly and Colin R. Fletcher*
A search for sums of powers	*Nigel Derby*
On the product $\prod \zeta(n)$	*Ferit Oktem*
A slowly converging sequence via Pythgoras	*Martin Griffiths*
Triangles around a given triangle	*Stan Dolan*
Artzt parabolae of a triangle	*John Sharp*
The carry-on-until-one-girl proportion	*Martin Griffiths*
An octet of circles	*Michael Sewell*
A Tripos surd	*Mark Villarino*
Ellipse – what else?	*Aldo Scimone*
Evaluating Fresnel-type integrals	*G. J. O. Jameson*
Differential equations and power series	*Raymond Beauregard and Vladimir Dobrushkin*
Variations on a Sangaku problem involving kissing spheres	*P. K. Aravind*

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