The left-hand-side of (2) is divisible by b^k . Therefore the right-hand-side should also be divisible by b. This is only possible when $b \mid t^k$ (since we already have the first part of the right-hand-side $\sum_{l=1}^{k-1} \binom{k}{l} b^l t^{k-l}$ divisible by b). But that is not possible for $b > 1$ since we have $\gcd(t, b) = 1$. We hence arrive at a contradiction. Therefore $\sqrt[k]{N}$ is an irrational number if N is not the k th power of any integer.

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106.45 A generalisation of a classical open-top box problem

Todhunter [1] in 1855 published his Box Problem by asking "If a rectangular piece of pasteboard, the sides of which are a and b, has a square cut out at each corner, find the sides of the square that the remainder may form a box of maximum content." Since then there have been several publications on either square-based or rectangular-based open-top box problems (e.g. see [2 – 9]). There are no records of any publication on a generalisation of this classical box problem to arbitrary irregular polygonal open-top prisms. This is due to the challenge involved in the identification of not so identical cuts required at various vertices of a given irregular polygon. Honouring a quote by Carl Gustav Jacob Jacobi (1804-1851), e.g. see [10], that "*Man muss immer generalisieren*" (One should always generalise), in this Note we intend to do the same.

We start with a regular n-sided polygon and then move on to a rather involved case of an irregular n-sided polygon. The real challenge is the identification of the vertices of the base of the irregular prism. We conclude the paper with two special case examples.

In the sequel, we assume all polygons are simple and convex and all prisms are upright and open-top. For simplicity, we also label all the vertices of the polygons in a counterclockwise manner.

Regular Prism Problem

A regular n-sided polygon with sides a is given. Right kites of sides x and h are cut out of each corner to make an open-top upright regular n-sided polygonal prism with height h. Find the best cuts that optimises the volume of the prism.

Solution

Consider the regular n-sided polygon whose first two vertices are A and B. To make an upright n-sided polygonal prism, we need to cut identical right kites of sides $BC = x$ by $CE = h = x \tan\frac{(n-2)\pi}{2n}$ from every corner

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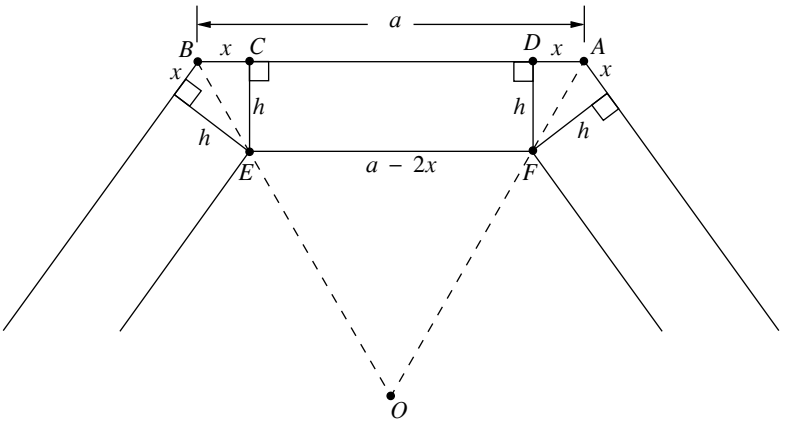


FIGURE 1

of the polygon. The base of the prism is then a regular n-gon with sides $a - 2x$. Connecting the centre of the base to the vertices of the base results in n isosceles triangles where the height of each triangle is

$$H = \tfrac{1}{2}(a - 2x)\tan\frac{(n-2)\pi}{2n}.$$

So, the area of each isosceles triangle is

$$A_T = \tfrac{1}{2}\,(\text{Base})(\text{Height}) = \tfrac{1}{4}(a - 2x)^2 \tan\frac{(n-2)\pi}{2n}.$$

Therefore, the volume V of the prism is equal to $V =$ (Area of the base)(height) or

$$V(x) = \frac{n}{4}\left(\tan\frac{(n-2)\pi}{2n}\right)^2 (4x^3 - 4ax^2 + a^2x).$$

By the first and second derivative test, we see that the absolute maximum volume is attained for

$$x = \frac{a}{6} \text{ and } h = \frac{a}{6}\tan\frac{(n-2)\pi}{2n}$$

and the maximum volume is

$$V\left(\frac{a}{6}\right) = \frac{n}{2}\tan^2\frac{(n-2)\pi}{2n}\left(\frac{a}{3}\right)^3.$$

Obviously, this for the square case yields the optimum prism $x = h = \frac{a}{6}$ and $V = \dfrac{2a^3}{27}$.

To overcome the challenge of identifying the vertices of the base of the irregular prism, we introduce an argument that is new and does not appear to have been used or appeared in any prior publications.

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Irregular Prism Problem

An n-sided polygon with vertices A_k; $1 \leqslant k \leqslant n$ and coordinates $A_k : (x_{A_k}, y_{A_k})$ is given. Right kites of sides h and t_k are cut from each respective corner A_k to make an open-top n-sided irregular polygonal prism with height h. Find the best cuts h and t_k off each corner A_k to optimise the volume of the desired prism.

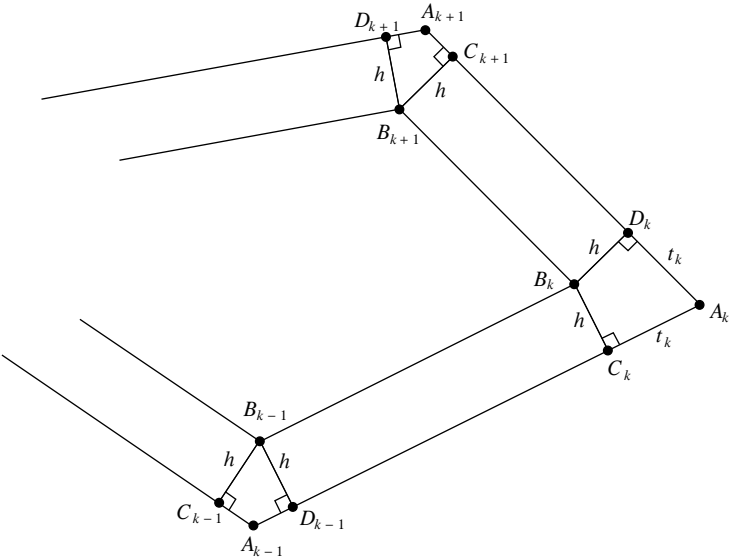


FIGURE 2

Consider the right kite $A_kD_kB_kC_k$ cut off the corner A_k; $1 \leqslant k \leqslant n$ where C_k is on the side $A_{k-1}A_k$ and D_k is on the side A_kA_{k+1} for which $A_{n+1} = A_1$. For $C_kB_k = B_kD_k = h$ and $C_kA_k = A_kD_k = t_k$ we have $A_{k-1}C_k = A_{k-1}A_k - t_k$ and $D_kA_{k+1} = A_kA_{k+1} - t_k$ where $A_0 = A_n$.

Because the polygon is convex with counterclockwise orientation, for each interior angle $\angle A_k$, the angle $\frac{1}{2}\angle A_k$ is an acute angle. Therefore, $t_k = h \cot\left(\frac{1}{2}\angle A_k\right)$.

Let (x_{C_k}, y_{C_k}) denote the coordinates of C_k and m_{k-1} be the slope of the side $A_{k-1}A_k$.

Then $x_{C_k} = x_{A_k} \pm \dfrac{t_k}{\sqrt{1+(m_{k-1})^2}}$ as well as $x_{C_k} = x_{A_{k-1}} \pm \dfrac{A_{k-1}A_k - t_k}{\sqrt{1+(m_{k-1})^2}}$.

Considering C_k lies between the vertices A_{k-1} and A_k, it follows from the above equations that

$$x_{C_k} = x_{A_k} - \frac{x_{A_k} - x_{A_{k-1}}}{|x_{A_k} - x_{A_{k-1}}|}\frac{t_k}{\sqrt{1+(m_{k-1})^2}} = x_{A_k} - \frac{x_{A_k} - x_{A_{k-1}}}{|x_{A_k} - x_{A_{k-1}}|}\frac{\cot\left(\frac{1}{2}\angle A_k\right)}{\sqrt{1+(m_{k-1})^2}}h.$$

We also have $y_{C_k} = y_{A_k} \pm \dfrac{|m_{k-1}|t_k}{\sqrt{1+(m_{k-1})^2}}$ and $y_{C_k} = y_{A_{k-1}} \pm \dfrac{|m_{k-1}|(A_{k-1}A_k - t_k)}{\sqrt{1+(m_{k-1})^2}}$

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yielding

$$y_{C_k} = y_{A_k} - \frac{y_{A_k} - y_{A_{k-1}}}{|y_{A_k} - y_{A_{k-1}}|} \frac{|m_{k-1}| t_k}{\sqrt{1 + (m_{k-1})^2}} = y_{A_k} - \frac{y_{A_k} - y_{A_{k-1}}}{|y_{A_k} - y_{A_{k-1}}|} \frac{|m_{k-1}| \cot\left(\frac{1}{2}\angle A_k\right)}{\sqrt{1 + (m_{k-1})^2}} h.$$

A similar argument applied to the coordinates of $D_k : \left(x_{D_k}, y_{D_k}\right)$ yields

$$x_{D_k} = x_{A_k} + \frac{x_{A_{k+1}} - x_{A_k}}{|x_{A_{k+1}} - x_{A_k}|} \frac{t_k}{\sqrt{1 + (m_k)^2}} = x_{A_k} + \frac{x_{A_{k+1}} - x_{A_k}}{|x_{A_{k+1}} - x_{A_k}|} \frac{\cot\left(\frac{1}{2}\angle A_k\right)}{\sqrt{1 + (m_k)^2}} h,$$

and

$$y_{D_k} = y_{A_k} + \frac{y_{A_{k+1}} - y_{A_k}}{|y_{A_{k+1}} - y_{A_k}|} \frac{|m_k| t_k}{\sqrt{1 + (m_k)^2}} = y_{A_k} + \frac{y_{A_{k+1}} - y_{A_k}}{|y_{A_{k+1}} - y_{A_k}|} \frac{|m_k| \cot\left(\frac{1}{2}\angle A_k\right)}{\sqrt{1 + (m_k)^2}} h.$$

Note that the slope of B_kC_k is $m_{B_kC_k} = -\frac{1}{m_{k-1}}$.

Therefore, the equation of the line B_kC_k is $y = -\frac{1}{m_{k-1}}\left(x - x_{C_k}\right) + y_{C_k}$.

Similarly, the equation of the line B_kD_k is $y = -\frac{1}{m_k}\left(x - x_{D_k}\right) + y_{D_k}$.

The intersection of the lines B_kC_k and B_kD_k yields the coordinates of $B : \left(x_{B_k}, y_{B_k}\right)$ as

$$x_{B_k} = \frac{1}{m_{k-1} - m_k}\left[m_{k-1}x_{D_k} - m_k x_{C_k} + m_{k-1}m_k\left(y_{D_k} - y_{C_k}\right)\right]$$

and

$$y_{B_k} = \frac{1}{m_{k-1} - m_k}\left[x_{C_k} - x_{D_k} + m_{k-1}y_{C_k} - m_k y_{D_k}\right].$$

In the case of any vertical side, L'Hôpital's Rule can easily be applied. Because the polygon is convex, no more than two sides can be vertical, and no more than two sides can be horizontal.

Therefore, by Green's Theorem [11 - 13], the area of the base $B_1B_2\ldots B_n$ is

$$A(h) = \frac{1}{2}\sum_{k=1}^{n}\left(x_{B_{k+1}} + x_{B_k}\right)\left(y_{B_{k+1}} - y_{B_k}\right)$$

where $A(h)$ is a function of h because x_{B_k}, $x_{B_{k+1}}$, y_{B_k} and $y_{B_{k=1}}$ are functions of h.

Since the height of the prism is h, the volume of the prism is given by

$$V(h) = \frac{h}{2}\sum_{k=1}^{n}\left(x_{B_{k+1}} + x_{B_k}\right)\left(y_{B_{k+1}} - y_{B_k}\right).$$

A first and second derivative test yields the optimum height h that maximises $V(h)$.

In the following we give two examples, the first of which uses L'Hôpital's Rule.

Example 1:

Consider the rectangle $A_1A_2A_3A_4$ with vertices $A_1 : (0, 0)$, $A_2 : (5, 0)$, $A_3 : (5, 3)$ and $A_4 : (0, 3)$. Let m_k represent the slope of the side $A_{k-1}A_k$;

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$1 \leqslant k \leqslant 4$ where $A_0 = A_4$. Therefore, $m_1 = m_3 = 0$ and $m_2 = m_4 = \infty$. Substituting these slopes in the coordinates of $B_k : \left(x_{B_k}, y_{B_k}\right)$ and applying L'Hôpital's Rule we obtain

$$x_{B_1} = \frac{1}{m_4 - 0}\left[m_4 x_{D_1} - 0 \cdot x_{C_1} + m_4 \cdot 0 \cdot \left(y_{D_1} - y_{C_1}\right)\right] = \frac{1}{m_4}\left[m_4 x_{D_1}\right] = x_{D_1} = h$$

and

$$y_{B_1} = \frac{1}{m_4 - 0}\left[x_{C_1} - x_{D_1} + m_4 y_{C_1} - 0 \cdot y_{D_1}\right] = \frac{1}{m_4}\left[x_{C_1} - x_{D_1} + m_4 y_{C_1}\right] = y_{C_1} = h.$$

Similarly, $B_2 : \left(x_{B_2}, y_{B_2}\right) = (5 - h, h)$, $B_3 : \left(x_{B_3}, y_{B_3}\right) = (5 - h, 3 - h)$ and $B_4 : \left(x_{B_4}, y_{B_4}\right) = (h, 3 - h)$. Therefore, the volume of the rectangular prism is

$$V(h) = \frac{h}{2}\sum_{k=1}^{4}\left(x_{B_{k+1}} + x_{B_k}\right)\left(y_{B_{k+1}} - y_{B_k}\right) = 4h^3 - 16h^2 + 15h.$$

We observe that $h^* = \dfrac{8 - \sqrt{19}}{6}$ maximises the volume $V(h)$ since $V'\left(h^*\right) = 0$ and $V''\left(h^*\right) < 0$.

The second example beautifully demonstrates the process involved in the determination of the coordinates of the vertices of the base of an irregular pentagonal prism.

Example 2:

Consider the irregular pentagon $A_1A_2A_3A_4A_5$ with vertices $A_1 : (2, -6)$, $A_2 : (4, 1)$, $A_3 : (1,4)$, $A_4 : (3, 3)$ and $A_5 : (-5, -2)$. Right kites of side-lengths h and $t_k : 1 \leqslant k \leqslant 5$ are cut from each respective corner A_k to make an open-top irregular pentagonal upright prism with height h. The slopes of the sides of the pentagon $A_1A_2A_3A_4A_5$ are given by $m_1 = m_{A_1A_2} = 7/2$, $m_2 = m_{A_2A_3} = -1$, $m_3 = m_{A_3A_4} = 1/4$, $m_4 = m_{A_4A_5} = 5/2$ and $m_5 = m_{A_5A_1} = -4/7$. Because the polygon is convex and the orientation is anticlockwise, we have $0 < A_k < \pi$ which means that

$$\cot\frac{\angle A_k}{2} = \cot\left(\frac{1}{2}\cos^{-1}\frac{1}{\sqrt{1 + (\tan\angle A_k)^2}}\right) > 0; \qquad 1 \leqslant k \leqslant 5.$$

Calculating the coordinates of C_k and D_k and substituting in $B_k : \left(x_{B_k}, y_{B_k}\right)$ yields

$$\begin{aligned}
B_1 &: \left(x_{B_1}, y_{B_1}\right) = (2 - 0.6111623458h, -6 + 1.500986732h),\\
B_2 &: \left(x_{B_2}, y_{B_2}\right) = (4 - 3.244109645h, 1 - 0.8406332474h),\\
B_3 &: \left(x_{B_3}, y_{B_3}\right) = (1 - 0.9568838194h, 4 - 3.454654205h),\\
B_4 &: \left(x_{B_4}, y_{B_4}\right) = (-3 + 2.825055800h, 3 - 3.236435714h) \text{ and}\\
B_5 &: \left(x_{B_5}, y_{B_5}\right) = (-5 + 1.653062275h, -2 + 0.5765260817h).
\end{aligned}$$

So, the area of the base $B_1B_2B_3B_4B_5$ can be easily calculated using Green's

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Theorem. Multiplying the area of the base by the height h yields the volume

$$V(h) = 10.68996601h^3 - 37.97764981h^2 + 35.5h.$$

The optimum volume $V(h^*) = 9.97$ is attained for $h^* = 0.6407005769$.

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