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LOWER DEVIATIONS FOR BRANCHING PROCESSES WITH IMMIGRATION

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Abstract

Let $\{Y_n, n \geq 1\}$ be a critical branching process with immigration having finite variance for the offspring number of particles and finite mean for the immigrating number of particles. In this paper we study lower deviation probabilities for Y_n. More precisely, assuming that $k, n \to \infty$ so that $k = o(n)$, we investigate the asymptotics of $\mathbb{P}(Y_n \leq k)$ and $\mathbb{P}(Y_n = k)$. Our results clarify the role of the moment conditions in the local limit theorem for Y_n proved by Mellein (1982).

Keywords: Branching process; local limit theorem; lower deviations

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1. Introduction, main results, and discussion

Let $\{Y_n\}$ be a branching process with the offspring distribution $\{p_k\}$ and with immigration governed by the distribution $\{q_k\}$. A mathematically rigorous description can be as follows. Let $\{\xi_{n,j}\}$ be independent random variables with distribution $\{p_k\}$ and let $\{\eta_n\}$ be independent random variables with distribution $\{q_k\}$. We assume also that the sequences $\{\xi_{n,j}\}$ and $\{\eta_n\}$ are independent from each other. Then the process $\{Y_n\}$ can be defined by the following recursive equation:

$$Y_{n+1} = \sum_{j=1}^{Y_n} \xi_{n+1,j} + \eta_{n+1}, \quad n \geq 0. \tag{1}$$

The starting point Y_0 can be deterministic or random; one only assumes that Y_0 is independent of the sequences $\{\xi_{n,j}\}$ and $\{\eta_n\}$.

In this paper we shall consider only critical processes, that is,

$$\sum_{k=1}^{\infty} kp_k = 1. \tag{2}$$

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Furthermore, we shall always assume that

$$B := \sum_{k=1}^{\infty} k(k-1)p_k \in (0, \infty) \quad \text{and} \quad \lambda := \sum_{k=1}^{\infty} kq_k \in (0, \infty). \tag{3}$$

It is worth mentioning that the strict positivity of $\sum_{k=1}^{\infty} kq_k$ means that the random variables $\{\eta_n\}$ are non-degenerate and, consequently, standard Galton–Watson processes are excluded.

Seneta [8] has shown that if (2) and (3) are valid then $2Y_n/(Bn)$ converges weakly towards a gamma distribution. More precisely, for every $x > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{2Y_n}{Bn} \le x\right) = \frac{1}{\Gamma(\gamma)} \int_0^x u^{\gamma-1} \mathrm{e}^{-u}\, \mathrm{d}u, \tag{4}$$

where

$$\gamma := \frac{2\lambda}{B}.$$

Seneta considers the case $Y_0 = 1$ only, but it is very easy to check that his arguments work for all starting distributions.

Mellein [5] has proved the corresponding local limit theorem. Assuming that

$$\sum_{k=1}^{\infty} (k^2 \log k)p_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} (k \log k)q_k < \infty, \tag{5}$$

he has shown that, for every fixed i,

$$\mathbb{P}(Y_n = k \mid Y_0 = i) \sim \frac{1}{\Gamma(\gamma)} \left(\frac{2}{B}\right)^{\gamma} \frac{k^{\gamma-1}}{n^{\gamma}} \mathrm{e}^{-2k/(Bn)} \tag{6}$$

as $n, k \to \infty$ and k/n remains bounded.

Since the moments in (5) do not show up in the asymptotics (6), it is natural to ask whether these conditions are needed for the local limit theorem. This question is our main motivation to consider lower deviation probabilities for Y_n. More precisely, we are going to determine the asymptotic behaviour of the probabilities $\mathbb{P}(Y_n \le k)$ and $\mathbb{P}(Y_n = k)$ in the case when $k \to \infty$ but $k = \mathrm{o}(n)$ for processes satisfying (2) and (3) only. Comparing asymptotics for local probabilities with the corresponding values of the limiting density, we can then conclude whether the extra condition (5) is really needed for the validity of (6).

Theorem 1. *Assume that* (2) *and* (3) *hold. Then there exists a slowly varying function* $L(x)$ *such that, for every fixed* $i \ge 0$,

$$\mathbb{P}(Y_n \le k \mid Y_0 = i) \sim \frac{1}{\Gamma(\gamma+1)} \left(\frac{2}{B}\right)^{\gamma} \frac{k^{\gamma} L(k)}{n^{\gamma} L(n)} \tag{7}$$

for $k \to \infty$ *and* $k = \mathrm{o}(n)$. *If, additionally,* $\{p_k\}$ *and* $\{q_k\}$ *are aperiodic, then, for every fixed* $i \ge 0$,

$$\mathbb{P}(Y_n = k \mid Y_0 = i) \sim \frac{1}{\Gamma(\gamma)} \left(\frac{2}{B}\right)^{\gamma} \frac{k^{\gamma-1} L(k)}{n^{\gamma} L(n)} \tag{8}$$

for $k \to \infty$ *such that* $k = \mathrm{o}(n)$. *Finally, for every fixed* $k \ge 1$ *there exists* $\mu_k \in [0, \infty)$ *such that, as* $n \to \infty$,

$$n^{\gamma} L(n)\, \mathbb{P}(Y_n = k \mid Y_0 = i) \to \mu_k. \tag{9}$$

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The function $L(x)$ converges to a positive constant provided that (5) holds. If

$$\sum_{k=1}^{\infty}(k^2\log k)p_k<\infty \quad \textit{and} \quad \sum_{k=1}^{\infty}(k\log k)q_k=\infty,$$

then $L(n)$ tends to zero. Furthermore, if

$$\sum_{k=1}^{\infty}(k^2\log k)p_k=\infty \quad \textit{and} \quad \sum_{k=1}^{\infty}(k\log k)q_k<\infty,$$

then $L(n)$ tends to infinity.

This result shows that (6) can fail if the moment conditions in (5) are not valid. This is quite different from the case of critical Galton–Watson processes. Nagaev and Wachtel [6] have shown that the existence of the second moment is sufficient for the local limit theorem.

We conjecture that the domain of lower deviations is the only zone where one needs (5) for the local asymptotics (6). In other words, we believe that the following statement holds: if (2) and (3) are valid, then for every fixed $\varepsilon>0$ we have

$$\sup_{k\ge\varepsilon n}\left|n\mathbb{P}(Y_n=k)-\frac{1}{\Gamma(\gamma)}\left(\frac{2k}{Bn}\right)^{\gamma-1}\mathrm{e}^{-2k/(Bn)}\right|\to 0 \quad \text{as } n\to\infty. \tag{10}$$

Noting that $u^{\gamma-1}\mathrm{e}^{-u}\to 0$ as $u\to 0$ for $\gamma>1$ and combining (10) with the fact that $\mathbb{P}(Y_n=k)=\mathrm{o}(n^{-1})$ for $k=\mathrm{o}(n)$ (see Remark 1 at the end of the paper for the corresponding uniform bound), we get for $\gamma>1$ the most standard version of the local limit theorem:

$$\sup_{k\ge 0}\left|n\mathbb{P}(Y_n=k)-\frac{1}{\Gamma(\gamma)}\left(\frac{2k}{Bn}\right)^{\gamma-1}\mathrm{e}^{-2k/(Bn)}\right|\to 0 \quad \text{as } n\to\infty. \tag{11}$$

(Note that this relation is not stronger than (6), because for $k=\mathrm{o}(n)$ we get only $\mathbb{P}(Y_n=k)=\mathrm{o}(n^{-1})$.)

Pakes [7] has shown (9) under the condition (5). Therefore Theorem 1 generalizes the results of Mellein [5] and Pakes [7] to the whole class of processes satisfying (3).

Our Theorem 1 can be used to derive asymptotics for the so-called harmonic moments $\mathbb{E}[Y_n^{-r};Y_n>0]$ in a rather simple way. This has been done by other means in Li and Zhang [4].

Our approach to lower deviation probabilities differs from that in the above-mentioned papers by Mellein and by Pakes. We first determine an optimal strategy which leads to atypically small values of Y_n. This allows us to reduce our problem on small deviations to a problem on normal deviations, where one can use results known in the literature. To describe this optimal strategy, we introduce some additional notation. Let $\{Z_n^{(i)}\}$, $i\ge 1$ be a sequence of independent Galton–Watson processes with offspring distribution $\{p_k\}$ and with $\mathbb{P}\big(Z_0^{(i)}=k\big)=q_k$, $k\ge 0$. It is immediate from (1) that if $Y_0=0$ then one can define Y_n and $\{Z_n^{(i)}\}$ on the same probability space so that

$$Y_n=\sum_{i=1}^{n}Z_{n-i}^{(i)} \quad \text{almost surely.} \tag{12}$$

Now define

$$\theta_n:=\inf\big\{i\le n: Z_{n-i}^{(i)}>0\big\}. \tag{13}$$

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In words, θ_n is the first generation where immigrants have descendants at time n. We know from (4) that Y_n is typically of size n and that a Galton–Watson process Z_n conditioned on non-extinction is also of order n. Thus it is quite plausible to assume that if we want to have only k particles in the nth generation, then θ_n should be such that $n - \theta_n$ is of order k. In the course of the proof of Theorem 1, we show that this strategy is indeed optimal.

A strategy that keeps the size of the process as small as possible, up to an appropriate time moment, is quite standard for branching processes. This type of behaviour is also optimal for lower deviations of supercritical Galton–Watson processes; see e.g. [2].

Our paper is organized as follows. In Section 2 we provide some results about upper bounds for lower deviation probabilities. Section 3 contains estimates for the concentration function, and Section 4 gathers the proof of the main theorem.

In what follows, we shall use the symbols c, c_k and C, C_k to denote numbers which may depend on distributions $\{p_k\}$ and $\{q_k\}$ only and may change their values from place to place.

2. Analysis of generating functions and some preliminary probabilistic estimates

Define $f(s) := \sum_{k=0}^{\infty} p_k s^k$ and $h(s) := \sum_{k=0}^{\infty} q_k s^k$; here and in what follows, $s \in \mathbb{C}$. In other words, $f(s)$ is the offspring generating function of the processes $\{Y_n\}$ and $\{Z_n^{(i)}\}$, and $h(s)$ is the generating function of the number of immigrants in $\{Y_n\}$ and of $Z_0^{(i)}$. Let $f_n(s)$ denote the nth iteration of f.

In this section we shall *always* assume that $Y_0 = 0$.

It is immediate from (12) that

$$H_n(s) := \mathbb{E}\big[s^{Y_n}\big] = \prod_{k=0}^{n-1} h(f_k(s)), \quad n \geq 1.$$

For the random variable θ_n defined in (13), we then have, for every $l \leq n$,

$$\mathbb{P}(\theta_n > l) = \prod_{i=1}^{l} \mathbb{P}\big(Z_{n-i}^{(i)} = 0\big) = \prod_{i=1}^{l} h(f_{n-i}(0)) = \prod_{k=n-l}^{n-1} h(f_k(0))$$

and

$$\mathbb{P}(\theta_n = l) = \mathbb{P}(\theta_n > l-1) - \mathbb{P}(\theta_n > l) = (1 - h(f_{n-l}(0))) \prod_{k=n-l+1}^{n-1} h(f_k(0)). \tag{14}$$

Define

$$F(0) := 1 \quad \text{and} \quad F(n) := \prod_{k=0}^{n-1} h(f_k(0)), \quad n \geq 1.$$

Then

$$\mathbb{P}(\theta_n > l) = \frac{F(n)}{F(n-l)}, \quad l \in \{0, 1, \dots, n\}$$

and

$$\mathbb{P}(\theta_n = l) = (1 - h(f_{n-l}(0))) \frac{F(n)}{F(n-l+1)}, \quad l \in \{1, 2, \dots, n\}.$$

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The first purpose of this section is to determine the asymptotic behaviour of $F(n)$. We show that this function is regularly varying at infinity and derive some properties of the corresponding slowly varying function.

We begin with a simple lemma, which is an easy consequence of the Taylor formula.

Lemma 1. *For $|s| \le 1$, it holds that*

$$\log h(s) = -h'(1)(1-s) + \mathrm{o}(s-1), \quad s \to 1, \tag{15}$$

where log *denotes the principal value of the logarithm.*

Proof. We start by noting that for $|s| < 1$,

$$\frac{h(1)-h(s)}{1-s} = \sum_{k=1}^{\infty} q_k \frac{1-s^k}{1-s} = \sum_{k=1}^{\infty} q_k \sum_{j=0}^{k-1} s^j.$$

By the triangle inequality,

$$\left| \frac{h(1)-h(s)}{1-s} - h'(1) \right| \le \sum_{k=1}^{\infty} q_k \left| \sum_{j=0}^{k-1} s^j - k \right|.$$

For every fixed k, we have

$$\left| \sum_{j=0}^{k-1} s^j - k \right| \to 0 \quad \text{as } s \to 1.$$

Furthermore, for every k we have the bound

$$\left| \sum_{j=0}^{k-1} s^j - k \right| \le 2k, \quad |s| \le 1.$$

The assumption $\sum_{k=1}^{\infty} k q_k < \infty$ allows us to apply the dominated convergence theorem, which gives

$$\left| \frac{h(1)-h(s)}{1-s} - h'(1) \right| \to 0 \quad \text{as } s \to 1.$$

In other words,

$$h(s) = h(1) - h'(1)(1-s) + \mathrm{o}((1-s)).$$

Taking the logarithm, we get the desired representation. □

The following lemma describes the asymptotic behaviour of $F(n)$.

Lemma 2. *There exists a slowly varying function $L(x)$ such that*

$$\frac{1}{n^{\gamma} F(n)} \sim L(n) \quad \text{as } n \to \infty.$$

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Proof. According to the representation theorem for slowly varying functions, it suffices to show that $1/(n^\gamma F(n))$ is asymptotically equivalent to a function $L(n)$ of the following form:

$$L(n) = c \exp\left\{\int_1^n \frac{\tau(y)}{y}\, \mathrm{d}y\right\}, \tag{16}$$

where τ is a bounded function such that $\tau(y) \to 0$ as $y \to \infty$, and $c > 0$.

We infer from (15) that

$$\log h(s) = -h'(1)(1-s) + (s-1)\alpha(s), \quad |s| \le 1, \tag{17}$$

where $\alpha(s) \to 0$ as $s \to 1$ and $|s| \le 1$.

Due to the condition $B < \infty$,

$$1 - f_j(0) = \frac{2}{Bj} + \frac{\epsilon_j}{j}, \tag{18}$$

where $\epsilon_j \to 0$ as $j \to \infty$. Combining this with (17), we have, as $n \to \infty$,

$$\begin{aligned}
n^\gamma F(n) &= n^\gamma \exp\left\{\sum_{j=0}^{n-1} \log h(f_j(0))\right\} \\
&= n^\gamma \exp\left\{-h'(1)\sum_{j=0}^{n-1}(1-f_j(0)) - \sum_{j=1}^{n}(1-f_j(0))\alpha(f_j(0))\right\} \\
&= n^\gamma \exp\left\{-\frac{2h'(1)}{B}\sum_{j=1}^{n}\frac{1}{j}\right\} \exp\left\{\sum_{j=1}^{n}\left(\frac{-h'(1)\epsilon_j}{j} - \frac{(2/B+\epsilon_j)\alpha(f_j(0))}{j}\right)\right\} \\
&\sim n^\gamma C \exp\{-\gamma \log n\} \exp\left\{-\sum_{j=1}^{n}\frac{(h'(1)\epsilon_j + (2/B+\epsilon_j)\alpha(f_j(0)))}{j}\right\} \\
&= C \exp\left\{-\sum_{j=1}^{n}\frac{(h'(1)\epsilon_j + (2/B+\epsilon_j)\alpha(f_j(0)))}{j}\right\},
\end{aligned}$$

where C is a positive constant.

Define

$$\tau(y) = y\frac{h'(1)\epsilon_j + (2/B+\epsilon_j)\alpha(f_j(0))}{j}$$

for all $y \in [j-1, j)$ and every $j \ge 2$. Then the representation (16) holds with this function τ and with $c = C\exp\{-h'(1)\epsilon_1 + (2/B+\epsilon_1)\alpha(f(0))\}$. This completes the proof of Lemma 2. □

Lemma 3. *If $k, n \to \infty$ and $k < n$, then*

$$\prod_{j=k}^{n-1} h(f_j(0)) \sim \left(\frac{k}{n}\right)^\gamma \frac{L(k)}{L(n)}.$$

Furthermore, there exists a constant C such that

$$\frac{1}{C}\left(\frac{k}{n}\right)^\gamma \frac{L(k)}{L(n)} \le \prod_{j=k}^{n-1} h(f_j(0)) \le C\left(\frac{k}{n}\right)^\gamma \frac{L(k)}{L(n)} \quad \textit{for all } 1 \le k < n.$$

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Proof. This is an immediate consequence of Lemma 2. □

Let $\{a_k\}$ be a sequence of positive numbers and let $A(s)$ denote its generating function. If $A(1) < \infty$, then

$$B(s) := \frac{A(1) - A(s)}{1 - s} = \sum_{j=0}^{\infty} s^j \sum_{k=j+1}^{\infty} a_k, \quad |s| < 1, \tag{19}$$

and

$$B(1) := \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} a_k.$$

It is also clear that $B(1) = A'(1) := \lim_{s\uparrow 1} A'(s)$. If $A'(1)$ is finite then we can apply (19) to the generating function $B(s)$. As a result, we have

$$\frac{A'(1) - \frac{A(1)-A(s)}{1-s}}{1 - s} = \sum_{j=0}^{\infty} s^j \sum_{k=j+1}^{\infty} \sum_{i=k+1}^{\infty} a_i, \quad |s| < 1. \tag{20}$$

These equalities will be used in the following two statements, which describe the asymptotics of $L(n)$ in the case when exactly one moment condition in (5) is valid. (The situation when both moment conditions are valid has been studied by Mellein, who showed in [5] that $L(n)$ is asymptotically constant in that case.)

Lemma 4. *Assume that* (3) *holds. If*

$$\sum_{k=1}^{\infty} (k^2 \log k) p_k < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} (k \log k) q_k = \infty,$$

then

$$L(n) \to 0 \quad \textit{as } n \to \infty. \tag{21}$$

Proof. First, applying (20) to the function $h(s)$, we get

$$1 - h(s) = (1 - s)h'(1) - (1 - s)^2 \sum_{j=0}^{\infty} s^j \sum_{i=j+1}^{\infty} R_i, \tag{22}$$

where $R_j = \sum_{k=j+1}^{\infty} q_k$.

By the series representation for the function $\log(1 - x)$, we have

$$\log(1 - x) = -x - \beta(x), \quad x \in (0, 1),$$

where

$$0 \le \beta(x) := \sum_{k=2}^{\infty} \frac{x^k}{k} \le \frac{x^2}{2(1 - x)}, \quad x \in (0, 1).$$

This implies that

$$\log h(f_j(0)) = -(1 - h(f_j(0))) - \beta_j,$$

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where

$$0 \le \beta_j := \beta(1 - h(f_j(0))) \le \frac{(1 - h(f_j(0)))^2}{2h(f_j(0))} \le \frac{(1 - h(f_j(0)))^2}{2h(p_0)}.$$

Noting that (18) implies that

$$(1 - h(f_j(0)))^2 \le (h'(1))^2(1 - f_j(0))^2 \le Cj^{-2},$$

we conclude that the sequence β_j is summable. Consequently

$$\begin{aligned} F(n) &= \exp\left\{\sum_{j=0}^{n-1} \log h(f_j(0))\right\} \\ &= \exp\left\{-\sum_{j=0}^{n-1} [1 - h(f_j(0)) + \beta_j]\right\} \\ &\sim C_0 \exp\left\{-\sum_{j=1}^{n-1} (1 - h(f_j(0)))\right\} \quad \text{as } n \to \infty. \end{aligned} \tag{23}$$

Now applying the representation (22), we have

$$\sum_{j=1}^{n-1} (1 - h(f_j(0))) = h'(1) \sum_{j=1}^{n-1} (1 - f_j(0)) - \sum_{j=1}^{n-1} (1 - f_j(0))^2 \sum_{l=0}^{\infty} f_j^l(0) \sum_{i=l+1}^{\infty} R_i. \tag{24}$$

If $\sum_{k\ge1} (k^2 \log k)p_k$ is finite and if $\{p_k\}$ is aperiodic, then, due to Lemma 8 in [3],

$$\sum_{j=1}^{n-1} (1 - f_j(0)) = \sum_{j=1}^{n-1} \frac{2}{Bj} + C + \mathrm{o}(1) \quad \text{as } n \to \infty. \tag{25}$$

We now show that this relation holds also for periodic distributions. Assume that $d := \gcd\{k : p_k > 0\} > 1$. Then, following [3], we introduce an auxiliary generating function

$$g(s) := [f(s^{1/d})]^d.$$

The distribution corresponding to $g(s)$ is aperiodic. It is easy to see that $g'(1) = 1$ and $g''(1) = B/d$. Furthermore, the iterations of g satisfy

$$g_n(s) = [f_n(s^{1/d})]^d.$$

In particular, $f_n(0) = (g_n(0))^{1/d}$. This implies that

$$1 - f_n(0) = 1 - (g_n(0))^{1/d} = \frac{1}{d}(1 - g_n(0)) + \psi_n,$$

where $\psi_n = \mathrm{O}((1 - g_n(0))^2)$ as $n \to \infty$. Since $g'(1) = 1$ and $g''(1) < \infty$, we may apply (18) to the iterations of this new generating function. This gives

$$1 - g_n(0) \sim \frac{2}{g''(1)n} \quad \text{as } n \to \infty.$$

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Therefore the sequence ψ_n is summable and

$$\sum_{j=1}^{n-1}(1-f_j(0)) = \frac{1}{d}\sum_{j=1}^{n-1}(1-g_j(0)) + C' + \mathrm{o}(1) \quad \text{as } n\to\infty.$$

Recalling that $g(s)$ corresponds to an aperiodic distribution, we may apply (25) to the sum $\sum_{j=1}^{n-1}(1-g_j(0))$. This completes the proof of (25) in the general case.

Equality (25) implies that

$$F(n) \sim \frac{1}{n^{\gamma}L(n)},$$

where

$$L(n) = C_1^{-1}\exp\left\{-\sum_{j=1}^{n-1}(1-f_j(0))^2\sum_{l=0}^{\infty}f_j^l(0)\sum_{i=l+1}^{\infty}R_i\right\} \quad \text{and} \quad C_1>0.$$

It is now obvious that (21) is equivalent to

$$\lim_{n\to\infty}\sum_{j=1}^{n-1}(1-f_j(0))^2\sum_{l=0}^{\infty}f_j^l(0)\sum_{i=l+1}^{\infty}R_i = \infty. \tag{26}$$

Set $\overline{R}_l := \sum_{i=l+1}^{\infty}R_i$. It follows from (18) that $f_j^l(0)\ge c_0$ for all $1\le l\le j$ and $(1-f_j(0))^2 \ge c_1 j^{-2}$ for all $j\ge 1$. Using these estimates and interchanging the order of summation, we get

$$\begin{aligned}
\sum_{j=1}^{n-1}(1-f_j(0))^2\sum_{l=0}^{\infty}f_j^l(0)\sum_{i=l+1}^{\infty}R_i &= \sum_{i=1}^{n-1}(1-f_j(0))^2\sum_{l=0}^{\infty}f_j^l(0)\overline{R}_l \\
&\ge c_1^2\sum_{j=1}^{n-1}\frac{1}{j^2}\sum_{l=1}^{j}c_0\overline{R}_l \\
&\ge c_2\sum_{l=1}^{n/2}\overline{R}_l\sum_{j=l}^{n-1}\frac{1}{j^2} \\
&\ge \frac{c_2}{4}\sum_{l=1}^{n/2}\overline{R}_l\frac{1}{l} \\
&\ge \frac{c_2}{4}\sum_{l=1}^{n/2}\frac{1}{l}\sum_{i=l+1}^{n/2}R_i \\
&= \frac{c_2}{4}\sum_{i=2}^{n/2}R_i\sum_{l=1}^{i-1}\frac{1}{l} \\
&\ge c_3\sum_{i=2}^{n/2}R_i\log i \to \infty \quad \text{as } n\to\infty,
\end{aligned}$$

where in the last step we used the assumption $\sum_{k=1}^{\infty}(k\log k)q_k=\infty$. This proves (26) and thus (21). □

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Lemma 5. *Assume that* (3) *holds. If*

$$\sum_{k=1}^{\infty}(k^2\log k)p_k=\infty \quad \textit{and} \quad \sum_{k=1}^{\infty}(k\log k)q_k<\infty,$$

then

$$L(n)\to+\infty \quad \textit{as } n\to\infty. \tag{27}$$

Proof. Changing the order of summation, we have

$$\begin{aligned}\sum_{j=1}^{\infty}(1-f_j(0))^2\sum_{l=0}^{\infty}f_j^l(0)\sum_{i=l+1}^{\infty}R_i &= \sum_{j=1}^{\infty}(1-f_j(0))^2\sum_{i=1}^{\infty}R_i\sum_{l=0}^{i-1}f_j^l(0)\\ &= \sum_{j=1}^{\infty}(1-f_j(0))\sum_{i=1}^{\infty}R_i\big[1-f_j^i(0)\big]\\ &= \sum_{i=1}^{\infty}R_i\sum_{j=1}^{\infty}(1-f_j(0))\big[1-f_j^i(0)\big].\end{aligned}$$

Using bounds $1-f_j(0)\le C/j$ and $1-f_j^i(0)\le i(1-f_j(0))$, we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}(1-f_j(0))\big[1-f_j^i(0)\big] &\le \sum_{j=1}^{i}(1-f_j(0))+\sum_{j=i+1}^{\infty}i(1-f_j(0))^2\\ &\le \sum_{j=1}^{i}\frac{C}{j}+\sum_{j=i+1}^{\infty}\frac{C^2 i}{j^2}\\ &\le C_1\log i, \quad i\ge 2.\end{aligned}$$

Then, according to the assumption $\sum_{k\ge1}(k\log k)q_k<\infty$,

$$\sum_{j=1}^{\infty}(1-f_j(0))^2\sum_{l=0}^{\infty}f_j^l(0)\sum_{i=l+1}^{\infty}R_i<\infty.$$

Combining this with (23) and with (24), we establish

$$F(n)\sim C_2\exp\left\{-h'(1)\sum_{j=1}^{n-1}(1-f_j(0))\right\} \quad \text{as } n\to\infty. \tag{28}$$

Following [6], we define

$$u(s)=\frac{1}{1-f(s)}-\frac{1}{1-s}.$$

We also define

$$\delta(s)=\frac{B}{2}-u(s).$$

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Then we have

$$\begin{aligned}\frac{1}{1-f_n(0)}-1&=\sum_{k=0}^{n-1}\left(\frac{1}{1-f_{k+1}(0)}-\frac{1}{1-f_k(0)}\right)\\&=\sum_{k=0}^{n-1}u(f_k(0))\\&=\frac{Bn}{2}-\sum_{k=0}^{n-1}\delta(f_k(0)).\end{aligned}\tag{29}$$

Thus

$$1-f_n(0)=\left(1+\frac{Bn}{2}-\sum_{k=0}^{n-1}\delta(f_k(0))\right)^{-1}.$$

Let $\varepsilon(s)$ be such that

$$f(s)=s+\left(\frac{B}{2}-\varepsilon(s)\right)(1-s)^2.$$

To derive an explicit expression for $\varepsilon(s)$, we first expand $f(s)-s$ by using the definition of $f(s)$:

$$f(s)-s=p_0(1-s)+\sum_{w=2}^{\infty}p_w(s^w-s)=p_0(1-s)-(1-s)\sum_{w=2}^{\infty}p_w\sum_{j=1}^{w-1}s^j.$$

Next, using the identity

$$s^j=1-j(1-s)+(1-s)\sum_{v=1}^{j-1}(1-s^v)$$

and the criticality assumption $f'(1)=\sum_{w=1}^{\infty}wp_w=1$, we get

$$\begin{aligned}f(s)-s&=p_0(1-s)-(1-s)\sum_{w=2}^{\infty}p_w\sum_{j=1}^{w-1}\left(1-j(1-s)+(1-s)\sum_{v=1}^{j-1}(1-s^v)\right)\\&=(1-s)\left(p_0-\sum_{w=2}^{\infty}(w-1)p_w\right)+(1-s)^2\sum_{w=2}^{\infty}p_w\sum_{j=1}^{w-1}\left(j-\sum_{v=1}^{j-1}(1-s^v)\right)\\&=(1-s)^2\sum_{w=2}^{\infty}p_w\frac{w(w-1)}{2}-(1-s)^2\sum_{w=3}^{\infty}p_w\sum_{j=2}^{w-1}\sum_{v=1}^{j-1}(1-s^v).\end{aligned}$$

Noting that

$$\sum_{w=2}^{\infty}\frac{w(w-1)}{2}p_w=\frac{B}{2},$$

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we finally conclude that

$$\varepsilon(s) = \sum_{w=3}^{\infty} p_w \sum_{j=2}^{w-1} \sum_{v=1}^{j-1} (1 - s^v), \quad |s| \le 1. \tag{30}$$

It is also clear that $\varepsilon(s) \in [0, B/2]$ for $s \in [0, 1]$.

We next notice that

$$\begin{aligned}
\delta(s) &= \frac{B}{2} - u(s) \\
&= \frac{B}{2} - \left(\frac{1}{1 - f(s)} - \frac{1}{1 - s} \right) \\
&= \frac{B}{2} - \frac{f(s) - s}{(1 - s)(1 - f(s))} \\
&= \frac{B}{2} - \frac{B/2 - \varepsilon(s)}{1 - (1 - s)(B/2 - \varepsilon(s))} \\
&= \frac{\varepsilon(s) - \frac{B}{2}(1 - s)(B/2 - \varepsilon(s))}{1 - (1 - s)(B/2 - \varepsilon(s))} \\
&=: \varepsilon(s) + \gamma(s)
\end{aligned}$$

and $\gamma(s) = \mathrm{O}((1 - s))$ as $s \to 1$. Plugging this representation of $\delta(s)$ into (29), we obtain

$$\frac{1}{1 - f_n(0)} = 1 + \frac{Bn}{2} - \sum_{k=0}^{n-1} \varepsilon(f_k(0)) - \sum_{k=0}^{n-1} \gamma(f_k(0)).$$

Combining $\gamma(s) = \mathrm{O}((1 - s))$ and (18), we obtain

$$\sum_{k=0}^{n-1} \gamma(f_k(0)) = \mathrm{O}(\log n), \quad n \to \infty.$$

Noting that $\varepsilon(f_k(0)) \to 0$ as $k \to \infty$, we infer that $\sum_{k=0}^{n-1} \varepsilon(f_k(0)) = \mathrm{o}(n)$. Combining this with the equality

$$\frac{1}{a_n + b_n} = \frac{1}{a_n} + \mathrm{O}\left(\frac{b_n}{a_n^2} \right), \quad n \to \infty,$$

which is valid for all sequences $\{a_n\}$, $\{b_n\}$ satisfying $a_n \to \infty$ and $b_n = \mathrm{o}(a_n)$, we conclude that

$$1 - f_n(0) = \left(1 + \frac{Bn}{2} - \sum_{k=0}^{n-1} \varepsilon(f_k(0)) \right)^{-1} + \mathrm{O}\left(\frac{\log n}{n^2} \right), \quad n \to \infty.$$

Plugging this into (28), we obtain

$$F(n) \sim C_3 \exp\left\{ -h'(1) \sum_{j=1}^{n-1} \left(1 + \frac{Bj}{2} - \sum_{k=0}^{j-1} \varepsilon(f_k(0)) \right)^{-1} \right\}, \quad n \to \infty. \tag{31}$$

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Thus it suffices to analyse the asymptotic behaviour of the sum

$$h'(1)\sum_{j=1}^{n-1}\left(1+\frac{Bj}{2}-\sum_{k=0}^{j-1}\varepsilon(f_k(0))\right)^{-1}.$$

Since $\varepsilon(s)$ is non-negative on the interval $[0,1]$, applying the inequality

$$\frac{1}{a-b}\geq\frac{1}{a}+\frac{b}{a^2},\quad 0\leq b\leq a,$$

we obtain

$$\left(1+\frac{Bj}{2}-\sum_{k=0}^{j-1}\varepsilon(f_k(0))\right)^{-1}\geq\left(1+\frac{Bj}{2}\right)^{-1}+\left(1+\frac{Bj}{2}\right)^{-2}\sum_{k=0}^{j-1}\varepsilon(f_k(0)).$$

Consequently

$$\begin{aligned}
&h'(1)\sum_{j=1}^{n-1}\left(1+\frac{Bj}{2}-\sum_{k=0}^{j-1}\varepsilon(f_k(0))\right)^{-1}\\
&\geq h'(1)\sum_{j=1}^{n-1}\left(1+\frac{Bj}{2}\right)^{-1}+\sum_{j=1}^{n-1}\frac{1}{(1+Bj/2)^2}\sum_{k=0}^{j-1}\varepsilon(f_k(0))\\
&\geq\gamma\log n-c_1+c_2\sum_{j=1}^{n-1}\frac{1}{j^2}\sum_{k=0}^{j-1}\varepsilon(f_k(0)).
\end{aligned}$$

Combining this with (31), we see that the convergence $L(n)\to\infty$ will be proved if we show that the latter sum grows unboundedly. To this end, we first notice that

$$\sum_{j=1}^{n-1}\frac{1}{j^2}\sum_{k=0}^{j-1}\varepsilon(f_k(0))=\sum_{k=0}^{n-2}\varepsilon(f_k(0))\sum_{j=k+1}^{n-1}\frac{1}{j^2}\geq\frac{n/2-1}{2n}\sum_{k=1}^{n/2}\frac{1}{k}\varepsilon(f_k(0)),$$

where in the last step we used the inequalities

$$\sum_{j=k+1}^{n-1}\frac{1}{j^2}\geq\int_{k+1}^{n}\frac{\mathrm{d}x}{x^2}=\frac{1}{k+1}-\frac{1}{n}\geq\frac{n/2-1}{2nk},\quad 1\leq k\leq n/2.$$

Thus we need to show that

$$\sum_{k=1}^{\infty}\frac{\varepsilon(f_k(0))}{k}=\infty. \tag{32}$$

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Changing the order of summation in (30), we have

$$\begin{aligned}\varepsilon(f_k(0)) &= \sum_{u=3}^{\infty} p_u \sum_{j=2}^{u-1} \sum_{v=1}^{j-1} \big(1 - f_k^v(0)\big) \\ &= \sum_{j=2}^{\infty} \sum_{u=j+1}^{\infty} p_u \sum_{v=1}^{j-1} \big(1 - f_k^v(0)\big) \\ &= \sum_{j=2}^{\infty} K_j \sum_{v=1}^{j-1} \big(1 - f_k^v(0)\big),\end{aligned}$$

where $K_j = \sum_{u=j+1}^{\infty} p_u$. Interchanging the order of the remaining two sums, we finally get

$$\varepsilon(f_k(0)) = \sum_{v=1}^{\infty} \big(1 - f_k^v(0)\big) \sum_{j=v+1}^{\infty} K_j = \sum_{v=1}^{\infty} \big(1 - f_k^v(0)\big)\overline{K_v},$$

where $\overline{K_v} = \sum_{j=v+1}^{\infty} K_j$. This representation implies that

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k} \varepsilon(f_k(0)) &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{v=1}^{\infty} \overline{K_v} \big(1 - f_k^v(0)\big) \\ &= \sum_{v=1}^{\infty} \overline{K_v} \sum_{k=1}^{\infty} \frac{1}{k} \big(1 - f_k^v(0)\big) \\ &\geq \sum_{v=1}^{\infty} \overline{K_v} \sum_{k=1}^{v} \frac{1 - f_k^v(0)}{k}.\end{aligned}$$

Noting that $f_k(0)$ is increasing in k and using (18), we obtain

$$\sum_{k=1}^{v} \frac{1 - f_k^v(0)}{k} \geq \big(1 - f_v^v(0)\big) \sum_{k=1}^{v} \frac{1}{k} \geq c \log v.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{k} \varepsilon(f_k(0)) \geq c \sum_{v=1}^{\infty} \overline{K_v} \log v.$$

Now it remains to notice that the assumption $\sum_{k=1}^{\infty} (k^2 \log k) p_k = \infty$ is equivalent to

$$\sum_{v=1}^{\infty} \overline{K_v} \log v = \infty.$$

This concludes the proof of (32), which, in turn, implies (27). □

To investigate the asymptotic behaviour of lower deviations for Y_n, we need the following lemma, which deals with lower and upper bounds for $\mathbb{P}(Y_n \leq k)$. Recall that in this section we always assume that $Y_0 = 0$.

Lemma 6. *There exist positive constants c_1 and c_2 such that*

$$\mathbb{P}(Y_n \leq k) \geq c_1 \prod_{j=k}^{n-1} h(f_j(0)), \quad k_0 \leq k \leq n \tag{33}$$

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and

$$\mathbb{P}(Y_n \le k) \le c_2 \prod_{j=k}^{n-1} h(f_j(0)), \quad 1 \le k \le n, \tag{34}$$

where $k_0 = \min\{k \ge 0 : q_k > 0\} \vee 1$.

Proof. We first prove (34). By the Markov inequality,

$$\mathbb{P}(Y_n \le k) = \mathbb{P}\big(s^{Y_n} \ge s^k\big) \le \frac{H_n(s)}{s^k}, \quad s \in (0, 1).$$

Letting $s = f_k(0)$, we get

$$\mathbb{P}(Y_n \le k) \le \frac{H_n(f_k(0))}{(f_k(0))^k}.$$

Notice that (18) implies that $(f_k(0))^k \ge c_3 > 0$ for all $k \ge 0$. Therefore

$$\mathbb{P}(Y_n \le k) \le \frac{H_n(f_k(0))}{c_3}.$$

Observe that

$$H_n(f_k(0)) = \prod_{j=0}^{n-1} h(f_j(f_k(0))) = \prod_{j=k}^{n+k-1} h(f_j(0)).$$

To complete the proof of (34), it remains to notice that

$$\prod_{j=k}^{n+k-1} h(f_j(0)) \le \prod_{j=k}^{n-1} h(f_j(0)).$$

To prove (33), we notice that if all immigrants from the first $n-k$ generations have no descendants at time n, then Y_n has the same distribution as Y_k. Therefore

$$\mathbb{P}(Y_n \le k) \ge \mathbb{P}(Y_k \le k) \prod_{i=1}^{n-k} \mathbb{P}\big(Z_{n-i}^{(i)} = 0\big).$$

Due to the limit theorem (4), there exists a positive number c such that

$$\mathbb{P}(Y_k \le k) \ge c > 0$$

uniformly in $k \ge k_0 := \min\{j \ge 0 : q_j > 0\} \vee 1$. Furthermore, it is immediate from the definition of processes $Z_{n-i}^{(i)}$ that

$$\prod_{i=1}^{n-k} \mathbb{P}\big(Z_{n-i}^{(i)} = 0\big) = \prod_{j=k}^{n-1} h(f_j(0)).$$

Thus the lower bound is also proved. □

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3. Estimates for concentration functions

The aim of this section is to provide some estimates for local probabilities of random variables Y_n and $Z_{n-i}^{(i)}$, which will be used in the proof of local asymptotics (8) and (9). Since these two statements use the aperiodicity condition on the distributions $\{p_k\}$ and $\{q_k\}$, we shall always assume in the present section that this condition holds.

We begin with bounds for the concentration function of Y_n.

Lemma 7. *For every $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that for all $n > N$ and $|s| \le 1$,*

$$\left|\frac{H_n(s)}{h(s)}\right| \le \exp\left\{-(h'(1) - B\varepsilon)\sum_{j=N}^{n-1} \mathrm{Re}(1 - f_j(s)) + \varepsilon \sum_{j=N}^{n-1} |\mathrm{Im}(1 - f_j(s))|\right\}.$$

Proof. The proof is the same as the proof of Lemma 6 in [6]. For readers' convenience and completeness, we provide the proof below.

Since $|h(f_j(s))| \le 1$, we obtain for all $n > N \ge 1$ the inequality

$$\left|\frac{H_n(s)}{h(s)}\right| = \left|\prod_{j=1}^{n-1} h(f_j(s))\right| \le \left|\prod_{j=N}^{n-1} h(f_j(s))\right| = \exp\left\{\sum_{j=N}^{n-1} \mathrm{Re}\log h(f_j(s))\right\}. \tag{35}$$

In view of (17), for $|s| \le 1$, we have

$$\mathrm{Re}\log h(f_j(s)) \le -(h'(1) - |\mathrm{Re}\,\alpha(f_j(s))|)\,\mathrm{Re}(1 - f_j(s)) + |\mathrm{Im}\,\alpha(f_j(s))|\,|\mathrm{Im}(1 - f_j(s))|. \tag{36}$$

It is immediate from the triangle inequality that

$$|1 - f_j(s)| \le \sum_{k=1}^{\infty} \mathbb{P}(Z_j = k)|1 - s^k| \le 2\mathbb{P}(Z_j > 0).$$

This implies that $f_j(s) \to 1$ as $j \to \infty$ uniformly in s from the unity disk. Thus one can choose $N = N(\varepsilon)$ such that for all $j \ge N$,

$$|\mathrm{Re}\,\alpha(f_j(s))| \le \varepsilon B, \quad |\mathrm{Im}\,\alpha(f_j(s))| \le \varepsilon.$$

Substituting the above bounds into (36) and recalling (35), we arrive at the conclusion of the lemma. □

Lemma 8. *For each $\varepsilon \in (0, 2\gamma)$ there exists a number $A(\varepsilon) > 0$ such that, for any $n \ge 1$ and $|t| \le \pi/2$,*

$$|H_n(\mathrm{e}^{\mathrm{i}t})| \le \left|\frac{H_n(\mathrm{e}^{\mathrm{i}t})}{h(\mathrm{e}^{\mathrm{i}t})}\right| \le A(\varepsilon)(n|t|)^{-2((\gamma-2\varepsilon)(1-\varepsilon)-2\varepsilon^2/B)}.$$

In particular, there exists a constant c^* such that, for every $n \ge 1$ and all $\frac{1}{n} \le |t| \le \pi/2$,

$$|H_n(\mathrm{e}^{\mathrm{i}t})| \le \left|\frac{H_n(\mathrm{e}^{\mathrm{i}t})}{h(\mathrm{e}^{\mathrm{i}t})}\right| \le c^*(n|t|)^{-\theta},$$

where $\theta = \min\{\gamma, 1/2\}$.

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Proof. Consider the function

$$\psi(x) = \frac{2(Bx/2+1)}{(Bx/2+1)^2 + c^2}, \quad x \geq 0.$$

It is immediate from Lemma 7 in [6] that

$$\left| \sum_{k=0}^{n-1} \psi(k) - \frac{2}{B} \log\left(1 + \frac{(Bn/2+1)^2}{c^2}\right) \right| \leq 1. \tag{37}$$

Further, by Lemma 8 in [6], there exist N and constants $a = a(\varepsilon, N)$ and $b = b(\varepsilon, N)$ such that for all $|t| \leq \pi/2$,

$$\sum_{j=N}^{n-1} \mathrm{Re}\big(1 - f_j(\mathrm{e}^{\mathrm{i}t})\big) \geq \frac{2(1-\varepsilon)}{B} \log\left(1 + (Bn/2+1)^2 \tan^2 \frac{t}{2}\right) - a, \tag{38}$$

$$\sum_{j=N}^{n-1} |\mathrm{Im}\big(1 - f_j(\mathrm{e}^{\mathrm{i}t})\big)| \leq \frac{2\varepsilon}{B} \log\left(1 + (Bn/2+1)^2 \tan^2 \frac{t}{2}\right) + b. \tag{39}$$

Using Lemma 8 and applying then (37)–(39) and $|x| \leq |\tan x|$, we conclude that

$$\begin{aligned}\left|\frac{H_n(\mathrm{e}^{\mathrm{i}t})}{h(\mathrm{e}^{\mathrm{i}t})}\right| &\leq A_1(\varepsilon) \exp\left\{\left(-(\gamma - 2\varepsilon)(1-\varepsilon) + \frac{2\varepsilon^2}{B}\right) \log\left(1 + (Bn/2+1)^2 \tan^2 \frac{t}{2}\right)\right\} \\ &\leq A_2(\varepsilon)(n^2 \tan^2(t/2))^{-((\gamma-2\varepsilon)(1-\varepsilon) - 2\varepsilon^2/B)} \\ &\leq A_3(\varepsilon)(n|t|)^{-2((\gamma-2\varepsilon)(1-\varepsilon) - 2\varepsilon^2/B)},\end{aligned}$$

where $A_1(\varepsilon)$, $A_2(\varepsilon)$, and $A_3(\varepsilon)$ are finite positive numbers. This gives the first estimate. Choosing ε so small that

$$2\left((\gamma - 2\varepsilon)(1-\varepsilon) - \frac{2\varepsilon^2}{B}\right) \geq \theta$$

and using the restriction $n|t| \geq 1$, we obtain the second estimate. □

Lemma 9. *There exists a constant c such that for all $n, m \geq 1$,*

$$\mathbb{P}(Y_n = m) \leq \frac{c}{m}.$$

Proof. We make use of the following bound for the concentration function (see e.g. the proof of Lemma 4 in [6]):

$$\sup_x \mathbb{P}(\xi = x) \leq \left(\frac{96}{95}\right)^2 \frac{1}{a} \int_{-a}^{a} |\varphi(t)| \, \mathrm{d}t, \tag{40}$$

where $\varphi(t)$ is the characteristic function of the random variable ξ, and $a > 0$.

Now let ξ be a random variable with the generating function $H_n'(s)/H_n'(1)$. Then, for any $n \geq 1$,

$$\sup_k k\mathbb{P}(Y_n = k) \leq \left(\frac{96}{95}\right)^2 \frac{1}{a} \int_{-a}^{a} |H_n'(\mathrm{e}^{\mathrm{i}t})| \, \mathrm{d}t. \tag{41}$$

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Observe that for $|s| \le 1$, we have

$$H_n'(s) = \sum_{j=0}^{n-1} h'(f_j(s)) f_j'(s) \prod_{0 \le l \le n-1,\ l \ne j} h(f_l(s)) = H_n(s) \sum_{l=0}^{n-1} \frac{h'(f_l(s))}{h(f_l(s))} f_l'(s).$$

We shall estimate integrals of

$$\left| H_n(\mathrm{e}^{\mathrm{i}t}) \frac{h'\big(f_l(\mathrm{e}^{\mathrm{i}t})\big)}{h\big(f_l(\mathrm{e}^{\mathrm{i}t})\big)} f_l'(\mathrm{e}^{\mathrm{i}t}) \right|$$

separately. If $l = 0$, then $f_l'(s) \equiv 1$ and the function $h'(f_l(\mathrm{e}^{\mathrm{i}t}))$ is bounded. Noting that

$$\left| \frac{H_n(\mathrm{e}^{\mathrm{i}t})}{h(\mathrm{e}^{\mathrm{i}t})} \right|$$

is bounded as well, we get

$$\int_{1/n < |t| < a} \left| H_n(\mathrm{e}^{\mathrm{i}t}) \frac{h'(\mathrm{e}^{\mathrm{i}t})}{h(\mathrm{e}^{\mathrm{i}t})} \right| \mathrm{d}t \le C. \tag{42}$$

Assume now that $l \ge 1$. As we have noticed before, $f_l(s) \to 1$ as $l \to \infty$ uniformly in $|s| \le 1$. This property allows us to conclude that if a is sufficiently small then

$$|h\big(f_l(\mathrm{e}^{\mathrm{i}t})\big)| \ge \frac{1}{2} \quad \text{for all } l \ge 1 \text{ and all } t \in [-a, a].$$

Combining this with the bounds $|f_l'(\mathrm{e}^{\mathrm{i}t})| \le 1$, $|h'\big(f_l(\mathrm{e}^{\mathrm{i}t})\big)| \le h'(1)$ and applying Lemma 8, we obtain

$$\int_{1/n < |t| < \min\{1/l, a\}} \left| H_n(\mathrm{e}^{\mathrm{i}t}) \frac{h'\big(f_l(\mathrm{e}^{\mathrm{i}t})\big)}{h\big(f_l(\mathrm{e}^{\mathrm{i}t})\big)} f_l'(\mathrm{e}^{\mathrm{i}t}) \right| \mathrm{d}t \le c^* \int_{1/n < |t| < 1/l} (n|t|)^{-\theta}\, \mathrm{d}t \le \frac{c^*}{1-\theta} \frac{l^{\theta-1}}{n^\theta}. \tag{43}$$

For $|t| \ge 1/l$, we additionally use the inequality

$$|f_l'(\mathrm{e}^{\mathrm{i}t})| \le \frac{c}{(l|t|)^{3/2}},$$

which was derived in the proof of Lemma 9 from [6]. As a result, we have

$$\begin{aligned} &\int_{\min\{1/l, a\} < |t| < a} \left| H_n(\mathrm{e}^{\mathrm{i}t}) \frac{h'\big(f_l(\mathrm{e}^{\mathrm{i}t})\big)}{h\big(f_l(\mathrm{e}^{\mathrm{i}t})\big)} f_l'(\mathrm{e}^{\mathrm{i}t}) \right| \mathrm{d}t \\ &\quad \le c_1 \int_{\min\{1/l, a\} < |t| < a} (n|t|)^{-\theta} (l|t|)^{-3/2}\, \mathrm{d}t \\ &\quad \le c_2 \frac{l^{\theta-1}}{n^\theta}. \end{aligned} \tag{44}$$

Summing over l the estimates in (43) and (44) and taking into account (42), we obtain

$$\int_{1/n < |t| < a} |H_n'(\mathrm{e}^{\mathrm{i}t})|\, \mathrm{d}t \le c.$$

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Using the elementary bound

$$|H_n'(\mathrm{e}^{\mathrm{i}t})| \le h'(1)n,$$

we get

$$\int_{|t|<1/n} |H_n'(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t \le 2h'(1).$$

Consequently

$$\int_{-a}^{a} |H_n'(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t \le c.$$

Plugging this into (41), we complete the proof of the lemma. □

For the proof of our main result, we also need a concentration function bound for the Galton–Watson process $\{Z_n^{(l)}\}$, which starts with a random number of particles.

Lemma 10. *There exists a constant c such that for all $1 \le l < n$ we have*

$$\mathbb{P}\big(Z_{n-l}^{(l)} > 0\big) \le \frac{c}{n-l}$$

and

$$\max_{k\ge 1} \mathbb{P}\big(Z_{n-l}^{(l)} = k\big) \le \frac{c}{(n-l)^2}.$$

The first inequality holds without the aperiodicity assumption.

Proof. By the definition of $Z_{n-l}^{(l)}$,

$$\mathbb{P}\big(Z_{n-l}^{(l)} > 0\big) = 1 - h(f_{n-l}(0)) \le h'(1)(1 - f_{n-l}(0)) \le \frac{c}{n-l},$$

where in the last step we used (18).

Let Z_n denote a critical Galton–Watson process. It has been shown in the proposition in [6] that

$$\max_{k\ge 1} \mathbb{P}(Z_n = k \mid Z_0 = 1) \le \frac{c_0}{(n+1)^2}.$$

By the branching property, for all $k \ge 1$ and $j \ge 2$,

$$\begin{aligned}\mathbb{P}(Z_n = k \mid Z_0 = j) &= \sum_{l=0}^{k} \mathbb{P}(Z_n = l \mid Z_0 = 1)\,\mathbb{P}(Z_n = k-l \mid Z_0 = j-1)\\ &\le \mathbb{P}(Z_n = k \mid Z_0 = j-1) + \max_{l\ge 1} \mathbb{P}(Z_n = l \mid Z_0 = 1).\end{aligned}$$

Taking the maximum over k, we then have

$$\begin{aligned}\max_{k\ge 1} \mathbb{P}(Z_n = k \mid Z_0 = j) &\le \max_{k\ge 1} \mathbb{P}(Z_n = k \mid Z_0 = j-1) + \max_{k\ge 1} \mathbb{P}(Z_n = k \mid Z_0 = 1)\\ &\le j \max_{k\ge 1} \mathbb{P}(Z_n = k \mid Z_0 = 1), \quad j \ge 1.\end{aligned}$$

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Therefore

$$\max_{k\geq 1}\mathbb{P}(Z_n=k\mid Z_0=j)\leq\frac{c_0 j}{(n+1)^2} \tag{45}$$

for all $j\geq 1$. Combining this estimate with the total probability law, we have

$$\begin{aligned}\mathbb{P}\big(Z_{n-l}^{(l)}=k\big)&=\sum_{j=1}^{\infty}q_j\mathbb{P}(Z_{n-l}=k\mid Z_0=j)\\&\leq\frac{c_0}{(n-l)^2}\sum_{j=1}^{\infty}jq_j.\end{aligned}$$

Since the right-hand side does not depend on k and since the expectation of $\{q_j\}$ is finite, we obtain the desired bound. □

Lemma 11. *There exists a constant c such that for all $1\leq l<n$ we have*

$$\mathbb{P}\big(Z_{n-l}^{(l)}=k\big)\leq\frac{c}{k(n-l)},\quad k\geq 1.$$

Proof. The generating function of $Z_{n-l}^{(l)}$ equals $h(f_{n-l}(s))$. Therefore the generating function of the sequence

$$\big\{k\mathbb{P}\big(Z_{n-l}^{(l)}=k\big)\big\}_{k\geq 1}$$

equals $h'(f_{n-l}(s))f'_{n-l}(s)$. Applying (40) with $a=\pi/2$, we have

$$k\mathbb{P}\big(Z_{n-l}^{(l)}=k\big)\leq c\int_{-\pi/2}^{\pi/2}|h'\big(f_{n-l}(\mathrm{e}^{\mathrm{i}t})\big)f'_{n-l}(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t\leq ch'(1)\int_{-\pi/2}^{\pi/2}|f'_{n-l}(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t.$$

It has been shown in the proof of Lemma 9 in [6] that the integral on the right-hand side is bounded by $c/(n-l)$. Thus the proof is finished. □

The next result is a simple generalization of the local limit theorem for critical Galton–Watson processes proved in [6].

Lemma 12. *Let Z_n be a critical Galton–Watson process with an offspring law having finite variance. Let $g(s)$ be a generating function of Z_0. If $g'(1)$ is finite then*

$$\mathbb{P}(Z_n=j)=\frac{4g'(1)}{B^2n^2}\exp\left\{-\frac{2j}{Bn}\right\}(1+\mathrm{o}(1))$$

for $j,n\to\infty$ in such a way that the ratio j/n remains bounded.

Proof. Let $\{g_k\}$ denote the coefficients of $g(s)$. Then we have

$$\begin{aligned}\mathbb{P}(Z_n=j)&=\sum_{k=1}^{\infty}g_k\mathbb{P}(Z_n=j\mid Z_0=k)\\&=\sum_{k=1}^{\infty}g_k\sum_{t=1}^{k}\binom{k}{t}(1-f_n(0))^t f_n^{k-t}(0)\,\mathbb{P}\left(\sum_{i=1}^{t}\xi_i^{(n)}=j\right),\end{aligned}$$

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where $\{\xi_m^{(n)}\}$ is a sequence of independent and identically distributed random variables with joint distribution

$$\mathbb{P}\big(\xi_m^{(n)} = j\big) = \mathbb{P}(Z_n = j \mid Z_0 = 1, Z_n > 0).$$

Separating the summands corresponding to $t = 1$, we have

$$\begin{aligned}\mathbb{P}(Z_n = j) &= \sum_{k=1}^{\infty} k g_k f_n^{k-1}(0)\, \mathbb{P}(Z_n = j \mid Z_0 = 1) \\ &\quad + \sum_{k=1}^{\infty} g_k \sum_{t=2}^{k} \binom{k}{t} (1 - f_n(0))^t f_n^{k-t}(0)\, \mathbb{P}\left(\sum_{i=1}^{t} \xi_i^{(n)} = j\right). \end{aligned} \tag{46}$$

According to Lemma 11 in [6],

$$\max_{j \geq 1} \mathbb{P}\left(\sum_{i=1}^{t} \xi_i^{(n)} = j\right) \leq \frac{c}{n\sqrt{t}}.$$

Therefore

$$\begin{aligned}&\sum_{k=1}^{\infty} g_k \sum_{t=2}^{k} \binom{k}{t} (1 - f_n(0))^t f_n^{k-t}(0)\, \mathbb{P}\left(\sum_{i=1}^{t} \xi_i^{(n)} = j\right) \\ &\quad \leq \frac{c}{n} \sum_{k=1}^{\infty} g_k \sum_{t=2}^{k} \binom{k}{t} (1 - f_n(0))^t f_n^{k-t}(0) \\ &\quad = \frac{c}{n} \sum_{t=2}^{\infty} \frac{1}{t!} (1 - f_n(0))^t \sum_{k=t}^{\infty} g_k \frac{k!}{(k-t)!} f_n^{k-t}(0) \\ &\quad = \frac{c}{n} \sum_{t=2}^{\infty} \frac{1}{t!} (1 - f_n(0))^t g^{(t)}(f_n(0)) \\ &\quad = \frac{c}{n} (1 - g(f_n(0)) - g'(f_n(0))(1 - f_n(0))). \end{aligned}$$

In the last step we used the Taylor series representation for g at point $f_n(0)$. The assumption $g'(1) < \infty$ implies that $g'(r_n) \to g'(1)$ for any sequence $\{r_n\}$ with values in [0,1] such that $r_n \to 1$. Combining this property with the mean value theorem, we obtain

$$1 - g(f_n(0)) - g'(f_n(0))(1 - f_n(0)) = o(1 - f_n(0)) = o(n^{-1}) \quad \text{as } n \to \infty.$$

Consequently

$$\sum_{k=1}^{\infty} g_k \sum_{t=2}^{k} \binom{k}{t} (1 - f_n(0))^t f_n^{k-t}(0)\, \mathbb{P}\left(\sum_{i=1}^{t} \xi_i^{(n)} = j\right) = o(n^{-2}) \quad \text{as } n \to \infty \tag{47}$$

uniformly in j. It remains to notice that the theorem in [6] implies that

$$\begin{aligned}\sum_{k=1}^{\infty} k g_k f_n^{k-1}(0)\, \mathbb{P}(Z_n = j \mid Z_0 = 1) &= g'(f_n(0))\, \mathbb{P}(Z_n = j \mid Z_0 = 1) \\ &= \frac{4g'(1)}{B^2 n^2} \exp\left\{-\frac{2j}{Bn}\right\}(1 + o(1))\end{aligned}$$

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provided that $j, n \to \infty$ in such a way that the ratio j/n remains bounded. Plugging this relation and (47) into (46), we get the desired asymptotic equivalence. □

4. Asymptotics for lower deviation probabilities: Proof of Theorem 1

Notice that the properties of the slowly varying function $L(n)$ mentioned in Theorem 1 are proved in Lemmas 4 and 5. Thus we have to show the asymptotic relations (7)–(9). Recall also that the local asymptotics (8) and (9) use the aperiodicity condition and that (7) should be proved without this condition.

We start by proving (8) and (9). Let us first show that the case $Y_0 = i$ with arbitrary $i \geq 0$ can be reduced to the case $Y_0 = 0$. By the branching property,

$$\begin{aligned}
&\mathbb{P}(Y_n = k \mid Y_0 = i) \\
&\quad = \sum_{j=0}^{k} \mathbb{P}(Y_n = j \mid Y_0 = 0)\, \mathbb{P}(Z_n = k - j \mid Z_0 = i) \\
&\quad = \mathbb{P}(Y_n = k \mid Y_0 = 0)(f_n(0))^i + \sum_{j=0}^{k-1} \mathbb{P}(Y_n = j \mid Y_0 = 0)\, \mathbb{P}(Z_n = k - j \mid Z_0 = i). \qquad (48)
\end{aligned}$$

Using (45), we obtain

$$\sum_{j=0}^{k-1} \mathbb{P}(Y_n = j \mid Y_0 = 0)\, \mathbb{P}(Z_n = k - j \mid Z_0 = i) \leq C \frac{i}{n^2} \mathbb{P}(Y_n \leq k \mid Y_0 = 0).$$

Furthermore, combining Lemma 3 with the upper bound from Lemma 6, we have

$$\mathbb{P}(Y_n \leq k \mid Y_0 = 0) \leq C \frac{k^\gamma L(k)}{n^\gamma L(n)}, \quad 1 \leq k \leq n.$$

Combining these estimates with the representation (48), we obtain

$$|\mathbb{P}(Y_n = k \mid Y_0 = i) - \mathbb{P}(Y_n = k \mid Y_0 = 0)(f_n(0))^i| \leq Ci \frac{k^\gamma L(k)}{n^{\gamma+2} L(n)}, \quad 1 \leq k \leq n.$$

Since, for every fixed i, the right-hand side in this inequality is negligibly small in comparison with right-hand sides in (8) and (9), we get the desired reduction to the case of starting state 0.

From now on, we shall assume that $Y_0 = 0$. By the law of total probability, for every $k \geq 1$,

$$\mathbb{P}(Y_n = k) = \sum_{l=1}^{n} \mathbb{P}(Y_n = k, \theta_n = l). \qquad (49)$$

Furthermore, for every l we have

$$\begin{aligned}
\mathbb{P}(Y_n = k, \theta_n = l) &= \mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k, Z_{n-l}^{(l)} > 0\big) \prod_{j=1}^{l-1} \mathbb{P}\big(Z_{n-j}^{(j)} = 0\big) \\
&= \mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k, Z_{n-l}^{(l)} > 0\big) \prod_{j=n-l+1}^{n-1} h(f_j(0)). \qquad (50)
\end{aligned}$$

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We split the sum in (49) into three parts. More precisely, we fix some $\varepsilon \in (0, 1]$ and consider separately the sums over $[1, n - \varepsilon^{-1}k)$, $[n - \varepsilon^{-1}k, n - \varepsilon k)$, and $[n - \varepsilon k, n]$.

To estimate the sum over $[1, n - \varepsilon^{-1}k)$, we first apply Lemma 10 to get

$$\begin{aligned}\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k, Z_{n-l}^{(l)} > 0\big) &= \sum_{m=1}^{k} \mathbb{P}\big(Z_{n-l}^{(l)} = m\big)\, \mathbb{P}(Y_{n-l} = k - m) \\ &\le \frac{c}{(n-l)^2} \sum_{m=1}^{k} \mathbb{P}(Y_{n-l} = k - m) \\ &\le \frac{c}{(n-l)^2} \mathbb{P}(Y_{n-l} \le k).\end{aligned}$$

Now applying Lemmas 3 and 6, we conclude that

$$\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k, Z_{n-l}^{(l)} > 0\big) \le c_1 \frac{1}{(n-l)^2} \frac{k^\gamma L(k)}{(n-l)^\gamma L(n-l)}.$$

Combining this with (50) and once again using Lemma 3, we obtain

$$\begin{aligned}\mathbb{P}(Y_n = k, \theta_n = l) &\le c_1 \frac{1}{(n-l)^2} \frac{k^\gamma L(k)}{(n-l)^\gamma L(n-l)} \prod_{j=n-l+1}^{n-1} h(f_j(0)) \\ &\le c_2 \frac{1}{(n-l)^2} \frac{k^\gamma L(k)}{(n-l)^\gamma L(n-l)} \frac{(n-l)^\gamma L(n-l)}{n^\gamma L(n)} \\ &= c_2 \frac{1}{(n-l)^2} \frac{k^\gamma L(k)}{n^\gamma L(n)}.\end{aligned}$$

This bound implies that

$$\begin{aligned}\sum_{l \le n - \varepsilon^{-1}k} \mathbb{P}(Y_n = k, \theta_n = l) &\le c_2 \frac{k^\gamma L(k)}{n^\gamma L(n)} \sum_{l \le n - \varepsilon^{-1}k} \frac{1}{(n-l)^2} \\ &\le c_3 \varepsilon \frac{k^{\gamma-1} L(k)}{n^\gamma L(n)}. \qquad (51)\end{aligned}$$

Combining Lemmas 9, 10, and 11, we have, for all $l < n$,

$$\begin{aligned}&\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k, Z_{n-l}^{(l)} > 0\big) \\ &= \sum_{m=1}^{k} \mathbb{P}\big(Z_{n-l}^{(l)} = m\big)\, \mathbb{P}(Y_{n-l} = k - m) \\ &= \sum_{m=1}^{k/2} \mathbb{P}\big(Z_{n-l}^{(l)} = m\big)\, \mathbb{P}(Y_{n-l} = k - m) + \sum_{m \in (k/2, k]} \mathbb{P}\big(Z_{n-l}^{(l)} = m\big)\, \mathbb{P}(Y_{n-l} = k - m) \\ &\le \frac{c_1}{k} \mathbb{P}\big(Z_{n-l}^{(l)} > 0\big) + \frac{c_2}{k(n-l)} \mathbb{P}(Y_{n-l} \le k/2) \\ &\le \frac{c_3}{k(n-l)}.\end{aligned}$$

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Recall that, as usual, numbers c_j do not depend on k, n, l, and ε, and may change their values from place to place.

Combining (18) with the bound $1-h(f_{n-l}(0)) \geq h'(f_{n-l}(0))(1-f_{n-l}(0))$, we conclude that $1-h(f_{n-l}(0)) \geq c_4/(n-l)$. Therefore

$$\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k,\, Z_{n-l}^{(l)} > 0\big) \leq c\frac{1-h(f_{n-l}(0))}{k}, \quad l < n.$$

In the case $l = n$ we have, by the Markov inequality,

$$\mathbb{P}\big(Z_0^{(n)} + Y_0 = k,\, Z_0^{(n)} > 0\big) = q_k \leq \frac{\lambda}{k} \leq c\frac{1-h(0)}{k}.$$

Applying these bounds to the right-hand side of (50) and taking into account (14), we obtain

$$\begin{aligned}\sum_{l\geq n-\varepsilon k} \mathbb{P}(Y_n = k,\, \theta_n = l) &\leq \frac{c}{k}\sum_{l\geq n-\varepsilon k}(1-h(f_{n-l}(0)))\prod_{j=n-l+1}^{n-1} h(f_j(0)) \\ &= \frac{c}{k}\sum_{l\geq n-\varepsilon k}\mathbb{P}(\theta_n = l) \\ &\leq \frac{c}{k}\prod_{j=\varepsilon k}^{n-1} h(f_j(0)).\end{aligned}$$

Now applying Lemma 3, we conclude that

$$\sum_{l\geq n-\varepsilon k}\mathbb{P}(Y_n = k,\, \theta_n = l) \leq c_1\frac{1}{k}\frac{(\varepsilon k)^\gamma L(\varepsilon k)}{n^\gamma L(n)} \leq c_2\varepsilon^\gamma\frac{k^{\gamma-1}L(k)}{n^\gamma L(n)}. \tag{52}$$

Thus it remains to consider the sum over the interval $I_k := [n-k/\varepsilon,\, n-\varepsilon k]$. Fix a sequence $\{M_k\}$ such that $M_k \to \infty$ and $M_k = \mathrm{o}(k)$. Applying Lemma 12, we have, uniformly in $l \in I_k$ and $m \in (M_k,\, k]$,

$$\mathbb{P}\big(Z_{n-l}^{(l)} = m\big) = h'(1)\frac{4}{B^2(n-l)^2}\exp\left\{-\frac{2m}{B(n-l)}\right\} + \mathrm{o}\left(\frac{1}{(n-l)^2}\right) \quad \text{as } n\to\infty.$$

Therefore, as $n \to \infty$,

$$\begin{aligned}&\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k,\, Z_{n-l}^{(l)} > M_k\big) \\ &\quad= \sum_{m=M_k+1}^{k}\mathbb{P}\big(Z_{n-l}^{(l)} = m\big)\,\mathbb{P}(Y_{n-l} = k-m) \\ &\quad= \frac{4h'(1)}{B^2(n-l)^2}\mathbb{E}\left[\exp\left(-\frac{2(k-Y_{n-l})}{B(n-l)}\right);\, Y_{n-l} < k - M_k\right] + \mathrm{o}\left(\frac{1}{(n-l)^2}\right).\end{aligned} \tag{53}$$

Furthermore, using Lemma 10, we have

$$\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k,\, 0 < Z_{n-l}^{(l)} \leq M_k\big) \leq \frac{c}{(n-l)^2}\mathbb{P}(Y_{n-l} \in [k-M_k,\, k)).$$

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Combining this with (53) and noting that the limit theorem (4) implies that

$$\mathbb{P}(Y_{n-l} \in [k - M_k, k)) = \mathrm{o}(1) \quad \text{as } k, n \to \infty,$$

we obtain the representation

$$\begin{aligned}&\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k, Z_{n-l}^{(l)} > 0\big)\\ &\quad = \frac{4h'(1)}{B^2(n-l)^2}\mathbb{E}\bigg[\exp\bigg(-\frac{2(k - Y_{n-l})}{B(n-l)}\bigg); Y_{n-l} < k\bigg] + \mathrm{o}\bigg(\frac{1}{(n-l)^2}\bigg) \quad \text{as } n \to \infty.\end{aligned}$$

According to (4), $2Y_{n-l}/(B(n-l))$ converges weakly to the gamma distribution with density $\Gamma(\gamma)^{-1}u^{\gamma-1}\mathrm{e}^{-u}$. For every $R \geq 0$, the functional $F_R(u) := \mathrm{e}^u 1\{u < R\}$ is bounded and its discontinuity has probability zero according to the limiting gamma distribution. This implies that

$$\mathbb{E}\bigg[F_R\bigg(\frac{2Y_{n-l}}{B(n-l)}\bigg)\bigg] \to \int_0^R \mathrm{e}^u \frac{1}{\Gamma(\gamma)}u^{\gamma-1}\mathrm{e}^{-u}\,\mathrm{d}u = \frac{R^\gamma}{\Gamma(\gamma+1)}$$

for every $R \geq 0$. Noting that both sides in this relation are increasing in R and that the right-hand side is continuous in R, we conclude that the convergence is uniform on compact subsets of $[0, \infty)$. For every $l \in I_k$, we have

$$\frac{2k}{B(n-l)} \leq \frac{2}{B\varepsilon}.$$

Then, by the uniform convergence on compact sets, as $n \to \infty$,

$$\mathbb{E}\bigg[\exp\bigg(\frac{2Y_{n-l}}{B(n-l)}\bigg); Y_{n-l} < k\bigg] = \frac{1}{\Gamma(\gamma+1)}\bigg(\frac{2k}{B(n-l)}\bigg)^\gamma + \mathrm{o}(1).$$

Consequently, uniformly in $l \in I_k$,

$$\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} = k, Z_{n-l}^{(l)} > 0\big) = \frac{h'(1) + \mathrm{o}(1)}{\Gamma(\gamma+1)}\frac{1}{k^2}\bigg(\frac{2k}{B(n-l)}\bigg)^{\gamma+2}\exp\bigg(-\frac{2k}{B(n-l)}\bigg). \tag{54}$$

We know from Lemma 3 that

$$\prod_{j=n-l+1}^{n-1} h(f_j(0)) \sim \frac{(n-l)^\gamma L(n-l)}{n^\gamma L(n)} \quad \text{as } k, n \to \infty, \tag{55}$$

uniformly in $l \in I_k$.

Plugging (54) and (55) into (50) and summing over l, we deduce

$$\begin{aligned}\sum_{l \in I_k}\mathbb{P}(Y_n = k, \theta_n = l) &= \frac{h'(1) + \mathrm{o}(1)}{\Gamma(\gamma+1)n^\gamma L(n)}\frac{1}{k^2}\bigg(\frac{2k}{B}\bigg)^{\gamma+2}\sum_{l \in I_k}\frac{L(n-l)}{(n-l)^2}\exp\bigg(-\frac{2k}{B(n-l)}\bigg)\\ &= \frac{h'(1) + \mathrm{o}(1)}{\Gamma(\gamma+1)}\bigg(\frac{2}{B}\bigg)^{\gamma+2}\frac{k^\gamma L(k)}{n^\gamma L(n)}\sum_{l \in I_k}\frac{\exp\big(-\frac{2k}{B(n-l)}\big)}{(n-l)^2}.\end{aligned} \tag{56}$$

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Approximating the sum by the integral, we obtain

$$\begin{aligned}\sum_{l\in I_k}\frac{\exp\bigl(-\frac{2k}{B(n-l)}\bigr)}{(n-l)^2}&=\frac{1}{k}\sum_{\varepsilon k\le j\le k/\varepsilon}\left(\frac{k}{j}\right)^2\exp\left(-\frac{2k}{Bj}\right)\frac{1}{k}\\&=\frac{1+\mathrm{o}(1)}{k}\int_{\varepsilon}^{1/\varepsilon}u^{-2}\mathrm{e}^{-2/Bu}\,\mathrm{d}u\\&=\frac{B+\mathrm{o}(1)}{2k}\bigl(\mathrm{e}^{-2\varepsilon/B}-\mathrm{e}^{-2/(B\varepsilon)}\bigr).\end{aligned}$$

Plugging this into (56) and recalling that $\gamma=2h'(1)/B$, we have

$$\sum_{l\in I_k}\mathbb{P}(Y_n=k,\,\theta_n=l)=\frac{1+\mathrm{o}(1)}{\Gamma(\gamma)}\left(\frac{2}{B}\right)^{\gamma}\frac{k^{\gamma-1}L(k)}{n^{\gamma}L(n)}\bigl(\mathrm{e}^{-2\varepsilon/B}-\mathrm{e}^{-2/(B\varepsilon)}\bigr).\tag{57}$$

Now combining (51), (52), (57) and letting $\varepsilon\to 0$, we get (8).

To establish (9), we shall again use (49). Since (51) is valid for all k it remains to consider the case when $l\ge n-k/\varepsilon$. In view of (50), we have to determine the asymptotic behaviour of

$$\sum_{m\le k/\varepsilon}\mathbb{P}\bigl(Z_m^{(1)}+Y_m=k,\,Z_m^{(1)}>0\bigr)\prod_{j=m+1}^{n-1}h(f_j(0)).$$

Since this sum can be rewritten as

$$\begin{aligned}&\sum_{m\le k/\varepsilon}\mathbb{P}\bigl(Z_m^{(1)}+Y_m=k,\,Z_m^{(1)}>0\bigr)\prod_{j=m+1}^{n-1}h(f_j(0))\\&\quad=\prod_{l=0}^{n-1}h(f_l(0))\sum_{m\le k/\varepsilon}\frac{\mathbb{P}\bigl(Z_m^{(1)}+Y_m=k,\,Z_m^{(1)}>0\bigr)}{\prod_{j=0}^{m}h(f_j(0))},\end{aligned}$$

we infer from Lemma 2 that, as $n\to\infty$,

$$\sum_{l>n-\varepsilon^{-1}k}\mathbb{P}(Y_n=k,\,\theta_n=l)\sim\frac{1}{n^{\gamma}L(n)}\sum_{m\le k/\varepsilon}\frac{\mathbb{P}\bigl(Z_m^{(1)}+Y_m=k,\,Z_m^{(1)}>0\bigr)}{\prod_{j=0}^{m}h(f_j(0))}.$$

Combining this with (51) and letting $\varepsilon\to 0$, we get, for every fixed k,

$$\lim_{n\to\infty}n^{\gamma}L(n)\,\mathbb{P}(Y_n=k)=\sum_{m\ge 0}\frac{\mathbb{P}\bigl(Z_m^{(1)}+Y_m=k,\,Z_m^{(1)}>0\bigr)}{\prod_{j=0}^{m}h(f_j(0))}=:\mu_k.\tag{58}$$

It follows from Lemma 10 that

$$\begin{aligned}\mathbb{P}\bigl(Z_m^{(1)}+Y_m=k,\,Z_m^{(1)}>0\bigr)&=\sum_{j=1}^{k}\mathbb{P}\bigl(Z_m^{(1)}=j\bigr)\,\mathbb{P}(Y_m=k-j)\\&\le\frac{c}{m^2}\sum_{j=1}^{k}\mathbb{P}(Y_m=k-j)\\&=\frac{c}{m^2}\mathbb{P}(Y_m\le k)\end{aligned}$$

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for all $m, k \geq 1$. Furthermore, by Lemma 6,

$$\mathbb{P}(Y_m \leq k) \leq c \prod_{j=k}^{m-1} h(f_j(0)), \quad m > k.$$

Therefore

$$\frac{\mathbb{P}\big(Z_m^{(1)} + Y_m = k, Z_m^{(1)} > 0\big)}{\prod_{j=0}^{m} h(f_j(0))} \leq \frac{c}{m^2} \frac{1}{h(f_m(0)) \prod_{j=0}^{k-1} h(f_j(0))}, \quad m > k \geq 1.$$

This implies that the series on the right-hand side of (58) is finite: $\mu_k < \infty$ for every k. This completes the proof of (9).

To complete the proof of Theorem 1, it remains to show (7). Its proof is quite similar to that of (8) and we will only sketch it, omitting some details. We start by noting that, as $n \to \infty$,

$$\mathbb{P}(Y_n = 0) = \prod_{l=0}^{n-1} h(f_k(0)) \sim \frac{1}{n^\gamma L(n)}.$$

Thus it remains to find asymptotics for $\mathbb{P}(1 \leq Y_n \leq k)$. For this probability, we have

$$\begin{aligned}\mathbb{P}(1 \leq Y_n \leq k) &= \sum_{l=1}^{n} \mathbb{P}(1 \leq Y_n \leq k, \theta_n = l) \\ &= \sum_{l=1}^{n} \mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} \leq k, Z_{n-l}^{(l)} > 0\big) \prod_{j=n-l+1}^{n-1} h(f_j(0)). \end{aligned} \tag{59}$$

Again we fix some $\varepsilon \in (0, 1]$ and split the sum into three parts, as was done in the proof of (8). In the case $l < n - \varepsilon^{-1}k$, we apply the bound

$$\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} \leq k, Z_{n-l}^{(l)} > 0\big) \leq \mathbb{P}\big(1 \leq Z_{n-l}^{(l)} \leq k\big)\mathbb{P}(Y_{n-l} \leq k).$$

There exists a constant c such that

$$\mathbb{P}\big(1 \leq Z_{n-l}^{(l)} \leq k\big) \leq c \frac{k}{(n-l)^2}. \tag{60}$$

It is worth mentioning that Lemma 10 cannot be used here, since we want to prove (7) without the aperiodicity assumption on the distributions $\{p_k\}$ and $\{q_k\}$.

To prove (60), we recall that if $Z_0^{(l)} = m \geq 2$, then $Z_{n-l}^{(l)}$ is the sum of m independent copies of the process starting from one ancestor. Decomposing according to the value of one fixed summand, we have

$$\begin{aligned}&\mathbb{P}\big(1 \leq Z_{n-l}^{(l)} \leq k \mid Z_0^{(l)} = m\big) \\ &\quad = \sum_{j=0}^{k} \mathbb{P}\big(Z_{n-l}^{(l)} = j \mid Z_0^{(l)} = 1\big)\mathbb{P}\big(1 - j \leq Z_{n-l}^{(l)} \leq k - j \mid Z_0^{(l)} = m - 1\big) \\ &\quad \leq \mathbb{P}\big(1 \leq Z_{n-l}^{(l)} \leq k \mid Z_0^{(l)} = m - 1\big) + \sum_{j=1}^{k} \mathbb{P}\big(Z_{n-l}^{(l)} = j \mid Z_0^{(l)} = 1\big) \\ &\quad = \mathbb{P}\big(1 \leq Z_{n-l}^{(l)} \leq k \mid Z_0^{(l)} = m - 1\big) + \mathbb{P}\big(1 \leq Z_{n-l}^{(l)} \leq k \mid Z_0^{(l)} = 1\big) \\ &\quad = m\mathbb{P}\big(1 \leq Z_{n-l}^{(l)} \leq k \mid Z_0^{(l)} = 1\big).\end{aligned}$$

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Now taking the expectation, we get

$$\mathbb{P}\big(1 \le Z_{n-l}^{(l)} \le k\big) \le h'(1)\mathbb{P}\big(1 \le Z_{n-l}^{(l)} \le k \mid Z_0^{(l)} = 1\big).$$

Applying Lemma 1 from [6], we arrive at (60).

Combining (60) with Lemma 6, we get

$$\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} \le k,\, Z_{n-l}^{(l)} > 0\big) \le ck\frac{k^\gamma L(k)}{(n-l)^{\gamma+2}L(n-l)}.$$

Now recalling that

$$\prod_{j=n-l+1}^{n-1} h(f_j(0)) \le c\frac{(n-l)^\gamma L(n-l)}{n^\gamma L(n)},$$

we conclude that

$$\begin{aligned}\sum_{l=1}^{n-\varepsilon^{-1}k} \mathbb{P}(1 \le Y_n \le k,\, \theta_n = l) &\le c_1 k\frac{k^\gamma L(k)}{n^\gamma L(n)} \sum_{l=1}^{n-\varepsilon^{-1}k} (n-l)^{-2} \\ &\le c_2\varepsilon\frac{k^\gamma L(k)}{n^\gamma L(n)}. \end{aligned} \tag{61}$$

In the case $l > n - \varepsilon k$, we shall apply the following rather simple bound:

$$\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} \le k,\, Z_{n-l}^{(l)} > 0\big) \le \mathbb{P}\big(Z_{n-l}^{(l)} > 0\big) \le \frac{c}{n-l+1},$$

where in the last step we used Lemma 10. This implies that

$$\begin{aligned}\sum_{l\in(n-\varepsilon k,n]} \mathbb{P}(1 \le Y_n \le k,\, \theta_n = l) &\le c_1 \sum_{l\in(n-\varepsilon k,n]} \frac{(n-l+1)^{\gamma-1}L(n-l+1)}{n^\gamma L(n)} \\ &\le c_2\varepsilon^\gamma \frac{k^\gamma L(k)}{n^\gamma L(n)}.\end{aligned}$$

We know that $2Y_{n-l}/(B(n-l))$ converges weakly, as $n-l\to\infty$, to the gamma distribution with density $\Gamma(\gamma)^{-1}u^{\gamma-1}\mathrm{e}^{-u}$, $u>0$. Furthermore, the conditional distribution of $2Z_{n-l}^{(l)}/(B(n-l))$ conditioned on $Z_{n-l}^{(l)} > 0$ converges weakly to the exponential distribution with density e^{-v}, $v>0$. (This convergence is a simple consequence of the classical Yaglom theorem. One can just repeat the arguments used in the proof of Lemma 12.) Combining these two convergences and recalling that Y_{n-l} and $Z_{n-l}^{(l)}$ are independent, we conclude that the sum

$$\frac{2Y_{n-l}}{B(n-l)} + \frac{2Z_{n-l}^{(l)}}{B(n-l)}$$

conditioned on $Z_{n-l}^{(l)} > 0$ converges to the convolution of the above gamma distribution and exponential distribution. This limiting distribution then has the density

$$\int_0^x \frac{u^{\gamma-1}\mathrm{e}^{-u}}{\Gamma(\gamma)}\mathrm{e}^{-(x-u)}\,\mathrm{d}u = \frac{x^\gamma \mathrm{e}^{-x}}{\Gamma(\gamma+1)}, \quad x>0.$$

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Therefore, for every l in I_k,

$$\mathbb{P}\big(Z_{n-l}^{(l)} + Y_{n-l} \le k \mid Z_{n-l}^{(l)} > 0\big) - \phi\left(\frac{2k}{B(n-l)}\right) = \mathrm{o}(1) \quad \text{as } n \to \infty,$$

where

$$\phi(v) = \frac{1}{\Gamma(\gamma+1)} \int_0^v u^\gamma \mathrm{e}^{-u}\, \mathrm{d}u.$$

Furthermore,

$$\mathbb{P}\big(Z_{n-l}^{(l)} > 0\big) \sim \frac{\gamma}{n-l}.$$

Combining these relations with Lemma 3, we infer that

$$\begin{aligned}\sum_{l\in I_k} \mathbb{P}(1 \le Y_n \le k,\, \theta_n = l) &\sim \frac{\gamma}{n^\gamma L(n)} \sum_{l\in I_k} (n-l)^{\gamma-1} L(n-l)\phi\left(\frac{2k}{B(n-l)}\right) \\ &\sim \gamma \frac{k^\gamma L(k)}{n^\gamma L(n)} \sum_{l\in I_k} \left(\frac{n-l}{k}\right)^{\gamma-1} \phi\left(\frac{2k}{B(n-l)}\right)\frac{1}{k} \\ &\sim \gamma \left(\frac{2}{B}\right)^\gamma \frac{k^\gamma L(k)}{n^\gamma L(n)} \int_{B\varepsilon/2}^{B/(2\varepsilon)} u^{\gamma-1}\phi(1/u)\, \mathrm{d}u. \end{aligned} \tag{62}$$

Plugging (61)–(62) into (59) and letting $\varepsilon \to 0$, we conclude that

$$\mathbb{P}(1 \le Y_n \le k) \sim \gamma \left(\frac{2}{B}\right)^\gamma \left(\int_0^\infty u^{\gamma-1}\phi(1/u)\, \mathrm{d}u\right) \frac{k^\gamma L(k)}{n^\gamma L(n)}.$$

Changing the variables and then applying Fubini's theorem, we obtain

$$\begin{aligned}\int_0^\infty u^{\gamma-1}\phi(1/u)\, \mathrm{d}u &= \int_0^\infty v^{-1-\gamma}\phi(v)\, \mathrm{d}v \\ &= \int_0^\infty v^{-1-\gamma} \left(\frac{1}{\Gamma(\gamma+1)} \int_0^v u^\gamma \mathrm{e}^{-u}\, \mathrm{d}u\right) \mathrm{d}v \\ &= \frac{1}{\Gamma(\gamma+1)} \int_0^\infty u^\gamma \mathrm{e}^{-u} \left(\int_u^\infty v^{-1-\gamma}\, \mathrm{d}v\right) \mathrm{d}u \\ &= \frac{1}{\gamma\Gamma(\gamma+1)} \int_0^\infty \mathrm{e}^{-u}\, \mathrm{d}u \\ &= \frac{1}{\gamma\Gamma(\gamma+1)}. \end{aligned}$$

Remark 1. We finish the paper by showing that (10) yields (11). Assume that $k \le \delta n$ with some $\delta \in (0, 1)$. Letting $\varepsilon = 1$ in (51) and in (52), we get

$$\mathbb{P}(Y_n = k) \le c \frac{k^{\gamma-1} L(k)}{n^\gamma L(n)}, \quad k \le \delta n.$$

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Since $L(x)$ is slowly varying, for every $\gamma > 1$ we have (see Proposition 1.5.8 in [1])

$$x^{\gamma-1}L(x) \sim \frac{1}{\gamma-1}\int_0^x y^{\gamma-2}L(y)\,\mathrm{d}y \quad \text{as } x \to \infty.$$

The function on the right-hand side is monotone increasing. Therefore

$$\begin{aligned}\max_{k\le\delta n} k^{\gamma-1}L(k) &\le c_1 \max_{k\le\delta n} \frac{1}{\gamma-1}\int_0^k y^{\gamma-2}L(y)\,\mathrm{d}y \\ &= \frac{1}{\gamma-1}\int_0^{\delta n} y^{\gamma-2}L(y)\,\mathrm{d}y \\ &\le c_2(\delta n)^{\gamma-1}L(\delta n).\end{aligned}$$

This implies that

$$\begin{aligned}\limsup_{n\to\infty}\max_{k\le\delta n} n\mathbb{P}(Y_n = k) &\le c_3 \limsup_{n\to\infty} \frac{\max_{k\le\delta n} k^{\gamma-1}L(k)}{n^{\gamma-1}L(n)} \\ &\le c_2c_3 \limsup_{n\to\infty} \frac{(\delta n)^{\gamma-1}L(\delta n)}{n^{\gamma-1}L(n)} \\ &= c_2c_3\delta^{\gamma-1}.\end{aligned}$$

Since the right-hand side goes to zero as $\delta \to 0$, we see that (10) yields (11).

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