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Dirichlet forms on partial *-algebras

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1. *Introduction*

Dirichlet forms and their associated function spaces have been studied by a number of authors [**4**, **6**, **7**, **12**, **15–18**, **22**, **25**, **26**]. Important motivation for the study has been the connection of Dirichlet forms with Markov processes [**16–18**, **25**, **26**]: for example, to every regular symmetric Dirichlet form, there is an associated Hunt process [**13**, **20**]. This makes the theory of Dirichlet forms a convenient source of examples of Hunt processes. In the non-commutative setting, Markov fields have been studied by several authors [**1–3**, **14**, **19**, **24**, **28**]. It is therefore interesting to develop a non-commutative extension of the theory of Dirichlet forms and to study their connection with non-commutative Markov processes.

To the best of this author's knowledge, Dirichlet forms in the non-commutative context have not been systematically studied. The only study we are aware of is that by Albeverio and Høegh-Krohn [**5**] which deals with Dirichlet forms and semigroups of completely positive maps on certain C*-algebras. But even in this C*-algebraic setting, the connection between Dirichlet forms and non-commutative Markov processes is yet to be clarified.

In this paper we begin the study of Dirichlet forms in a partial*-algebraic setting. The choice of setting has been dictated by the fact that in the most physically interesting contexts, the appropriate algebraic structure is a partial *-algebra rather than a C*-algebra [**8**, **21**]. The paper is devoted solely to the study of Dirichlet forms and their associated Markovian semigroups of operators. Work on the connection of Dirichlet forms with non-commutative Markov processes is in progress.

The contents of the rest of the paper are as follows. In §2 we introduce the notion of a bitrace $(\tau, \mathscr{B})$ on a semi-associative partial *-algebra $\mathscr{A}$. The pair $(\tau, \mathscr{B})$ consists of an ideal $\mathscr{B}$ of $\mathscr{A}$, a certain sesqui-linear form τ on $\mathscr{B}$ and some technical assumptions. This enables us to formulate the definition of a non-commutative L^2-space over $(\mathscr{A}, \mathscr{B}, \tau)$, denoted by $L^2(\mathscr{A}, \mathscr{B}, \tau)$, and then to introduce and discuss Lipschitzian maps in §3. These are certain mappings of the self-adjoint portion of $L^2(\mathscr{A}, \mathscr{B}, \tau)$ into itself. In §4 we introduce Dirichlet forms. Their definition involves the notion of Lipschitzian maps. We discuss the connection of Dirichlet forms with Markovian endomorphisms of $L^2(\mathscr{A}, \mathscr{B}, \tau)$. The last section deals with normal contractions and their relationship with Dirichlet forms. The results of this paper are generalizations of the corresponding results in [**5**].

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2. *Bitraces on partial *-algebras*

In this section, we introduce bitraces on partial *-algebras and describe certain structures associated with them.

We begin with the notion of a partial *-algebra: see [**8**, **10**].

*Definition. A partial *-algebra* is a pair $(\mathscr{A}, \Gamma)$ consisting of a complex linear space $\mathscr{A}$ equipped with an anti-linear involution $x \mapsto x^*$ of $\mathscr{A}$ into itself and a subset Γ of $\mathscr{A} \times \mathscr{A}$ with the following properties:

(i) $(x, y_1), (x, y_2) \in \Gamma$ and $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ imply $(x, \lambda_1 y_1 + \lambda_2 y_2) \in \Gamma$, where $\mathbb{C}$ denotes the field of complex numbers;

(ii) $(x, y) \in \Gamma$ implies the existence of a member $x.y$ in $\mathscr{A}$; and the map $(u, v) \mapsto u.v$ of Γ into $\mathscr{A}$ is such that if (x, y) and (x, z) lie in Γ, then $x.(\alpha y + \beta z) = \alpha(x.y) + \beta(x.z)$, for all $(\alpha, \beta) \in \mathbb{C}^2$;

(iii) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$ and $(x.y)^* = y^*.x^*$.

Remark 1. Henceforth we write $x.y$ simply as xy.

Remark 2. If $(\mathscr{A}, \Gamma)$ is a partial *-algebra and $(x, y) \in \Gamma$, then x (resp. y) is called a *left multiplier* (resp. a *right multiplier*) of y (resp. x).

Remark 3. For $x \in \mathscr{A}$, let $M_L(x)$ (resp. $M_R(x)$) denote the set of all left multipliers (resp. right multipliers) of x. Then, for any subset $\mathscr{C}$ of $\mathscr{A}$, we define $M_L(\mathscr{C})$, $M_R(\mathscr{C})$, $M^L(\mathscr{C})$ and $M^R(\mathscr{C})$ as follows:

$$M_L(\mathscr{C}) = \{x \in \mathscr{A} : x \in M_L(y), \quad \text{for all} \quad y \in \mathscr{C}\},$$

$$M_R(\mathscr{C}) = \{x \in \mathscr{A} : x \in M_R(y), \quad \text{for all} \quad y \in \mathscr{C}\},$$

$$M^L(\mathscr{C}) = \mathscr{C} \cap M_L(\mathscr{A}) \quad \text{and} \quad M^R(\mathscr{C}) = \mathscr{C} \cap M_R(\mathscr{A}).$$

Notice that $M^L(\mathscr{A}) = M_L(\mathscr{A})$, $M^R(\mathscr{A}) = M_R(\mathscr{A})$ and that $M^L(\mathscr{C})$ (resp. $M^R(\mathscr{C})$) consists of *universal* left (resp. right) multipliers [**8**] of $\mathscr{C}$.

Remark 4. Since $(x, y) \in \Gamma$ if and only if either $x \in M_L(y)$ or $y \in M_R(x)$, it is evident that the concept of a partial *-algebra may be formulated in terms of left or right multipliers. In that case, one may refer simply to $\mathscr{A}$, rather than $(\mathscr{A}, \Gamma)$ as a partial *-algebra and we do so henceforth.

Remark 5. The following notion of associativity has been introduced in [**10**].

Definition. A partial *-algebra $\mathscr{A}$ is called *semi-associative* if and only if $x, y \in \mathscr{A}$, with $y \in M_R(x)$, implies (i) $yz \in M_R(x)$ for all $z \in M_R(\mathscr{A})$, and (ii) $x(yz) = (xy)z$, for all $z \in M_R(\mathscr{A})$.

For examples of partial *-algebras, including semi-associative partial *-algebras, see [**8**, **10**].

A member $\mathbb{1} \in \mathscr{A}$ is called a *unit* of $\mathscr{A}$ (and $\mathscr{A}$ is said to be *unital*) if and only if $\mathbb{1} \in M_L(\mathscr{A}) \cap M_R(\mathscr{A})$, $\mathbb{1}^* = \mathbb{1}$ and $\mathbb{1}x = x = x\mathbb{1}$ for all $x \in \mathscr{A}$.

Definition. (i) We say that a *-invariant linear subspace $\mathscr{B}$ of $\mathscr{A}$ is an *ideal* if and only if $x \in \mathscr{A}$, $y \in \mathscr{B}$ and $x \in M_L(y)$ imply $xy \in \mathscr{B}$. (ii) We say that $\mathscr{B}$ is an ideal with unit e if e is a maximal idempotent member of $M^L(\mathscr{B}) \cap M^R(\mathscr{B})$ (in the sense that if e' is

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any other idempotent member of $M^L(\mathscr{B}) \cap M^R(\mathscr{B})$, then $ee' = e' = e'e$) satisfying $e^* = e$ and $ea = a = ae$, for all $a \in \mathscr{B}$.

If $\mathscr{B}$ is an ideal with unit e, then e does not coincide with the unit $\mathbb{1}$ of $\mathscr{A}$, unless $\mathscr{B} = \mathscr{A}$. We note too that it may happen that the only ideal of $\mathscr{A}$ with unit is $\mathscr{A}$ itself; but even this particular case is clearly still of interest in the present partial *-algebraic setting.

To show how an ideal with unit e can arise, let $\mathscr{H}$ be a Hilbert space, $\mathscr{D}$ some fixed dense linear manifold of $\mathscr{H}$,

$$\mathfrak{C}(\mathscr{D}, \mathscr{H}) = \{\text{closable } A : \mathscr{D} \subset D(A) \cap D(A^*)\},$$

where $D(B)$ denotes the domain of B and A^* is the operator adjoint of A, and

$$\mathfrak{C}(\mathscr{D}) = \{\text{closed } A \in \mathfrak{C}(\mathscr{D}, \mathscr{H}) : A = \overline{A \restriction \mathscr{D}}\}.$$

Define the product $A \square B$ of $A, B \in \mathfrak{C}(\mathscr{D})$, whenever it exists, as in (3·7) of [**10**].

Now, suppose that $(\mathscr{A}, \square) \subset (\mathfrak{C}(\mathscr{D}), \square)$ is a semi-associative, unital *-symmetric partial Op-*-algebra with unit $\mathbb{1}$, the identity operator, and involution $\dagger$ as defined in (3·5) of [**10**]. Set $\mathscr{A} \cap B(\mathscr{H}) = \mathscr{A}_b$, where $B(\mathscr{H})$ is the Banach algebra of all endomorphisms of $\mathscr{H}$, and define the commutant $\mathscr{A}'_\square$ of $\mathscr{A}$ as in (2·2) of [**11**]. Then, by proposition 5·3 of [**11**], $\mathscr{A}'_\square \cap B(\mathscr{H})$ is a von Neumann algebra and $\mathscr{A}'_\square \cap B(\mathscr{H}) = (\mathscr{A}_b)'$, where $(\mathscr{A}_b)'$ is the usual commutant of $\mathscr{A}_b$. Let e be any projection lying in $\mathscr{A}_b \cap (\mathscr{A}_b)'$. Then $\mathscr{B} \equiv \mathscr{A} \square e$ is an ideal of $\mathscr{A}$ with unit e.

Finally, if $(\mathscr{A}, \square)$ as above is *factorial*, i.e. if $\mathscr{A}_b \cap (\mathscr{A}_b)' = \mathbb{1}$, then we can only have $e = \mathbb{1}$, whence $\mathscr{B} = \mathscr{A}$.

Assumption. Throughout the rest of this paper we assume that the partial *-algebra $\mathscr{A}$ is *semi-associative* and *unital*, with unit $\mathbb{1}$.

Since $\mathscr{A}$ is semi-associative, $M_L(\mathscr{A})$ and $M_R(\mathscr{A})$ (or, equivalently, $M^L(\mathscr{A})$ and $M^R(\mathscr{A})$) are algebras but in general not *-algebras. Furthermore, if $\mathscr{B}$ is an ideal, then $M^L(\mathscr{B})$ (resp. $M^R(\mathscr{B})$) is a subalgebra of $M^L(\mathscr{A})$ (resp. $M^R(\mathscr{A})$).

Let $\mathscr{L}$ be a complex linear space and $\tau : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$ be a positive sequi-linear form. Then τ is called *faithful* if and only if $x \in \mathscr{L}$ and $\tau(x, x) = 0$ imply $x = 0$.

Suppose that $\tau : \mathscr{L} \times \mathscr{L} \to \mathbb{C}$ is a faithful, positive sesqui-linear form. Define $\langle .,. \rangle_\tau$: $\mathscr{L} \times \mathscr{L} \to \mathbb{C}$ and $\| \cdot \|_\tau : \mathscr{L} \to [0, \infty)$ by $\langle x, y \rangle_\tau = \tau(x, y)$ and $\|x\|_\tau = (\langle x, x \rangle_\tau)^{\frac{1}{2}}$. Then, in the sequel, we call $\langle .,. \rangle_\tau$ and $\| \cdot \|_\tau$ the inner product and the norm, respectively, induced by τ on $\mathscr{L}$.

Definition 2·1. By a *bitrace* on $\mathscr{A}$, we understand a pair $(\tau, \mathscr{B})$ consisting of

(i) an ideal $\mathscr{B}$ of $\mathscr{A}$, with unit e, called the ideal of definition of τ; and

(ii) a faithful positive sesqui-linear form $\tau : \mathscr{B} \times \mathscr{B} \to \mathbb{C}$ which is linear in y and conjugate-linear in x, with the following properties: (*a*) $\tau(y, x) = \tau(x^*, y^*)$ for $x, y \in \mathscr{B}$; (*b*) $(z^*x, y) = \tau(x, zy)$, for $x, y, z \in \mathscr{B}$ with $z \in M_L(y)$ and $z^* \in M_L(x)$; (*c*) τ is such that $\{xy \in \mathscr{B} : (x, y) \in \mathscr{B} \times \mathscr{B}$ with $x \in M_L(y)\}$ is dense in $\mathscr{B}$ in the norm-topology induced on $\mathscr{B}$ by τ; (*d*) τ is such that for each $y \in M_L(\mathscr{A})$ (resp. $y \in M_R(\mathscr{A})$), the map $x \mapsto yx$ (resp. $x \mapsto xy$), of $\mathscr{B}$ into $\mathscr{B}$ is continuous in the norm-topology induced on $\mathscr{B}$ by τ.

Henceforth, we fix the pair $(\tau, \mathscr{B})$ and refer simply to τ as a bitrace. Notice that a bitrace is an *h-form* (see [**9**]), but not conversely since h-forms need not satisfy (2·1) (ii) (*a*), (*c*) or (*d*).

If $\mathscr{H}$ is a Hilbert space, we write $B(\mathscr{H})$ for the Banach algebra of all

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endomorphisms of $\mathscr{H}$ and $\mathfrak{C}(\mathscr{H})$ for the set of all densely defined closed linear operators on $\mathscr{H}$, equipped with the operations of strong addition and strong product, the latter denoted simply by juxtaposition.

In the sequel, we write $L^2(\mathscr{A}, \mathscr{B}, \tau)$ for the completion of $\mathscr{B}$ in the norm-topology determined by $\|\cdot\|_\tau$. We denote the extension of the involution of $\mathscr{A}$ to $L^2(\mathscr{A}, \mathscr{B}, \tau)$ again by *.

In view of (2·1) (ii) (*d*), $L^2(\mathscr{A}, \mathscr{B}, \tau)$ is a partial *-algebra. By (2·1) (ii) (*a*), the involution of $L^2(\mathscr{A}, \mathscr{B}, \tau)$ is $\|\cdot\|_\tau$-continuous. But $L^2(\mathscr{A}, \mathscr{B}, \tau)$ is in general not a quasi-*-algebra, since the pair $(\mathscr{B}, \|\cdot\|_\tau)$ is in general not a locally convex *-algebra. The construction of $L^2(\mathscr{A}, \mathscr{B}, \tau)$ from $\mathscr{B}$ given above is analogous to the construction of the non-commutative L^2-space associated with a von Neumann algebra which possesses a faithful, normal semi-finite trace: see [**23**, **27**].

Definition. We say that the bitrace τ is *regular* if and only if $M^R(\mathscr{B})$ is dense in $L^2(\mathscr{A}, \mathscr{B}, \tau)$.

Assumption. In the rest of this paper, we assume that τ *is a regular bitrace.*

Remark 2·2. (1) Since for each $y \in M_R(\mathscr{A})$, the map $x \mapsto xy$ is $\|\cdot\|_\tau$-continuous from $\mathscr{B}$ into $\mathscr{B}$, by hypothesis (2·1)(ii)(*d*), it extends to a bounded transformation of $L^2(\mathscr{A}, \mathscr{B}, \tau)$ into itself. Hence each $x \in L^2(\mathscr{A}, \mathscr{B}, \tau)$ induces a densely defined map $y \mapsto \pi_L^\circ(x)\, y$, $y \in M_R(\mathscr{A})$, of $L^2(\mathscr{A}, \mathscr{B}, \tau)$ into itself with $M^R(\mathscr{B})$ as its dense domain. Now, since

$$\langle z, xy\rangle_\tau = \langle x^* z, y\rangle_\tau$$

i.e.

$$\langle z, \pi_L^\circ(x)\, y\rangle_\tau = \langle \pi_L^\circ(x^*)\, z, y\rangle_\tau,$$

for all $(x, y, z) \in L^2(\mathscr{A}, \mathscr{B}, \tau) \times M^R(\mathscr{B}) \times M^R(\mathscr{B})$, by (2·1)(ii)(*b*), it follows that $\pi_L^\circ(x)$ has a densely defined adjoint for each $x \in L^2(\mathscr{A}, \mathscr{B}, \tau)$. Hence $\pi_L^\circ(x)$ is closable for each $x \in L^2(\mathscr{A}, \mathscr{B}, \tau)$ and we denote its closure by $\pi_L(x)$. Then π_L is an injection of $L^2(\mathscr{A}, \mathscr{B}, \tau)$ into $\mathfrak{C}(L^2(\mathscr{A}, \mathscr{B}, \tau))$ with the properties that $\pi_L(x)^* = \pi_L(x^*)$, (where we have denoted the involution of $B(L^2(\mathscr{A}, \mathscr{B}, \tau))$ again by *), $\pi_L(\lambda x) = \lambda \pi_L(x)$ and $\pi_L(x+y) \supseteq \pi_L(x) + \pi_L(y)$, for all $\lambda \in \mathbb{C}$, $x, y \in L^2(\mathscr{A}, \mathscr{B}, \tau)$. Furthermore, if $x, y \in L^2(\mathscr{A}, \mathscr{B}, \tau)$ are such that xy exists in $L^2(\mathscr{A}, \mathscr{B}, \tau)$, then $\pi_L(xy) = \pi_L(x)\, \pi_L(y)$. Thus π_L is a representation of the partial *-algebra $L^2(\mathscr{A}, \mathscr{B}, \tau)$ in the sense of Antoine [**9**].

In the sequel, we identify $L^2(\mathscr{A}, \mathscr{B}, \tau)$ with its range in $\mathfrak{C}(L^2(\mathscr{A}, \mathscr{B}, \tau))$ under π_L.

(2) Let $L^2(\mathscr{A}, \mathscr{B}, \tau)_s = \{x \in L^2(\mathscr{A}, \mathscr{B}, \tau) : x = x^*\}$. We order $L^2(\mathscr{A}, \mathscr{B}, \tau)$ as follows. For $x_1, x_2 \in L^2(\mathscr{A}, \mathscr{B}, \tau)_s$, write $x_1 \leqslant x_2$ if and only if $\pi_L(x_1) \leqslant \pi_L(x_2)$. The positive cone in $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ with respect to this order is denoted by $L^2(\mathscr{A}, \mathscr{B}, \tau)_+$. We write $[x, y]$ for the order interval in $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ with the end-points $x, y \in L^2(\mathscr{A}, \mathscr{B}, \tau)_s$. i.e. $[x, y] = \{z \in L^2(\mathscr{A}, \mathscr{B}, \tau)_s : x \leqslant z \leqslant y\}$.

(3) If $\mathscr{C}$ is a subset of $L^2(\mathscr{A}, \mathscr{B}, \tau)$, we write $\mathscr{C}_s$ for its self-adjoint portion.

3. *Lipschitzian maps on* $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$

A real-valued function f on $\mathbb{R}$ is called *Lipschitzian* if and only if there is a positive constant λ such that

$$(f(x) - f(y))^2 \leqslant \lambda^2 (x-y)^2, \quad \text{for all} \quad (x, y) \in \mathbb{R}^2.$$

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Now, since $f(x)\otimes 1-1\otimes f(y)$ may be identified with $f(x)-f(y)$ through the identification of $\mathbb{R}\otimes\mathbb{R}$ with $\mathbb{R}$, it follows that f is Lipschitzian if and only if there is a positive constant λ such that

$$(f(x)\otimes 1-1\otimes f(y))^2 \leqslant \lambda^2(x\otimes 1-1\otimes y)^2$$

for all $(x, y)\in\mathbb{R}^2$. This reformulation of the Lipschitzian condition is our point of departure in the sequel.

Definition. We say that $f: M^R(\mathscr{B})_s \to M^R(\mathscr{B})_s$ is *Lipschitzian* if and only if there is a positive constant λ such that the inequality

$$(f(x)\otimes e-e\otimes f(y))^2 \leqslant \lambda^2(x\otimes e-e\otimes y)^2 \tag{3·1}$$

holds in $M^R(\mathscr{B})_s\otimes M^R(\mathscr{B})_s$ for all $x, y\in M^R(\mathscr{B})_s$.

Remark. Observe that (3·1) makes sense since $M^R(\mathscr{B})$ is an algebra.

Notation. (i) We denote the set of all Lipschitzian maps from $M^R(\mathscr{B})_s$ into $M^R(\mathscr{B})_s$ by $\mathrm{Lip}\,(\mathscr{B})$.

(ii) Let $f\in \mathrm{Lip}\,(\mathscr{B})$. We define $\|f\|_{\mathrm{Lip}}$ by

$$\|f\|_{\mathrm{Lip}} = \inf\{\lambda>0: (f(x)\otimes e-e\otimes f(y))^2 \leqslant \lambda^2(x\otimes e-e\otimes y)^2 \text{ for all } x, y\in M^R(\mathscr{B})_s\}.$$

It follows that $f\in \mathrm{Lip}\,(\mathscr{B})$ if and only if

$$(f(x)\otimes e-e\otimes f(y))^2 \leqslant \|f\|^2_{\mathrm{Lip}}\,(x\otimes e-e\otimes y))^2,$$

for all $x, y\in M^R(\mathscr{B})_s$.

There is now the following characterization of the members of $\mathrm{Lip}\,(\mathscr{B})$.

THEOREM 3·2. *A map $f: M^R(\mathscr{B})_s\to M^R(\mathscr{B})_s$ lies in $\mathrm{Lip}\,(\mathscr{B})$ if and only if the following inequality holds*:

$$\|f(x)z-zf(y)\|_\tau \leqslant \|f\|_{\mathrm{Lip}}\,\|xz-zy\|_\tau \tag{3·2}^*$$

for all $(x, y, z)\in M^R(\mathscr{B})_s\times M^R(\mathscr{B})_s\times M^R(\mathscr{B})$.

Proof. For each $a\in M^R(\mathscr{B})$, define the maps $\pi^L(a): M^R(\mathscr{B})\to L^2(\mathscr{A}, \mathscr{B}, \tau)$ and $\pi^R(a): M^R(\mathscr{B})\to L^2(\mathscr{A}, \mathscr{B}, \tau)$ by

$$\pi^L(a)\,x = ax \quad\text{and}\quad \pi^R(a)\,x = xa$$

for $x\in M^R(\mathscr{B})$. Then, in view of (2·1)(ii)(d), $a\mapsto\pi^L(a)$ (resp. $a\mapsto\pi^R(a)$) extends to an injection of $M^R(\mathscr{B})$ into $B(L^2(\mathscr{A}, \mathscr{B}, \tau))$. Therefore the map $a\otimes b\mapsto\pi^L(a)\,\pi^R(b)$ defines an injection from $M^R(\mathscr{B})\otimes M^R(\mathscr{B})$ into $B(L^2(\mathscr{A}, \mathscr{B}, \tau))$. Hence a map $f: M^R(\mathscr{B})_s\to M^R(\mathscr{B})_s$ satisfies (3·1) if and only if

$$(\pi^L(f(x))-\pi^R(f(y)))^2 \leqslant \|f\|^2_{\mathrm{Lip}}(\pi^L(x)-\pi^R(y))^2$$

for all $(x, y)\in M^R(\mathscr{B})_s\times M^R(\mathscr{B})_s$. The last inequality holds if and only if

$$\tau(z, (\pi^L(f(x))-\pi^R(f(y)))^2z) \leqslant \|f\|^2_{\mathrm{Lip}}\,\tau(z, (\pi^L(x)-\pi^R(y))^2z),$$

for all $z\in M^R(\mathscr{B})$. Now, for $z\in M^R(\mathscr{B})$, we have

$$\tau(z, (\pi^L(x)-\pi^R(y))^2z) = \tau(z, (\pi^L(x)-\pi^R(y))\,(xz-zy)) = \tau(z, x(xz-zy)-(xz-zy)y),$$

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the expressions $x(xz-zy)$ and $(xz-zy)y$ being well-defined and lying in $\mathscr{B}$ in view of the semi-associativity of $\mathscr{A}$. Thus

$$\tau(z, (\pi^L(x)-\pi^R(y))^2 z) = \tau(z, x(xz-zy)) - \tau(z, (xz-zy)y)$$
$$= \tau(xz, xz-zy) - \tau(zy, xz-zy) = \tau(xz-zy, xz-zy),$$

by (2·1)(ii)(*b*) and since $x^* = x$ and $y^* = y$, and because of the $\mathbb{R}$-linearity of the map $u \mapsto \tau(u, .)$ for $u \in \mathscr{B}$.

Similarly

$$\tau(z, (\pi^L(f(x))-\pi^R(f(y)))^2 z) = \|f(x)z - zf(y)\|_\tau^2 \quad (x, y \in M^R(\mathscr{B})_s, z \in M^R(\mathscr{B})).$$

Hence f lies in $\mathrm{Lip}(\mathscr{B})$ if and only if

$$\|f(x)z - zf(y)\|_\tau^2 \leqslant \|f\|_{\mathrm{Lip}}^2 \|xz - zy\|_\tau^2 \quad (x, y \in M^R(\mathscr{B})_s, z \in M^R(\mathscr{B})).$$

This concludes the proof.

Remark. We can now make the following inference from the preceding result.

COROLLARY 3·3. *Let $f \in \mathrm{Lip}(\mathscr{B})$. Then f extends to a map, denoted again by f, of $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ into itself which satisfies*

$$\|f(x)z - zf(y)\|_\tau \leqslant \|f\|_{\mathrm{Lip}} \|xz - zy\|_\tau \tag{3·4}$$

for all $x, y \in L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ and $z \in M^R(\mathscr{B})$.

Proof. By setting $z = e$ in (3·2)*, we get

$$\|f(x) - f(y)\|_\tau \leqslant \|f\|_{\mathrm{Lip}} \|x - y\|_\tau \tag{3·5}$$

for all $x, y \in M^R(\mathscr{B})_s$. Hence f extends to a mapping, denoted again by f, of $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ into itself. This extension also satisfies (3·5) for all $x, y \in L^2(\mathscr{A}, \mathscr{B}, \tau)_s$.

Now let $\{x_\alpha\}$ and $\{y_\alpha\}$ be nets of members of $M^R(\mathscr{B})_s$ which converge to x and y respectively in $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$. Then, using the fact that the maps $u \mapsto uz$ and $u \mapsto zu$ of $M^R(\mathscr{B})$ into itself are $\|\cdot\|_\tau$ – continuous (by hypothesis (2·1)(ii)(*d*)), we get

$$\lim_\alpha \|x_\alpha z - zy_\alpha\|_\tau = \|xz - zy\|_\tau \quad (z \in M^R(\mathscr{B}))$$

and

$$\lim_\alpha \|f(x_\alpha)z - zf(y_\alpha)\|_\tau = \|f(x)z - zf(y)\|_\tau \quad (z \in M^R(\mathscr{B})).$$

Hence, since

$$\|f(x_\alpha)z - f(y_\alpha)z\|_\tau \leqslant \|f\|_{\mathrm{Lip}} \|x_\alpha z - zy_\alpha\|_\tau \quad (z \in M^R(\mathscr{B})),$$

the assertion follows by taking limits. ∎

Notation 1. We denote again by $\mathrm{Lip}(\mathscr{B})$ the set of all maps $f: L^2(\mathscr{A}, \mathscr{B}, \tau)_s \to L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ which satisfy (3·4) and refer to such maps again as *Lipschitzian*.

Notation 2. We define $\mathrm{Lip}(\mathscr{B}, 0)$ by

$$\mathrm{Lip}(\mathscr{B}, 0) = \{f \in \mathrm{Lip}(\mathscr{B}) : f(0) = 0 \quad \text{and} \quad (f(x))^2 \leqslant \|f\|_{\mathrm{Lip}}^2 x^2 \quad \text{for all} \quad x \in M^R(\mathscr{B})_s\}.$$

Remark. From the argument presented above we see that (3·4) and (3·5) are equivalent conditions. In trying to show that a given function is Lipschitzian, it is sometimes easier to check that a relation of the type (3·1) holds and then to apply Corollary 3·3.

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4. *Markovian operators and Dirichlet forms*

A positive linear map $P \in B(L^2(\mathscr{A}, \mathscr{B}, \tau))$ is called *Markovian* if and only if P maps the order interval $[0, e] \subset L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ into itself.

By a *Dirichlet form* on $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$, we understand a closed, symmetric, non-negative, densely-defined bilinear form $\mathscr{E}$, with domain $D(\mathscr{E})$, such that

(i) $D(\mathscr{E}) \supseteq f(D(\mathscr{E}))$, for each $f \in \mathrm{Lip}\,(\mathscr{B}, 0)$, and

(ii) $\mathscr{E}(f(x), f(x)) \leqslant \|f\|^2_{\mathrm{Lip}}\, \mathscr{E}(x, x)$, for all $x \in D(\mathscr{E})$ and $f \in \mathrm{Lip}\,(\mathscr{B}, 0)$.

Remark 1. The notions of Markovian operators and Dirichlet forms are related. To see this, let us first note the following property of certain symmetric bilinear forms associated with Markovian operators.

Remark 2. Let I denote the identity member of $B(L^2(\mathscr{A}, \mathscr{B}, \tau))$.

PROPOSITION 4·1. *Let $P \in B(L^2(\mathscr{A}, \mathscr{B}, \tau))$ be symmetric and Markovian. Then the map $(x, y) \mapsto \langle x, (I-P)\, y\rangle_\tau$ is a symmetric, densely-defined non-negative bilinear form on $L^2(\mathscr{A}, \mathscr{B}, \tau)_s \times L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ with the property that*

$$\langle f(x), (I-P) f(x)\rangle_\tau \leqslant \|f\|^2_{\mathrm{Lip}} \langle x, (I-P)\, x\rangle_\tau$$

for all $f \in \mathrm{Lip}\,(\mathscr{B}, 0)$ and $x \in L^2(\mathscr{A}, \mathscr{B}, \tau)_s$.

Proof. That $(x, y) \mapsto \langle x, (I-P)\, y\rangle_\tau$ is a symmetric, densely-defined, non-negative bilinear form on $L^2(\mathscr{A}, \mathscr{B}, \tau)_s \times L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ is obvious. So it only remains to prove the alleged property. To this end, let $f \in \mathrm{Lip}\,(\mathscr{B}, 0)$. Then since τ is a bitrace and P is symmetric we have

$$\tau(f(x), (I-P) f(x)) = \tau(f(x)^2, e-Pe) + \tfrac{1}{2}(\tau(f(x)^2, Pe) + \tau(Pe, f(x)^2) - 2\tau(f(x), Pf(x))),$$

for all $x \in M^R(\mathscr{B})_s$.

Now, let ν_P and μ_P be the functionals on $\mathscr{B}$ and $\bar{\mathscr{B}} \otimes \mathscr{B}$ defined by

$$\nu_P(x) = \tau(x, e-Pe) \quad (x \in \mathscr{B})$$

and

$$\mu_P(x \otimes y) = \tau(x, Py) \quad (x \otimes y \in \bar{\mathscr{B}} \otimes \mathscr{B}),$$

where $\bar{\mathscr{B}}$ is the *conjugate* space to $\mathscr{B}$. These functionals satisfy the following positivity relations:

$$\nu_P(x^2) \geqslant 0, \quad \text{for all} \quad x \in M^R(\mathscr{B})_s,$$

and

$$\mu_P((x \otimes y)^2) \geqslant 0, \quad \text{for all} \quad x, y \in M^R(\mathscr{B})_s.$$

In terms of the functionals ν_P and μ_P, we now get

$$\tau(f(x), (I-P) f(x)) = \nu_P(f(x)^2) + \tfrac{1}{2}\mu_P((f(x) \otimes e - e \otimes f(x))^2),$$

showing that $\tau(f(x), (I-P) f(x))$ is non-negative for all $f \in \mathrm{Lip}\,(\mathscr{B}, 0)$, $x \in M^R(\mathscr{B})_s$. Furthermore, since

$$\nu_P(x^2) + \tfrac{1}{2}\mu_P((x \otimes e - e \otimes x)^2) = \tau(x, (I-P)\, x),$$

$$(f(x) \otimes e - e \otimes f(x))^2 \leqslant \|f\|^2_{\mathrm{Lip}}\, (x \otimes e - e \otimes x)^2,$$

and

$$f(x)^2 \leqslant \|f\|^2_{\mathrm{Lip}}\, x^2, \quad (x \in M^R(\mathscr{B})_s, \quad f \in \mathrm{Lip}\,(\mathscr{B}, 0))$$

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it follows that

$$\tau(f(x), (I-P)f(x)) \leqslant \|f\|^2_{\mathrm{Lip}}(\nu_P(x^2)+\tfrac{1}{2}\mu_P((x\otimes e-e\otimes x)^2))$$
$$= \|f\|^2_{\mathrm{Lip}}\tau(x, (I-P)x), \quad (x\in M^R(\mathscr{B})_s, \quad f\in \mathrm{Lip}(\mathscr{B}, 0)).$$

Using (3·5) and the denseness of $M^R(\mathscr{B})_s$ in $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$, the result now follows by continuity. ∎

Remark. Using Proposition 4·1, we may relate the notions of a Markovian operator and a Dirichlet form as follows.

THEOREM 4·2. *Let $P\in B(L^2(\mathscr{A}, \mathscr{B}, \tau))$ be Markovian and symmetric. Then the map $(x, y)\mapsto \langle x, (I-P)y\rangle_\tau$ from $L^2(\mathscr{A}, \mathscr{B}, \tau)_s \times L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ into $\mathbb{R}$ is a Dirichlet form.*

Definition. We say that a semigroup $\{P_t : t[0, \infty)\} \subset B(L^2(\mathscr{A}, \mathscr{B}, \tau))$ is Markovian if and only if P_t is Markovian for each $t\in[0, \infty)$.

Remark. Using Theorem 4·2, the following result is proved exactly as in [5]:

THEOREM 4·3. *Let $\{P_t : t\in[0, \infty)\} \subset B(L^2(\mathscr{A}, \mathscr{B}, \tau))$ be a symmetric, Markovian semigroup with $-H$ as its infinitesimal generator. Then the sesqui-linear form $\mathscr{E}$ defined on $L^2(\mathscr{A}, \mathscr{B}, \tau)_s \times L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ by*

$$D(\mathscr{E}) = D(H^{\frac{1}{2}}),$$
$$\mathscr{E}(x, y) = \langle H^{\frac{1}{2}}x, H^{\frac{1}{2}}y\rangle_\tau, \quad (x, y\in D(\mathscr{E})),$$

is a Dirichlet form.

Remark 4·4. Dirichlet forms also arise in the following manner.

Let $\{z_i\}_{i=1}^n$ be a fixed subset of $M^R(\mathscr{B})$. Define the bilinear form $\mathscr{E}_0$ on $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ by

$$D(\mathscr{E}_0) = M^R(\mathscr{B})_s,$$
$$\mathscr{E}_0(x, y) = \sum_{i=1}^n \langle xz_i - z_i x, yz_i - z_i y\rangle_\tau \quad (x, y\in D(\mathscr{E}_0)).$$

Then $\mathscr{E}_0$ is closable. Let $\mathscr{E}$ be the closure of $\mathscr{E}_0$.

PROPOSITION 4·5. *With the notation of Remark* 4·4, *$\mathscr{E}$ is a Dirichlet form.*

Proof. Let $f\in \mathrm{Lip}(\mathscr{B})$. Then $f(M^R(\mathscr{B})_s) \subseteq M^R(\mathscr{B})_s$ and by (3·2)*

$$\mathscr{E}_0(f(x), f(x)) = \sum_{i=1}^n \|f(x)z_i - z_i f(x)\|^2_\tau$$
$$\leqslant \|f\|^2_{\mathrm{Lip}} \sum_{i=1}^n \|xz_i - z_i x\|^2_\tau = \|f\|^2_{\mathrm{Lip}}\mathscr{E}_0(x, x), \quad (x\in M^R(\mathscr{B})_s).$$

The assertion now follows by continuity.

Remark. We need the following result in connection with the next theorem:

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PROPOSITION 4·6. *Let* $f: L^2(\mathscr{A}, \mathscr{B}, \tau)_s \to L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ *be defined by*

$$f(x) = \begin{cases} e & (x \geqslant e) \\ x & (x \in [0, e]) \\ 0 & (0 \geqslant x). \end{cases}$$

Then $f \in \mathrm{Lip}\,(\mathscr{B}, 0)$.

Proof. First note that $f(0) = 0$ and $(f(x))^2 \leqslant x^2$, for all $x \in M^R(\mathscr{B})_s$. So it only remains to show that

$$\|f(x)z - zf(y)\|_\tau \leqslant \|f\|_{\mathrm{Lip}}\, \|xz - zy\|_\tau$$

for all $x, y \in L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ and $z \in M^R(\mathscr{B})$. We establish this as follows.

Now it is clear that for $x, y \in M^R(\mathscr{B})_s$, the relation

$$(f(x) \otimes e - e \otimes f(y))^2 \leqslant (x \otimes e - e \otimes y)^2 \qquad (*)$$

holds in the three cases (i) $0 \geqslant x$, $0 \geqslant y$, (ii) $x, y \in [0, e]$ and (iii) $x \geqslant e$, $y \geqslant e$. Furthermore, if $x \geqslant 0$ and $0 \geqslant y$, then since $f(y) = 0$ for $0 \geqslant y$,

$$\begin{aligned} (f(x) \otimes e - e \otimes f(y))^2 &= (f(x) \otimes e)^2 \\ &\leqslant x^2 \otimes e \leqslant x^2 \otimes e - 2x \otimes y + e \otimes y^2 \\ &= (x \otimes e - e \otimes y)^2, \end{aligned}$$

i.e. (*) holds.

Similarly (*) also holds if $0 \geqslant x$ and $y \geqslant 0$. Hence (*) holds for all $x, y \in M^R(\mathscr{B})_s$. Thus, by Theorem 3·2, the inequality (3·2)* holds, i.e.

$$\|f(x)z - zf(y)\|_\tau \leqslant \|xz - zy\|_\tau,$$

and, by Corollary 3·3, f indeed lies in $\mathrm{Lip}\,(\mathscr{B}, 0)$. This concludes the proof. ∎

Remark. We know that to each densely defined closed symmetric bilinear form $\mathscr{F}$ on $L^2(\mathscr{A}, \mathscr{B}, \tau)_s \times L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ there corresponds a densely defined, non-negative linear map $H_{\mathscr{F}}$ such that

$$D(\mathscr{F}) = D(H_{\mathscr{F}}^{\frac{1}{2}})$$

and

$$\mathscr{F}(x, y) = \langle H_{\mathscr{F}}^{\frac{1}{2}} x, H_{\mathscr{F}}^{\frac{1}{2}} y \rangle_\tau$$

for all $x, y \in D(\mathscr{F})$. We write simply H for $H_{\mathscr{F}}$, if no ambiguity can arise. To H we associate the contraction semigroup $\{P_t = e^{-tH} : t \in [0, \infty)\}$.

We can now establish the following converse to Theorem 4·3.

THEOREM 4·7. *Let* $\mathscr{E}$ *be a Dirichlet form on* $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ *and* H *be the non-negative operator determined by* $\mathscr{E}$. *Then the contraction semigroup* $\{P_t = e^{-tH} : t \in [0, \infty)\}$ *generated by* H *is a symmetric Markov semigroup on* $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$.

Proof. We know that $\mathscr{E}$ and H are related by

$$\mathscr{E}(x, y) = \langle H^{\frac{1}{2}} x, H^{\frac{1}{2}} y \rangle_\tau,$$

for $x, y \in D(\mathscr{E})$. For $\alpha > 0$ let $G_\alpha = (\alpha + H)^{-1}$ be the resolvent of H and let $\mathscr{E}_\alpha$ be defined by

$$\mathscr{E}_\alpha(x, y) = (x, y) + \alpha \langle x, y \rangle_\tau,$$

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for $x, y \in D(\mathscr{E})$. Then for $x, y \in D(\mathscr{E})$ we have

$$\begin{aligned}\mathscr{E}_\alpha(G_\alpha x, y) &= \mathscr{E}(G_\alpha x, y) + \alpha\langle G_\alpha x, y\rangle_\tau \\ &= \langle H^{\frac{1}{2}} G x, H^{\frac{1}{2}} y\rangle_\tau + \alpha\langle H^{-\frac{1}{2}} G_\alpha x, H^{\frac{1}{2}} y\rangle_\tau \\ &= \langle H^{\frac{1}{2}}(1+\alpha H^{-1}) G_\alpha x, H^{\frac{1}{2}} y\rangle_\tau \\ &= \langle H^{\frac{1}{2}} H^{-1} x, H^{\frac{1}{2}} y\rangle_\tau \\ &= \langle x, y\rangle_\tau,\end{aligned}$$

and hence

$$\mathscr{E}(x, x) + \alpha\|x - y\|_\tau^2 = \alpha\langle y, (I - \alpha G_\alpha) y\rangle_\tau + \mathscr{E}_\alpha(x - \alpha G_\alpha y, x - \alpha G_\alpha y).$$

Thus $\mathscr{E}(x, x) + \alpha\|x-y\|_\tau^2$ takes its minimum value if $x = \alpha G_\alpha y$. One now shows as in [5] that if y is such that $f(y) = y$, where $f \in \mathrm{Lip}(\mathscr{B}, 0)$ with $\|f\|_{\mathrm{Lip}} \leqslant 1$, then $\alpha G_\alpha y = f(\alpha G_\alpha y)$. Now define $f: L^2(\mathscr{A}, \mathscr{B}, \tau)_s \to L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ by

$$f(x) = \begin{cases} e & (x \geqslant e) \\ x & (0 \leqslant x \leqslant e) \\ 0 & (0 \geqslant x). \end{cases}$$

Then, by Proposition 4·6, $f \in \mathrm{Lip}(\mathscr{B}, 0)$, with $\|f\|_{\mathrm{Lip}} \leqslant 1$, and we get that αG_α is Markovian for all $\alpha > 0$. Since

$$P_t u = \lim_{\lambda \uparrow \infty} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} (\lambda G_\lambda)^n u,$$

for $u \in L^2(\mathscr{A}, \mathscr{B}, \tau)$, we see that P_t is also Markovian for each $t \in [0, \infty)$. This concludes the proof. ∎

5. *Normal contractions on* $\mathscr{A}$

The notion of a *normal contraction* plays a role in the classical theory of Dirichlet forms (see e.g. [18]). As in [5], and following our formulation of the concept of a Lipschitzian map, we introduce the notion in the present setting as follows:

Definition. Let $x, y \in M^R(\mathscr{B})_s$. Then x will be called a *normal contraction* of y if and only if

$$x^2 \leqslant y^2 \quad \text{and} \quad (x \otimes e - e \otimes x)^2 \leqslant (y \otimes e - e \otimes y)^2$$

hold in $M^R(\mathscr{B})_s$ and $M^R(\mathscr{B})_s \otimes M^R(\mathscr{B})_s$, respectively.

Notation. For each $z \in M^R(\mathscr{B})$, let $\|\cdot\|_{z,\tau}$ be the semi-norm on $M^R(\mathscr{B})$ defined by

$$\|x\|_{z,\tau} = \|[x, z]\|_\tau$$

for $x \in M^R(\mathscr{B})$, where

$$[x, z] = xz - zx$$

for $x, z \in M^R(\mathscr{B})$.

Remark. Arguing as in Theorem 3·2, we get the following result.

PROPOSITION (5·1). *Let $x, y \in M^R(\mathscr{B})_s$. Then x is a normal contraction of y if and only if the following two conditions hold*:

(i) $x^2 \leqslant y^2$, *and*

(ii) $\|x\|_{z,\tau} \leqslant \|y\|_{z,\tau}$, *for all* $z \in M^R(\mathscr{B})$.

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Remark. Another characterization of normal contractions is given as follows in terms of non-expansive mappings on $M^R(\mathscr{B})_s$ (i.e. Lipschitzian mappings f satisfying $\|f\|_{\mathrm{Lip}} \leqslant 1$).

THEOREM 5·2. *Let $x, y \in M^R(\mathscr{B})_s$. Then*

$$(x \otimes e - e \otimes x)^2 \leqslant (y \otimes e - e \otimes y)^2 \qquad (*)$$

in $M^R(\mathscr{B})_s \otimes M^R(\mathscr{B})_s$ if and only if there is a non-expansive map $f: M^R(\mathscr{B})_s \to M^R(\mathscr{B})_s$ such that $x = f(y)$.

Furthermore x is a normal contraction of y if and only if there is a non-expansive map $f: M^R(\mathscr{B})_s \to M^R(\mathscr{B})_s$ such that $f(0) = 0$ and $x = f(y)$.

Proof. Let $f: M^R(\mathscr{B})_s \to M^R(\mathscr{B})_s$ be non-expansive. Then

$$(f(u) \otimes e - e \otimes f(v))^2 \leqslant (u \otimes e - e \otimes v)^2,$$

for all $u, v \in M^R(\mathscr{B})_s$, and, in particular,

$$(f(y) \otimes e - e \otimes f(y))^2 \leqslant (y \otimes e - e \otimes y)^2,$$

for all $y \in M^R(\mathscr{B})_s$. From this, one sees that if $x, y \in M^R(\mathscr{B})_s$, with $x = f(y)$, then

$$(x \otimes e - e \otimes x)^2 \leqslant (y \otimes e - e \otimes y)^2,$$

and so x is a normal contraction of y.

Next, by Proposition 5·1, (*) holds if and only if

$$\|x\|_{z,\tau} \leqslant \|y\|_{z,\tau},$$

for all $z \in M^R(\mathscr{B})$. Now let $M^R(\mathscr{B})_y$ (resp. $M^R(\mathscr{B})_x$) denote the *-subalgebra of $M^R(\mathscr{B})$ generated by y (resp. x). If z lies in the commutant $M^R(\mathscr{B})'_y$ of $M^R(\mathscr{B})_y$ in $M^R(\mathscr{B})$, then $\|x\|_{z,\tau} = 0$, whence $[x, z] = 0$, in view of the faithfulness of τ. Hence z also lies in the commutant $M^R(\mathscr{B})'_x$ of $M^R(\mathscr{B})_x$ in $M^R(\mathscr{B})$, i.e. $M^R(\mathscr{B})'_y \subset M^R(\mathscr{B})'_x$. This means that $\pi_L(M^R(\mathscr{B})'_y) \subset \pi_L(M^R(\mathscr{B})'_x)$, where π_L is as in Remark 2·2 (1), whence $\pi_L(x)$ lies in $\pi_L(M^R(\mathscr{B})_y)$. It then follows that there is a real-valued function $\tilde{f}$ on $\mathbb{R}$ such that $\pi_L(x) = \tilde{f}(\pi_L(y))$. Hence, since π_L is an injection, one concludes that $x = f(y)$, where $f = \pi_L^{-1} \circ \tilde{f} \circ \pi_L$. In view of (*), f is a non-expansive map of $M^R(\mathscr{B})_s$ into itself. This ends the proof of the first part of the theorem.

To prove the 'furthermore' part, we note that, in view of the preceding part, it remains only to show that if $x = f(y)$, where x is a normal contraction of y, then 0 is a fixed point of f. But this follows from the fact that $x^2 \leqslant y^2$, whence $y = 0$ implies $x = 0$. This concludes the proof of the theorem. ∎

If we combine the above result with the definition of a Dirichlet form, we obtain the following theorem.

THEOREM 5·3. *Let $\mathscr{E}$ be a densely-defined closed, symmetric bilinear form on $L^2(\mathscr{A}, \mathscr{B}, \tau)_s$ with domain $D(\mathscr{E})$. Then $\mathscr{E}$ is a Dirichlet form if and only if $\mathscr{E}(x, x) \leqslant \mathscr{E}(y, y)$ whenever $(x, y) \in D(\mathscr{E}) \times D(\mathscr{E})$ and x is a normal contraction of y.*

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