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EXPANSIONS FOR THE DISTRIBUTION OF *M*-ESTIMATES WITH APPLICATIONS TO THE MULTI-TONE PROBLEM

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Abstract. We give a stochastic expansion for estimates $\widehat{\theta}$ that minimise the arithmetic mean of (typically independent) random functions of a known parameter θ. Examples include least squares estimates, maximum likelihood estimates and more generally M-estimates. This is used to obtain leading cumulant coefficients of $\widehat{\theta}$ needed for the Edgeworth expansions for the distribution and density of $n^{1/2}(\widehat{\theta}-\theta^0)$ to magnitude $n^{-3/2}$ (or to n^{-2} for the symmetric case), where θ^0 is the true parameter value and n is typically the sample size. Applications are given to least squares estimates for both real and complex models. An alternative approach is given when the linear parameters of the model are nuisance parameters. The methods are illustrated with the problem of estimating the frequencies when the signal consists of the sum of sinusoids of unknown amplitudes.

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1. Introduction and summary

Let $\widehat{\theta}$ denote an estimate of θ in $\mathcal{R}^p$ based on a random sample of size n. There is a large amount of work on expansions for $\widehat{\theta}-\theta^0$, where θ^0 is the true value of θ. However, most of the work to date are for the sample mean and functions of it. For example, Monti [5] obtains an expansion for the sample mean up to the second order by expanding the saddlepoint approximation. Booth *et al.* [1] give tilted expansions of a sample mean from a distribution on k points. Kakizawa and Taniguchi [4] obtain expansions for $P(\widehat{\theta}<x)$ under the assumption that $\widehat{\theta}$ has a cumulant expansion in powers of n^{-1}. Gatto and Ronchetti [3] provide approximations for $P(m(\bar{X})<x)$ up to $1+O(n^{-1})$ for $m(\cdot)$ a smooth function. For a comprehensive review of the known work, we refer the readers to [8].

The aim of this paper is to provide expansions for those $\widehat{\theta}$ that minimise the arithmetic mean of random functions of θ. Maximum likelihood estimates (MLEs), least squares estimates (LSEs), and more generally M-estimates are examples of $\widehat{\theta}$ which minimise a random mean function

$$\Lambda=\Lambda(\theta)=n^{-1}\sum_{N=1}^{n}\Lambda_N(\theta)$$

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for θ in $\mathcal{R}^p$. If $E\,\partial\Lambda/\partial\theta = 0$ and $E\,\partial^2\Lambda/\partial\theta\partial\theta' > 0$, then $\widehat{\theta}\to_p \theta^0$ as $n \to \infty$. (We use $\mathcal{R}$ and $\mathcal{C}$ to denote the real and complex numbers).

The contents of this paper are organized as follows. In Section 2 we give a stochastic expansion for $\widehat{\theta} - \theta^0$ of the form

$$\widehat{\theta} - \theta^0 \approx \sum_{a=1}^{\infty} \delta^a,$$

where $\delta^a = O_p(n^{-a/2})$. (The a in δ^a is a superscript not a power).

In Section 3 we use this to obtain the leading coefficients in the expansions for the cumulants of $\widehat{\theta}$:

$$\kappa(\widehat{\theta}_{i_1}, \ldots, \widehat{\theta}_{i_r}) \approx \sum_{j=r-1}^{\infty} k_j^{i_1 \ldots i_r} n^{-j} \tag{1.1}$$

for $r \geq 1$ with $k_0^i = \theta_i^0$. This implies that $Y_n = n^{1/2}(\widehat{\theta} - \theta^0) \to \mathcal{N}_p(0, V)$ with $V = (k_1^{i_1 i_2}) = A^{-1}$ for $A = E\,\partial^2\Lambda/\partial\theta\partial\theta'$. The leading bias and skewness coefficients $k_0^{i_1}$ and $k_2^{i_1 i_2 i_3}$ give the Edgeworth expansion of the distribution (and its derivatives) of Y_n to $O(n^{-1})$, while the coefficients $k_2^{i_1 i_2}$ and $k_3^{i_1 \ldots i_4}$ give the Edgeworth expansion of the distribution of Y_n to $O(n^{-3/2})$ and $P(Y_n \in S)$ to $O(n^{-2})$ for $S = -S \subset \mathcal{R}^p$.

Section 4 applies these results to the LSEs for the general signal plus noise model

$$Y_N = S_N(\theta) + e_N \text{ in } \mathcal{R} \text{ or } \mathcal{C},\ 1 \leq N \leq n \tag{1.2}$$

with $\Lambda_N(\theta) = |Y_N - S_N(\theta)|^2/2$, where the residuals $e_1, \ldots, e_n$ are assumed independent with mean zero. While the complex formulation can also be dealt with by the real formulation, there are some significant simplifications in staying with the complex model. The M-estimate with respect to a given convex function ρ on $\mathcal{R}$ or $\mathcal{C}$ for the model (1.2) is $\widehat{\theta}$ for $\Lambda_N(\theta) = \rho(Y_N - \rho(\theta))$. For smooth ρ the leading cumulant coefficients were essentially found by this method in the real case in [7].

Section 5 considers two examples on the signal frequency problem

$$y_k = s(\theta) + n_k \text{ in } \mathcal{C}^M,\ k = 1, \ldots, K, \tag{1.3}$$

where n_k is complex normal with covariance not depending on θ, and the mth component of $s(\theta)$ is

$$s_m(\theta) = \sum_{r=1}^{R} a_r \exp(j\phi_r + jw_r(m + m_0)/M) \text{ for } m = 0, 1, \ldots, M-1,$$

where $j = \sqrt{-1}$. The $p = 3R$ parameters are $\theta' = (a', \phi', w')$. So, (1.3) can be written in the form (1.2) with $n = 2kM$. Changing the covariance constant m_0 is equivalent to reparameterising ϕ_r: we shall see that if $R = 1$ then taking $m_0 = -(M-1)/2$ makes the asymptotic covariance of $\widehat{\theta}$ diagonal.

In Section 6 we give a variation of the method for the case when the linear parameters of the model are nuisance parameters.

Appendix A provides a list of summation notations used throughout the paper. Some technical details required for the two examples in Section 5 are given in Appendices B and C. The proofs of all theorems are given in Appendix D.

For x a complex matrix we shall use x' to denote its transpose, $\bar{x}$ its complex conjugate, and x^* the transpose of its complex conjugate.

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2. The stochastic expansion

Suppose $\{\Lambda_N(\theta)\}$ are real random functions of θ in $\mathcal{R}^p$. Here we show that $\widehat{\theta}$ minimising $\Lambda = \Lambda(\theta) = n^{-1}\sum_{N=1}^{n}\Lambda_N(\theta)$ in $\mathcal{R}^p$ has the stochastic expansion

$$\delta = \widehat{\theta} - \theta^0 = \sum_{a=1}^{\infty}\delta^a \tag{2.1}$$

with $\delta^a = O_p(n^{-a/2})$. To avoid excessive subscripts we shall fix $1 \le i_0, i_1, \ldots \le p$ and set

$$\begin{aligned} \delta_j^a &= (\delta^a)_{i_j},\ \partial_j = \partial/\partial\theta_{i_j}, \delta_j = (\delta)_{i_j}, \Lambda_{\cdot 12\ldots} = \partial_1\partial_2\ldots\Lambda(\theta), &(2.2)\\ A_{12\ldots} &= E\,\Lambda_{\cdot 12\ldots},\ \text{and}\ \Delta_{12\ldots} = \Lambda_{\cdot 12\ldots} - A_{12\ldots}. &(2.3)\end{aligned}$$

Theorem 2.1 gives the first three δ^a explicitly in terms of these Δ's and A's. By (2.1) this gives $\widehat{\theta}$ explicitly in terms of Δ's and A's to $O_p(n^{-2})$.

For $\widehat{\theta}$ to be a consistent estimate we need to assume that

$$\begin{aligned} A_1 &= E\Lambda_{\cdot 1} = 0, &(2.4)\\ \Delta_{1\ldots r} &= O_p(n^{-1/2})\ \text{as}\ n\to\infty. &(2.5)\end{aligned}$$

Typically the model contains a location parameter, and the constraint (2.4) effectively specifies how it is defined, as well as identifying the other parameters of the model. The constraint (2.5) generally follows by the Central Limit Theorem, if the $\{\Lambda_N(\theta)\}$ are independent or weakly dependent.

Theorem 2.1. *Suppose $\widehat{\theta}$ is the estimate as defined above satisfying (2.4) and (2.5). Suppose also that the eigenvalues of the $p\times p$ matrix $A = (A_{12} : 1 \le i_1,\ i_2 \le p)$ are bounded away from zero as $n\to\infty$, so A has bounded inverse $A^{-1} = (A^{12})$ as $n\to\infty$. Then (2.1) holds. Furthermore,*

$$\begin{aligned} \delta_0^1 &= -A^{01}\Delta_1, &(2.6)\\ \delta_0^2 &= A^{01}\Delta_{12}A^{23}\Delta_3 - B^{045}\Delta_4\Delta_5, &(2.7)\\ \delta_0^3 &= -A^{01}(\Delta_{12}\delta_2^2 - A_{123}\sum_{23}^{2}A^{24}\Delta_4\delta_3^2/2 + \Delta_{123}A^{24}\Delta_4A^{35}\Delta_5/2 - C_1^{567}\Delta_5\Delta_6\Delta_7),\\ &= -A^{01}\left\{\Delta_{12}(A^{23}\Delta_{34}A^{45}\Delta_5 - B^{234}\Delta_3\Delta_4)\right.\\ &\quad -2^{-1}A_{123}\sum_{23}^{2}A^{24}\Delta_4(A^{35}\Delta_{56}A^{67}\Delta_7 - B^{356}\Delta_5\Delta_6)\\ &\quad \left.+2^{-1}A^{24}A^{35}\Delta_{123}\Delta_4\Delta_5 - C_1^{234}\Delta_2\Delta_3\Delta_4\right\}, &(2.8)\end{aligned}$$

and so on, where $B^{123} = A^{14}A^{25}A^{36}A_{456}/2$ and $C_1^{567} = A_{1234}A^{25}A^{36}A^{47}/6$.

Note that B^{045}, B^{234}, B^{356} and C_1^{234} in (2.6)–(2.8) follow from the definitions given for B^{123} and C_1^{567}. For example, $B^{045} = A^{04}A^{45}A^{56}A_{456}/2$, $B^{234} = A^{24}A^{35}A^{46}A_{456}/2$ and $C_1^{234} = A_{1234}A^{22}A^{33}A^{44}/6$. The first three δ^a given will be sufficient to obtain the cumulant coefficients needed.

3. The leading cumulant coefficients

In this section we give the cumulant coefficients of (1.1) needed for the distribution of $n^{1/2}(\widehat{\theta}-\theta^0)$ to $O(n^{-3/2})$, namely $k_{\cdot 1}^{12}$, $k_{\cdot 1}^{0}$, $k_{\cdot 2}^{123}$, $k_{\cdot 2}^{12}$, $k_{\cdot 3}^{1234}$, where we extend the notation of (D.1), (2.3) by setting $k_{\cdot j}^{1\ldots r} = k_j^{i_1\ldots i_r}$. We

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now assume $\Lambda_1(\theta),\dots,\Lambda_n(\theta)$ independent. For π a sequence of integers in $\{1,\dots,p\}$, using the dot notation of (2.2) for partial derivatives,

$$\Lambda_{.\pi} = n^{-1}\sum_{N=1}^{n}\Lambda_{N\cdot\pi}.$$

So, for $\pi_1,\pi_2,\dots$ such sequences, the joint cumulants of the $\Lambda_{.\pi}$ are given by

$$\kappa(\Lambda_{.\pi_1},\dots,\Lambda_{.\pi_r}) = n^{1-r}[\pi_1,\dots,\pi_r], \tag{3.1}$$

where

$$[\pi_1,\dots,\pi_r] = n^{-1}\sum_{N=1}^{n}\kappa(\Lambda_{N\cdot\pi_1},\dots,\Lambda_{N\cdot\pi_r}). \tag{3.2}$$

For example, $[1\dots r] = E\Lambda_{.1\dots r} = A_{1\dots r}$. We shall give the leading cumulant coefficients we need in terms of these [.] functions.

Set $\delta_{rs}=1$ for $r=s$ and 0 otherwise, and $a_{1\dots r}^{b_1\dots b_r} = \kappa(\delta_1^{b_1},\dots,\delta_r^{b_r})$. This has an expansion of the form

$$a_{1\dots r}^{b_1\dots b_r} = \sum_{j\ge(b_1+\dots+b_r)/2} a_{1\dots r\cdot j}^{b_1\dots b_r}\, n^{-j}.$$

Also, the left hand side of (1.1) is equal to $\theta^0_{i_1}\delta_{r1}+\kappa(\delta_1,\dots,\delta_r)$. Substituting (2.1) into this gives

$$k_{.j}^{1\dots r} = \theta^0_{i_1}\delta_{r1}\delta_{j0} + \sum_{b_1+\dots+b_r\le 2j} a_{1\dots r\cdot j}^{b_1\dots b_r}, \tag{3.3}$$

where $b_1,\dots,b_r$ are positive integers.

Theorem 3.1. *The coefficients $k_{.1}^{12}$, $k_{.1}^{0}$, $k_{.2}^{123}$, $k_{.2}^{12}$ and $k_{.3}^{1234}$ of (3.3) can be expressed as*

$$k_{.1}^{12} = a_{12.1}^{11},\ k_{.1}^{0} = a_{1.1}^{2},\ k_{.2}^{123} = a_{123.2}^{111} + \sum_{112}^{3} a_{123.2}^{112}, \tag{3.4}$$

$$k_{.2}^{12} = \sum_{12}^{2} a_{12.2}^{12} + \sum_{13}^{2} a_{12.2}^{13} + a_{12.2}^{22} \tag{3.5}$$

and

$$k_{.3}^{1234} = a_{1234\cdot 3}^{1111} + \sum_{1112}^{4} a_{1234\cdot 3}^{1112} + \sum_{1113}^{4} a_{1234\cdot 3}^{1113} + \sum_{1122}^{6} a_{1234\cdot 3}^{1122}, \tag{3.6}$$

where

$$k_{.1}^{12} = a_{12.1}^{11} = A^{13}A^{24}[3,4], \tag{3.7}$$

$$k_{.1}^{0} = a_{1.1}^{2} = A^{01}A^{23}[12,3] - B^{045}[4,5], \tag{3.8}$$

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$$a^{111}_{123\cdot 2} = -A^{14}A^{25}A^{36}[4,5,6], \tag{3.9}$$

$$a^{112}_{123\cdot 2} = A^{14}A^{25}\left\{A^{36}A^{78}\sum_{45}^{2}[4,67]\,[5,8] - B^{367}\sum_{45}^{2}[4,6]\,[5,7]\right\}, \tag{3.10}$$

$$a^{12}_{12\cdot 2} = -A^{13}A^{24}A^{56}[3,45,6] + A^{13}B^{245}[3,4,5], \tag{3.11}$$

$$\begin{aligned} a^{13}_{12\cdot 2} \;=\; & A^{13}A^{24}\left\{A^{56}A^{78}\sum^{3}[3,45]\,[67,8] - B^{567}\sum^{3}[3,45]\,[6,7]\right. \\ & -2^{-1}A_{456}\sum_{56}^{2}A^{57}\left(A^{68}A^{9,10}\sum^{3}[3,7]\,[89,10] - B^{689}\sum^{3}[3,7]\,[8,9]\right) \\ & \left.+2^{-1}A^{57}A^{68}\sum^{3}[3,456]\,[7,8] - C_4^{567}\sum^{3}[3,5]\,[6,7]\right\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} a^{22}_{12\cdot 2} \;=\; & A^{13}A^{45}A^{26}A^{78}\sum_{34,5}^{2}[5,8]\,[34,67] - \sum_{12}^{2}A^{13}A^{45}B^{267}\sum_{6,7}^{2}[5,6]\,[7,34] \\ & +B^{134}B^{267}\sum_{34}^{2}[3,6]\,[4,7], \end{aligned} \tag{3.13}$$

$$a^{1111}_{1234\cdot 3} = A^{15}A^{26}A^{37}A^{48}[5,6,7,8], \tag{3.14}$$

$$a^{1112}_{1234\cdot 3} = A^{15}A^{26}A^{37}\left\{-A^{48}A^{9,10}\sum^{6}[89,10]\,[5,6,7] + B^{489}\sum^{6}[8,9]\,[5,6,7]\right\}, \tag{3.15}$$

$$\begin{aligned} a^{1113}_{1234\cdot 3} \;=\; & A^{15}A^{26}A^{37}A^{48}\left\{A^{9,10}A^{11,12}\sum^{15}[5,6]\,[7,89]\,[10.11,12]\right. \\ & -B^{9,10,11}\sum^{15}[5,6]\,[7,89]\,[10,11] \\ & -2^{-1}A_{89,10}\sum_{9,10}^{2}A^{9,11}\left(A^{10,12}A^{13,14}\sum^{15}[5,6]\,[7,11]\,[12.13,14]\right. \\ & \left.-B^{10,12,13}\sum^{15}[5,6]\,[7,11]\,[12,13]\right) \\ & +2^{-1}A^{9,11}A^{10,12}\sum^{15}[5,6]\,[7,89.10]\,[11,12] \\ & \left.-C_8^{9,10,11}\sum^{15}[5,6]\,[7,9]\,[10,11]\right\} \end{aligned} \tag{3.16}$$

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and

$$a^{1122}_{1234\cdot 3} = A^{15}A^{26}\left\{A^{37}A^{89}\left[A^{4,10}A^{11,12}\sum^{15}[5,6]\,[78,9]\,[10.11,12]\right.\right.$$
$$\left.-B^{4,10,11}\sum^{15}[5,6]\,[78,9]\,[10,11]\right]$$
$$-B^{378}\left[A^{4,10}A^{11,12}\sum^{15}[5,6]\,[7,8]\,[10.11,12]\right.$$
$$\left.\left.-B^{4,10,11}\sum^{15}[5,6]\,[7,8]\,[10,11]\right]\right\}. \tag{3.17}$$

4. Least squares estimates

Here we apply the previous section to LSEs for both real and complex models. We begin with the *real* model. Suppose we observe

$$Y = S(\theta) + e \text{ in } \mathcal{R}^n, \text{ that is } Y_N = S_N(\theta) + e_N \text{ in } \mathcal{R} \text{ for } 1 \le N \le n \tag{4.1}$$

with $e_1, \ldots, e_n$ independent and identically distributed (i.i.d.) with mean zero. Denote their rth cumulant by $\lambda_r = \kappa_r(e_1)$. The LSE is $\widehat{\theta}$ minimising

$$\Lambda = n^{-1}|Y - S(\theta)|^2/2 = n^{-1}\sum_{N=1}^{n}\Lambda_N(\theta), \tag{4.2}$$

where $\Lambda_N(\theta) = |Y_N - S_N(\theta)|^2/2$. So,

$$\Lambda_{N\cdot 1\ldots r} = \partial_1 \ldots \partial_r S_N^2/2 - Y_N S_{N\cdot 1\ldots r},\ A_{1\ldots r} = [1\ldots r] = n^{-1}\sum_{N=1}^{n}\{\partial_1 \ldots \partial_r S_N^2/2 - S_N S_{N\cdot 1\ldots r}\}.$$

For $\pi_1, \pi_2, \ldots$ sequences of integers in $1 \ldots p$, set

$$\langle \pi_1, \pi_2, \ldots\rangle = n^{-1}\sum_{N=1}^{n} S_{N\cdot\pi_1}S_{N\cdot\pi_2}\ldots \tag{4.3}$$

Theorem 4.1 uses the previous section to express the cumulant coefficients we need in terms of these $\langle\ \rangle$ functions. Theorem 4.2 notes what form these take for the special case when A is diagonal, having in mind the example of the next section for the case $R = 1$.

Theorem 4.1. *For the model given by (4.1), the cumulant coefficients of Theorem 3.1 are*

$$k_1^{i_1 i_2} = k_{\cdot 1}^{12} = \lambda_2 A^{12}, \tag{4.4}$$

$$k_1^{i_0} = k_{\cdot 1}^{0} = 2^{-1}\lambda_2 A^{01}A^{23}\{-\langle 1,23\rangle - \langle 2,13\rangle + \langle 3,12\rangle\}, \tag{4.5}$$

$$k_2^{i_1 i_2 i_3} = k_{\cdot 2}^{123} = A^{14}A^{25}A^{36}\left\{\lambda_3\langle 4,5,6\rangle + \lambda_2^2\sum_{456}^{3}\langle 4,56\rangle\right\} - 6\lambda_2^2 B^{123}, \tag{4.6}$$

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$$a^{12}_{12\cdot 2} = \lambda_3 A^{13}\left(A^{24}A^{56}\langle 3,45,6\rangle - B^{245}\langle 3,4,5\rangle\right), \tag{4.7}$$

$$\begin{aligned} a^{13}_{12\cdot 2}/\lambda_2^2 &= A^{13}A^{24}A^{56}\left(A^{78}\sum_{38}^{2}\langle 3,45\rangle\langle 8,67\rangle + \langle 34,56\rangle\right) \\ &\quad -2^{-1}A^{24}\langle 3,45\rangle A^{56}A_{678}(A^{13}A^{78}+2A^{18}A^{37}) \\ &\quad +A^{13}A^{24}\left(-2^{-1}\langle 3,45\rangle B^5 - \langle 6,45\rangle B_3^{56}\right. \\ &\quad +2A_{346}B^6 + 2^{-1}A_{456}A_{378}A^{57}A^{68} + 2^{-1}\langle 3,456\rangle A^{56} \\ &\quad \left.+\langle 6,345\rangle A^{56} - 2^{-1}A_{3456}A^{56}\right), \end{aligned} \tag{4.8}$$

$$a^{22}_{12\cdot 2}/\lambda_2^2 = A^{13}A^{26}(A^{47}\langle 34,67\rangle + A^{45}A^{78}\langle 34,8\rangle\langle 5,67\rangle) - 2\sum_{12}^{2}A^{13}B^{246}\langle 6,34\rangle + 2B_6^{13}B_3^{26}, \tag{4.9}$$

$$a^{1111}_{1234\cdot 3} = \lambda_4 A^{15}A^{26}A^{37}A^{48}\langle 5,6,7,8\rangle, \tag{4.10}$$

$$\begin{aligned} a^{1112}_{1234\cdot 3} &= [\lambda_2\lambda_3]\left\{-A^{15}A^{26}A^{37}A^{48}A^{9,10}\sum_{567}^{3}\langle 5,89\rangle\langle 10,6,7\rangle\right. \\ &\quad -A^{48}[A^{26}A^{37}A^{19}\langle 89,6,7\rangle + A^{15}A^{37}A^{29}\langle 89,5,7\rangle + A^{15}A^{26}A^{39}\langle 89,5,6\rangle] \\ &\quad \left.+A^{15}A^{26}A^{37}A^{48}A^{9,10}\sum_{567}^{3}A_{58,10}\langle 6,7,9\rangle\right\}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} a^{1113}_{1234\cdot 3} &= \lambda_2^3 A^{15}A^{26}A^{37}A^{48}\left\{A^{9,10}A^{11,12}\sum^{15}\langle 5,6\rangle\langle 7,89\rangle\langle 12,10.11\rangle\right. \\ &\quad -B^{9,10,11}\sum^{15}\langle 5,6\rangle\langle 7,89\rangle\langle 10,11\rangle \\ &\quad -2^{-1}A_{89,10}\sum_{9,10}^{2}A^{9,11}\left(A^{10,12}A^{13,14}\sum^{15}\langle 5,6\rangle\langle 7,11\rangle\langle 12.13,14\rangle\right. \\ &\quad \left.-B^{10,12,13}\sum^{15}\langle 5,6\rangle\langle 7,11\rangle\langle 12,13\rangle\right) + 2^{-1}A^{9,11}A^{10,12}\sum^{15}\langle 5,6\rangle\langle 7,89.10\rangle\langle 11,12\rangle \\ &\quad \left.-C_8^{9,10,11}\sum^{15}\langle 5,6\rangle\langle 7,9\rangle\langle 10,11\rangle\right\} \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} a^{1122}_{1234\cdot 3} &= \lambda_2^3 A^{15}A^{26}\left\{A^{37}A^{89}\left[A^{4,10}A^{11,12}\sum^{15}\langle 5,6\rangle\langle 78,9\rangle\langle 10.11,12\rangle\right.\right. \\ &\quad \left.-B^{4,10,11}\sum^{15}\langle 5,6\rangle\langle 78,9\rangle\langle 10,11\rangle\right] - B^{378}\left[A^{4,10}A^{11,12}\sum^{15}\langle 5,6\rangle\langle 7,8\rangle\langle 10.11,12\rangle\right. \\ &\quad \left.\left.-B^{4,10,11}\sum^{15}\langle 5,6\rangle\langle 7,8\rangle\langle 10,11\rangle\right]\right\}, \end{aligned} \tag{4.13}$$

where $B^1 = B^{123}A_{23} = 2^{-1}A^{12}A_{234}A^{34}$ *and* $B_1^{23} = B^{234}A_{14} = 2^{-1}A_{145}A^{24}A^{35}$.

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Theorem 4.2. *The corresponding expressions of Theorem 4.1 for A diagonal are*

$$k_1^{i_0} = k_{\cdot 1}^0 = -2^{-1}\lambda_2 A^{00}A^{22}\langle 0,22\rangle, \tag{4.14}$$

$$k_2^{i_1 i_2 i_3} = k_{\cdot 2}^{123} = A^{11}A^{22}A^{33}\left\{\lambda_3\langle 1,2,3\rangle + \lambda_2^2\left(\sum_{123}^{3}\langle 1,23\rangle - 3A_{123}\right)\right\}, \tag{4.15}$$

$$a_{12\cdot 2}^{12} = \lambda_3 A^{11}\left\{A^{22}A^{55}\langle 1,25,5\rangle - B^{245}\langle 1,4,5\rangle\right\}, \tag{4.16}$$

$$\begin{aligned} a_{12\cdot 2}^{13}/\lambda_2^2 &= A^{11}A^{22}A^{33}[A^{77}(\langle 1,25\rangle\langle 7,57\rangle + \langle 1,57\rangle\langle 7,25\rangle) + \langle 12,55\rangle] \\ &\quad -2^{-1}A^{11}A^{22}A^{55}(\langle 1,25\rangle A_{577}A^{77} \\ &\quad +2\langle 3,25\rangle A_{531}A^{33}) + A^{11}A^{22}[-2^{-1}\langle 1,25\rangle B^5 \\ &\quad -\langle 6,25\rangle B_1^{56} + 2A_{126}B^6 + 2^{-1}A_{156}A_{256}A^{55}A^{66} \\ &\quad +2^{-1}\langle 1,255\rangle A^{55} + \langle 5,125\rangle A^{55} - 2^{-1}A_{1255}A^{55}], \end{aligned} \tag{4.17}$$

$$a_{12\cdot 2}^{22}/\lambda_2^2 = A^{11}A^{22}A^{44}(\langle 14,24\rangle + A^{77}\langle 7,14\rangle\langle 4,27\rangle) - \sum_{12}^{2}A^{11}B^{246}\langle 6,14\rangle + 2B_6^{13}B_3^{26}, \tag{4.18}$$

$$a_{1234\cdot 3}^{1111} = \lambda_4 A^{11}A^{22}A^{33}A^{44}\langle 1,2,3,4\rangle, \tag{4.19}$$

$$\begin{aligned} a_{1234\cdot 3}^{1112}/(\lambda_2\lambda_3) &= A^{11}\ldots A^{44}\left\{-A^{55}\sum_{123}^{3}\langle 1,45\rangle\langle 2,3,5\rangle - \sum_{123}^{3}\langle 1,2,34\rangle\right. \\ &\quad \left. +A^{55}\sum_{123}^{3}A_{145}\langle 2,3,5\rangle\right\}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} a_{1234\cdot 3}^{1113}/\lambda_2^3 &= A^{11}A^{22}A^{33}A^{44}\left\{A^{99}A^{11,11}\sum^{15}\langle 1,2\rangle\langle 3,49\rangle\langle 11,9.11\rangle\right. \\ &\quad -B^{9,10,11}\sum^{15}\langle 1,2\rangle\langle 3,49\rangle\langle 10,11\rangle \\ &\quad -2^{-1}A_{49,10}\sum_{9,10}^{2}A^{9,9}\left(A^{10,10}A^{13,13}\sum^{15}\langle 1,2\rangle\langle 3,9\rangle\langle 10\cdot 13,13\rangle\right. \\ &\quad \left.-B^{10,12,13}\sum^{15}\langle 1,2\rangle\langle 3,9\rangle\langle 12,13\rangle\right) + 2^{-1}A^{99}A^{10,10}\sum^{15}\langle 1,2\rangle\langle 3,49.10\rangle\langle 9,10\rangle \\ &\quad \left.-C_4^{9,10,11}\sum^{15}\langle 1,2\rangle\langle 3,9\rangle\langle 10,11\rangle\right\} \end{aligned} \tag{4.21}$$

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and

$$\begin{aligned} a^{1122}_{1234\cdot 3}/\lambda_2^3 &= A^{11}A^{22}\Big\{A^{33}A^{88}\Big[A^{44}A^{11,11}\sum^{15}\langle 1,2\rangle\langle 38,8\rangle\langle 4.11,11\rangle \\ &\quad -B^{4,10,11}\sum^{15}\langle 1,2\rangle\langle 38,8\rangle\langle 10,11\rangle\Big]-B^{378}\Big[A^{44}A^{11,11}\sum^{15}\langle 1,2\rangle\langle 7,8\rangle\langle 4.11,11\rangle \\ &\quad -B^{4,10,11}\sum^{15}\langle 1,2\rangle\langle 7,8\rangle\langle 10,11\rangle\Big]\Big\}, \end{aligned} \tag{4.22}$$

where $B^{123}=A^{11}A^{22}A^{33}A_{123}/2$, $C_1^{234}=A_{1234}A^{22}A^{33}A^{44}/6$, *and*

$$B^1=2^{-1}A^{11}A^{22}A_{122},\ B_1^{23}=2^{-1}A_{123}A^{22}A^{33}. \tag{4.23}$$

Note that the implicit summations are over the i's not on the left hand side. For example, in (4.14) the summation is over i_2, not i_0.

Note that B^5, B^6, B_3^{56}, B_6^{13} and B_3^{26} in Theorem 4.1 follow from the definitions given for B^1 and B_1^{23}. For example, $B^5=B^{523}A_{23}$, $B^6=B^{623}A_{23}$, $B_3^{56}=B^{564}A_{34}$ and $B_6^{13}=B^{134}A_{64}$. Similar comments apply to Theorem 4.2.

Now suppose we replace the real model (4.1) by the *complex* model

$$Y=S(\theta)+e \text{ in } \mathcal{C}^n, \text{ that is } Y_N=S_N(\theta)+e_N \text{ in } \mathcal{C} \text{ for } 1\le N\le n \tag{4.24}$$

with $e_N=e_{N1}+je_{N2}$ for $j=\sqrt{-1}$ and $\{e_{N1},e_{N2}\}$ independent and identically distributed with mean zero and cumulants $\{\lambda_r\}$. The LSE is again given by (4.2). This can be put in the framework of (4.1) with n replaced by $2n$, (so that Λ and $A_{1\ldots r}$ are half what they are for the complex version with $e_{N1}=e_N$ of (4.1)), but it is simpler to adapt the preceding as follows:

$$\begin{aligned} \Lambda_{N\cdot 1\ldots r} &= \partial_1\ldots\partial_r\bar{S}_NS_N/2-\bar{Y}_NS_{N\cdot 1\ldots r}/2-Y_N\bar{S}_{N\cdot 1\ldots r}/2, \\ A_{1\ldots r} &= [1\ldots r]=n^{-1}\sum_{N=1}^{n}\{\partial_1\ldots\partial_r|S_N|^2/2-Re\,\bar{S}_NS_{N\cdot 1\ldots r}\} \\ &= n^{-1}(\partial_1\ldots\partial_r|S|^2/2-Re\,S^*S_{\cdot 1\ldots r}). \end{aligned}$$

(Recall that $\bar{Y}_N$ is the complex conjugate of Y_N, and $S^*=\bar{S}'$.) Let us extend the notation of (4.3) by writing

$$\langle\bar{\pi}_1,\pi_2,\pi_3\rangle=n^{-1}\sum_{N=1}^{n}\bar{S}_{N\cdot\pi_1}S_{N\cdot\pi_2}S_{N\cdot\pi_3}$$

and so on for $\pi_1,\pi_2,\ldots$ sequences of integers in $1\ldots p$. One obtains

$$\begin{aligned} n\Lambda_{\cdot 1} &= -Re\,e^*S_{\cdot 1},\ n\Lambda_{\cdot 12}=Re\,(S_{\cdot 1}^*S_{\cdot 2}-e^*S_{\cdot 12}), \\ A_{12} &= Re\,B_{12}=(B_{12}+\bar{B}_{12})/2 \text{ for } B_{12}=S_{\cdot 1}^*S_{\cdot 2}/n=\langle\bar{1},2\rangle, \\ \Lambda_{\cdot 1\ldots r} &= Re\,(B_{1\ldots r}-e^*S_{\cdot 1\ldots r}/n),\ A_{1\ldots r}=Re\,B_{1\ldots r}, \end{aligned} \tag{4.25}$$

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where

$$\begin{aligned} B_{123} &= \left(\sum_{123}^{3} S^*_{\cdot 1} S_{\cdot 23}\right) \Big/ n = \sum_{123}^{3} \langle \bar{1}, 23\rangle, \\ B_{1234} &= \left(\sum_{1234}^{4} S^*_{\cdot 1} S_{\cdot 234} + \sum_{123}^{3} S^*_{\cdot 12} S_{\cdot 34}\right) \Big/ n = \sum_{1234}^{4} \langle \bar{1}, 234\rangle + \sum_{123}^{3} \langle \bar{12}, 34\rangle. \end{aligned}$$

Note that $B_{1\ldots r}$ can be written down using the form for the partial Bell polynomial B_{r2} on page 307 of [2]; his $B_{22} = x_1^2$, his $B_{32} = 3x_1x_2$, his $B_{42} = 4x_1x_3 + 3x_2^2$, his $B_{52} = 5x_1x_4 + 10x_2x_3$ so that

$$B_{12345} = \left(\sum^{5} S^*_{\cdot 1} S_{\cdot 2345} + \sum^{10} S^*_{\cdot 12} S_{\cdot 345}\right) \Big/ n = \sum^{5} \langle \bar{1}, 2345\rangle + \sum^{10} \langle \overline{12}, 345\rangle,$$

and so on. We now give the complex form of (D.5). Set $\gamma_i = S_{\cdot \pi_i}$ and $T_i = e^* \gamma_i$. If $r > 1$ then

$$\begin{aligned} (-2)^r [\pi_1, \ldots, \pi_r] &= 2^r n^{-1} \kappa(Re\, T_1, \ldots, Re\, T_r) = n^{-1} \kappa(T_1 + \bar{T}_1, \ldots, T_r + \bar{T}_r) \\ &= \sum_{i=0}^{r} k(1^i \overline{1}^{r-i}) \sum^{\binom{r}{i}} \langle \overline{\pi}_1, \ldots, \overline{\pi}_i, \pi_{i+1}, \ldots, \pi_r\rangle, \end{aligned} \tag{4.26}$$

where $k(1^i\overline{1}^j) = \kappa(e_1, \cdots, e_1, \overline{e}_1, \cdots, \overline{e}_1)$, counting e_1 i times and $\overline{e}_1$ j times. Also the inner summation is over all such $(\overline{\pi}_1, \ldots, \overline{\pi}_i)$ giving different terms. These joint cumulants can be written in terms of the real cumulants $\{\lambda_r\}$: $k(1^i\overline{1}^{r-i}) = [1 + (-1)^{r-i} j^r]\lambda_r$ so that $k(1^2) = 0,\ k(1\bar{1}) = 2\lambda_2$. For example,

$$\begin{aligned} 4[\pi_1, \pi_2] &= E\left(T_1T_2 + \sum^{2} T_1\bar{T}_2 + \bar{T}_1\bar{T}_2\right) \Big/ n = 2\lambda_2 \sum^{2} \langle \overline{\pi}_1, \pi_2\rangle = 4\lambda_2\, \mathrm{Real}\left(\langle \overline{\pi}_1, \pi_2\rangle\right), \\ -8[\pi_1, \pi_2, \pi_3] &= 2\lambda_3\, \mathrm{Real}\left((1+j)\left\{\langle \pi_1, \pi_2, \pi_3\rangle + \sum^{3} \langle \pi_1, \bar{\pi}_2, \bar{\pi}_3\rangle\right\}\right), \\ 16[\pi_1, \pi_2, \pi_3, \pi_4] &= 4\lambda_4\, \mathrm{Real}\left(\left\{\langle \pi_1, \pi_2, \pi_3, \pi_4\rangle + \sum_{234}^{3} \langle \overline{\pi}_1, \overline{\pi}_2, \pi_3, \pi_4\rangle\right\}\right). \end{aligned}$$

So, by (D.5) the cumulant coefficients for the complex case are obtained from those for the real case by replacing

$$\begin{aligned} \langle \pi_1, \pi_2\rangle \quad &\text{by} \quad \mathrm{Real}(\langle \bar{\pi}_1, \pi_2\rangle), \\ \langle \pi_1, \pi_2, \pi_3\rangle \quad &\text{by} \quad \mathrm{Real}\left((1+j)\left\{\langle \pi_1, \pi_2, \pi_3\rangle + \sum^{3} \langle \pi_1, \bar{\pi}_2, \bar{\pi}_3\rangle\right\}\right)\Big/4, \\ \langle \pi_1, \pi_2, \pi_3, \pi_4\rangle \quad &\text{by} \quad \mathrm{Real}\left(\left\{\langle \pi_1, \pi_2, \pi_3, \pi_4\rangle + \sum_{234}^{3} \langle \overline{\pi}_1, \overline{\pi}_2, \pi_3, \pi_4\rangle\right\}\right)\Big/4, \end{aligned} \tag{4.27}$$

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and so on. A simpler way of seeing this – without having to involve the joint cumulants $k(1^i\overline{1}^{r-i})$, is as follows. Consider the complex numbers $a = a_1 + ja_2$, $b = b_1 + jb_2$, ... Then $a_1 = (a + \bar{a})/2$ and $a_2 = -(a - \bar{a})j/2$. So,

$$\begin{aligned} a_1b_1 + a_2b_2 &= \text{Real}(\overline{a}b), \\ a_1b_1c_1 + a_2b_2c_2 &= \text{Real}((1+j)\left(abc + \sum\nolimits^3 \overline{abc}\right)\Big/4, \\ a_1b_1c_1d_1 + a_2b_2c_2d_2 &= \text{Real}\left(abcd + \sum_{bcd}^3 ab\overline{cd}\right)\Big/4 \\ &= \text{Real}\left(abcd + ab\overline{cd} + ac\overline{bd} + ad\overline{bc}\right)\Big/4, \end{aligned} \tag{4.28}$$

and so on. Now take $a = S_{N.\pi_1}$, $b = S_{N.\pi_2}$, ... Then $n^{-1}\sum_{N=1}^n (a_1b_1c_1 + a_2b_2c_2)$ is twice the real version of $\langle \pi_1, \pi_2, \pi_3\rangle$ for the real version of the complex version with n replaced by $2n$. By (4.28) it equals the right hand side of (4.27). Similarly, one can write down $\langle \pi_1, \ldots, \pi_r\rangle$ for the real version of the complex model in terms of $\langle \pi_1, \ldots, \pi_r\rangle$ for the complex model. So, the real versions (4.4), (4.5), (4.6), imply the complex versions

$$\begin{aligned} k_1^{i_1i_2} &= k_{.1}^{12} = \lambda_2 A^{12},\ k_1^{i_0} = k_{.1}^0 = 2^{-1}\lambda_2 A^{01}A^{23}\,\text{Real}\left(\{-\langle \bar{1}, 23\rangle - \langle \bar{2}, 13\rangle + \langle \bar{3}, 12\rangle\}\right), \\ k_2^{i_1i_2i_3} &= k_{.2}^{123} = A^{14}A^{25}A^{36}\,\text{Real}\left(\lambda_3(1+j)\left\{\langle 4,5,6\rangle + \sum\nolimits^3 \langle 4, \bar{5}, \bar{6}\rangle\right\}\Big/4 + \lambda_2^2 \sum_{456}^3 \langle \bar{4}, 56\rangle\right) - 6\lambda_2^2 B^{123}. \end{aligned}$$

Similarly, $k_2^{i_1i_2} = k_{.2}^{12}$ can be written down from its real form given by (3.5), (3.11), (3.12), (3.13), and $k_3^{i_1i_2i_3i_4} = k_{.3}^{1234}$ can be written down from its real form given by (3.6), (3.14)–(3.17). Note that if e_1 is complex normal with components having the same variance, then (4.26) implies that $[\pi_1, \ldots, \pi_r] = 0$ for $r > 2$ just as this holds by (D.5) for the real case (4.1) if $e_1 \sim \mathcal{N}(0, \lambda_2)$.

5. Examples

We now drop the convention of Sections 2–4 of suppressing the i's to the usual convention that

$$S_{N\cdot i_1 \ldots i_r} = \partial_{i_1} \ldots \partial_{i_r} S_N$$

for $\partial_i = \partial/\partial\theta_i$. That is, we write $\langle i_1 \ldots i_r\rangle$, where we had $\langle 1 \ldots r\rangle$, $A_{i_1i_2} = \langle i_1, i_2\rangle$, where we had $A_{12} = \langle 1, 2\rangle$ in (D.3), $\langle i_1, i_2i_3\rangle$, where we had $\langle 1, 23\rangle$ in (D.4), and so on. So, now

$$A_{rs} = \langle r, s\rangle = n^{-1}\sum_{N=1}^n S_{N\cdot r}S_{N\cdot s}.$$

Example 5.1. Consider the R signal M frequency problem: observe

$$y_k = s(\theta) + n_k \text{ in } \mathcal{C}^M \text{ for } k = 1, \ldots, K,$$

where $n_1, \ldots, n_K$ are independent $\mathcal{CN}_M(0, vI_M)$, that is, with real and imaginary parts independent $\mathcal{N}_M(0, vI_M/2)$. (So, λ_2 of Sect. 4 is $v/2$ and $\lambda_r = 0$ for $r \neq 2$.) Counting $m = 0, 1, \ldots, M-1$, suppose that the mth component of $s(\theta)$ has the form

$$s_m(\theta) = \sum_{r=1}^R a_r \exp(j\alpha_{mr}) = s_{m1} + js_{m2}$$

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say, for $j = \sqrt{-1}$, where $\alpha_{mr} = \varphi_r + \nu_m w_r$ and $\nu_m = (m + m_0)/M$, and $\{a_r, \varphi_r, w_r\}$ are real so that $p = 3R$ and we can take $\theta' = (a', \varphi', w')$. The main parameter is w; (a, φ) are nuisance parameters. We shall obtain the leading cumulant coefficients firstly by using the *real* model (4.1), and then for comparison using the *complex* model (4.24). For the real model the MLE $\widehat{\theta}$ minimises

$$\begin{aligned}\sum_{k=1}^{K} |y_k - s(\theta)|^2 /2 &= \sum_{N=1}^{n} (Y_N - S_N(\theta))^2 /2, \text{ where } n = 2kM,\\ \left\{ \begin{pmatrix} Y_1 \\ S_1 \end{pmatrix}, \ldots, \begin{pmatrix} Y_n \\ S_n \end{pmatrix} \right\} &= \left\{ \begin{pmatrix} y_{km1} \\ s_{m1} \end{pmatrix}, \begin{pmatrix} y_{km2} \\ s_{m2} \end{pmatrix} \right\} : \ 0 \le m < M, \ 1 \le k \le K,\end{aligned}$$

and $y_{km} = y_{km1} + j y_{km2}$. This puts the problem into the *real* formulation of (4.1) with $e_N \sim \mathcal{N}(0, \lambda_2)$. The constant m_0 is arbitrary, since it reparameterises φ. Choose $m_0 = -(M-1)/2$, so that $\{\nu_m\}$ have arithmetic mean zero. For $1 \le r, s \le R$, $s_{m1\cdot r} = \cos \alpha_{mr}$, $s_{m1\cdot r+R} = -a_r \sin \alpha_{mr}$, $s_{m1\cdot r+2R} = -\nu_m a_r \sin \alpha_{mr}$, $s_{m2\cdot r} = \sin \alpha_{mr}$, $s_{m2\cdot r+R} = a_r \cos \alpha_{mr}$, $s_{m2\cdot r+2R} = \nu_m a_r \cos \alpha_{mr}$ and

$$A_{rs} = (2M)^{-1} \sum_{m=0}^{M-1} \sum_{j=1}^{2} s_{mj\cdot r} s_{mj\cdot s}.$$

Fix $1 \le r, \ s \le R$ and set

$$\varphi_{rs} = \varphi_r - \varphi_s, \ w_{rs} = w_r - w_s, \ \delta_m = \alpha_{mr} - \alpha_{ms} = \varphi_{rs} + \nu_m w_{rs}. \tag{5.1}$$

Then the elements of $A = \{A_{ab} : 1 \le a, \ b \le 3R\}$ can be identified as follows.

$$\langle a, a' \rangle : \ \ \langle r, s \rangle = (2M)^{-1} \sum_{m=0}^{M-1} \cos \delta_m,$$

So, $\langle r, \, r \rangle = 2^{-1}$ and for $r \ne s$,

$$\begin{aligned}\langle r, s \rangle &= 2^{-1} \int_{-1/2}^{1/2} \cos(\varphi_{rs} + x w_{rs}) \, \mathrm{d}x + O(M^{-1})\\ &= (\cos \varphi_{rs}) w_{rs}^{-1} \sin(w_{rs}/2) + O(M^{-1}) \text{ as } M \to \infty,\end{aligned}$$

$$\begin{aligned}\langle a, \varphi' \rangle : \langle r, s + R \rangle &= a_s (2M)^{-1} \sum_{m=0}^{M-1} \sin \delta_m = a_s 2^{-1} \int_{-1/2}^{1/2} \sin(\varphi_{rs} + x w_{rs}) \, \mathrm{d}x + O(M^{-1})\\ &= a_s (\sin \varphi_{rs}) \, w_{rs}^{-1} \sin(w_{rs}/2) + O(M^{-1}) \text{ if } r \ne s\\ &= 0 \text{ if } r = s,\end{aligned}$$

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$$\begin{aligned}
\langle a, w' \rangle : \langle r, s+2R \rangle &= a_s (2M)^{-1} \sum_{m=0}^{M-1} \nu_m \sin \delta_m \\
&= a_s 2^{-1} \int_{-1/2}^{1/2} x \sin(\varphi_{rs} + x w_{rs}) \, \mathrm{d}x + O(M^{-1}) \\
&= a_s 2^{-1} \cos \varphi_{rs} \left\{ -w_{rs}^{-1} \cos(w_{rs}/2) + 2 w_{rs}^{-2} \sin(w_{rs}/2) \right\} \\
&\quad + O(M^{-1}) \text{ if } r \neq s \\
&= 0 \text{ if } r = s,
\end{aligned}$$

$$\langle \varphi, \varphi' \rangle : \langle r+R, s+R \rangle = a_r a_s (2M)^{-1} \sum_{m=0}^{M-1} \cos \delta_m = a_r a_s \langle r, s \rangle,$$

$$\begin{aligned}
\langle \varphi, w' \rangle : \langle r+R, s+2R \rangle &= (2M)^{-1} a_r a_s \sum_{m=0}^{M-1} \nu_m \cos \delta_m \\
&= 2^{-1} a_r a_s \int_{-1/2}^{1/2} x \cos(\varphi_{rs} + x w_{rs}) \, \mathrm{d}x + O(M^{-1}) \\
&= a_r a_s \sin \varphi_{rs} \{ 2^{-1} w_{rs}^{-1} \cos(w_{rs}/2) - w_{rs}^{-2} \sin(w_{rs}/2) \} \\
&\quad + O(M^{-1}) \text{ if } r \neq s \\
&= 0 \text{ if } r = s,
\end{aligned}$$

and

$$\begin{aligned}
\langle w, w' \rangle : \langle r+2R, s+2R \rangle &= a_r a_s (2M)^{-1} \sum_{m=0}^{M-1} \nu_m^2 \cos \delta_m \\
&= a_r a_s 2^{-1} \int_{-1/2}^{1/2} x^2 \cos(\varphi_{rs} + x w_{rs}) \, \mathrm{d}x + O(M^{-1}) \\
&= a_r a_s \cos \varphi_{rs} \left\{ (4^{-1} w_{rs}^{-1} - 2 w_{rs}^{-3}) \sin(w_{rs}/2) \right. \\
&\quad \left. + w_{rs}^{-2} \cos(w_{rs}/2) \right\} + O(M^{-1}) \text{ if } r \neq s \\
&= a_r^2 v_2/2 \text{ if } r = s,
\end{aligned}$$

where

$$v_i = M^{-1} \sum_{m=0}^{M-1} \nu_m^i = \int_{-1/2}^{1/2} x^i \, \mathrm{d}x + O(M^{-1}).$$

In particular,

$$\begin{aligned}
v_2 &= (1-\alpha)/12, \; v_4 = (1-\alpha)(3-7\alpha)/240, \\
v_6 &= (1-\alpha)(3-18\alpha+31\alpha^2)/1344, \; v_8 = (1-\alpha)(5-55\alpha+239\alpha^2-381\alpha^3)/11520,
\end{aligned}$$

and $v_i = 0$ for i odd, where $\alpha = M^{-2}$. So, A is only diagonal if $R = 1$.

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The *complex* formulation, (4.24) with $n = Mk$, is simpler than the real formulation above: choose θ as above. We have

$$s_{m\cdot r} = s_{m\cdot r+R}/(ja_r) = s_{m\cdot r+2R}/(j\nu_m a_r) = \exp(j\alpha_{mr}).$$

Set

$$g_{Mr}(\delta) = M^{-1}\sum_{m=0}^{M-1}\nu_m^r \exp(j\nu_m\delta).$$

A closed form for g_{Mr} is given in Appendix C. For example, $g_{M0}(0) = 1$ and if $\delta \neq 0$, then

$$\begin{aligned} Mg_{M0}(\delta) &= \{\cos(a-b) - \cos(a+b)\}/(1-\cos 2b) = \sin a/\sin b, \\ g_0(\delta) &= \sin a/a \text{ for } a = \delta/2, b = \delta/(2M). \end{aligned}$$

Note that

$$g_{Mr}(\delta) = g_r(\delta) + O(M^{-1}),$$

where

$$g_r(\delta) = \int_{-1/2}^{1/2} x^r \exp(jx\delta)\mathrm{d}x = 2(-1)^r r!\delta^{-r-1}\sum_{k=0}^{r}(1/(r-k)!)(\delta/2)^{r-k}b_k$$

and $b_k = \sin((k\pi+\delta)/2)$ or $j\cos((k\pi+\delta)/2)$ for r even or odd. Since $\{-\nu_m\} = \{\nu_m\}$,

$$\overline{g}_{Mr}(\delta) = g_{Mr}(-\delta) = (-1)^r g_{Mr}(\delta),$$

so g_{Mr} is real for r even, and imaginary for r odd, that is, $h_{Mr}(\delta) = j^{-1}g_{Mr}(\delta)$ is real for r odd. The $A_{i_1\cdots i_r}$ can be conveniently written in terms of the real and imaginary parts of the function

$$G_r(d_1,d_2) = H_r(d_1,d_2) + jI_r(d_1,d_2) = \exp(jd_1)g_{Mr}(d_2) = M^{-1}\sum_{m=0}^{M-1}\nu_m^r\exp(j(d_1+\nu_m d_2)).$$

So, for r even,

$$H_r(d_1,d_2) = \cos d_1\ g_{Mr}(d_2),\ \ I_r(d_1,d_2) = \sin d_1\ g_{Mr}(d_2),$$

and for r odd,

$$H_r(d_1,d_2) = -\sin d_1\ h_{Mr}(d_2),\ \ I_r(d_1,d_2) = \cos d_1\ h_{Mr}(d_2).$$

For example, B of (4.25) is given by $B_{ab} = s^*_{\cdot a}s_{\cdot b}/M$. As before fix $1 \le r, s \le R$. Then

$$\begin{aligned} B_{rs} &= B_{r,s+R}/(ja_s) = B_{r+R,s+R}/(a_r a_s) = \exp(-j\varphi_{rs})g_{M0}(w_{rs}), \\ B_{r,s+2R}/(ja_s) &= B_{r+R,s+2R}/(a_r a_s) = \exp(-j\varphi_{rs})g_{M1}(-w_{rs}), \\ B_{r+2R,s+2R} &= a_r a_s\exp(-j\varphi_{rs})g_{M2}(w_{rs}). \end{aligned}$$

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So, $A = A'$ is given in terms of φ_{rs}, w_{rs} of (5.1) by

$$\begin{aligned}
A_{rs} &= A_{r+R,s+R}/(a_r a_s) = \cos\varphi_{rs}\ g_{M0}(w_{rs}),\ A_{r,s+R} = a_s \sin\varphi_{rs}\ g_{M0}(w_{rs}),\\
A_{r,s+2R} &= \mathrm{Re}\, j a_s \exp(-j\varphi_{rs})(-j) h_{M1}(w_{rs}) = a_s \cos\varphi_{rs}\ h_{M1}(w_{rs}),\\
A_{r+R,s+2R} &= -a_r a_s \sin\varphi_{rs}\ h_{M1}(w_{rs}),\ A_{r+2R,s+2R} = a_r a_s \cos\varphi_{rs}\ g_{M2}(w_{rs}).
\end{aligned}$$

Although A for the real and complex formulations look different, A and $A_{1\cdots r}$ for the complex formulation are exactly twice what they are for the real formulation, as noted in Section 4.

For both the real and complex models $A_{i_1\cdots i_r}$ requires 6, 10 and 16 formulas for $r = 2, 3$ and 4; we gave these six formulas for $r = 2$ above; we now give the ten formulas for $r = 3$ for the complex case. We use the notation $1 \le r, s, t \le R$, $r_i = r + (i-1)R$, $s_i = s + (i-1)R$ and $t_i = t + (i-1)R$. So, $B_{r_1 s_1 t_1} = 0$ and $A_{rst} = A_{r_1 s_1 t_1} = 0$. Similarly, in terms of the Kronecker delta function $\delta_{ij} = 1$ for $i = j$ and 0 for $i \ne j$, we have, for example,

$$\begin{aligned}
B_{r_2 s_2 t_3} &= \sum_{rs}^{2} \delta_{st} a_r a_s j G_1(\varphi_{sr}, w_{sr}) - \delta_{rs} a_r a_t j G_1(\varphi_{rt}, w_{rt}),\\
B_{r_2 s_3 t_3} &= \delta_{st} a_r a_s j G_2(\varphi_{sr}, w_{sr}) + \sum_{st}^{2} \delta_{rt} a_r a_s j G_2(\varphi_{rs}, w_{rs}),\\
B_{r_3 s_3 t_3} &= \sum_{rst}^{3} \delta_{st} a_r a_s j G_3(\varphi_{sr}, w_{sr}),
\end{aligned}$$

so that

$$\begin{aligned}
A_{r_1 s_1 t_2} &= \delta_{st} A_{rs} + \delta_{tr} A_{st},\\
A_{r_1 s_2 t_2} &= a_s(\delta_{rt} - \delta_{st}) \cos\varphi_{rs}\ g_{M0}(w_{rs}) + a_t \delta_{rs} \cos\varphi_{rt}\ g_{M0}(w_{rt}),\\
A_{r_2 s_2 t_2} &= \sum_{rst}^{3} a_r a_s \delta_{st} \sin\varphi_{rs}\ g_{M0}(w_{rs}),\\
A_{r_1 s_1 t_3} &= \cos\varphi_{rs} \sum_{rs}^{2} \delta_{st} h_{M1}(w_{rs})\\
&= 0 \text{ if } r = s,\\
A_{r_1 s_2 t_3} &= \delta_{st} a_s \sin\varphi_{rs}\ h_{M1}(w_{rs}) - \sum_{st}^{2} \delta_{rt} a_s \sin\varphi_{rs}\ h_{M1}(w_{rs}),\\
A_{r_1 s_3 t_3} &= \delta_{st} a_s \cos\varphi_{rs}\ g_{M2}(w_{rs}) + \sum_{st}^{2} \delta_{rt} a_s \cos\varphi_{ts}\ g_{M2}(w_{ts}),\\
A_{r_2 s_2 t_3} &= -\sum_{rs}^{2} \delta_{st} a_r a_s I_1(\varphi_{sr}, w_{sr}) + \delta_{rs} a_r a_t I_1(\varphi_{rt}, w_{rt}),\\
A_{r_2 s_3 t_3} &= -\delta_{st} a_r a_s I_2(\varphi_{sr}, w_{sr}) - \sum_{st}^{2} \delta_{rt} a_r a_s I_2(\varphi_{rs}, w_{rs}),\\
A_{r_3 s_3 t_3} &= -\sum_{rst}^{3} \delta_{st} a_r a_s I_3(\varphi_{sr}, w_{sr}).
\end{aligned}$$

We now specialise the above example to the case $R = 1$.

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Example 5.2. Consider the one signal case, $R = 1$, of the previous example. By the *real* formulation above, with $a = a_1$,

$$A = (1/2)\mathrm{diag}\left(1, a^2, a^2 v_2\right), \; A^{-1} = \mathrm{diag}\left(2, 2a^{-2}, 2a^{-2}v_2^{-1}\right).$$

By (4.4), $n\mathrm{covar}\,\widehat{\theta} = \lambda_2 A^{-1} + O(n^{-1})$. By (4.14),

$$k_1^{i_0} = -2^{-1}\lambda_2 A^{i_0 i_0} \sum_{i_2=1}^{3} A^{i_2 i_2}\langle i_0, i_2 i_2\rangle.$$

Also $s_{mj\cdot 11} = 0$, $s_{m1\cdot 12} = -\sin\alpha_m$, $s_{m1\cdot 13} = -\nu_m \sin\alpha_m$, $s_{m2\cdot 12} = \cos\alpha_m$, $s_{m2\cdot 13} = \nu_m \cos\alpha_m$, $s_{m1\cdot 22} = -a\cos\alpha_m$, $s_{m1\cdot 23} = -\nu_m a\cos\alpha_m$, $s_{m1\cdot 33} = -\nu_m^2 a\cos\alpha_m$, $s_{m2\cdot 22} = -a\sin\alpha_m$, $s_{m2\cdot 23} = -\nu_m a\sin\alpha_m$ and $s_{m2\cdot 33} = \nu_m^2 a\sin\alpha_m$, where $\alpha_m = \alpha_{m1}$. So, the non-zero $\langle i, jk\rangle$ are $\langle 2, 12\rangle = -\langle 1, 22\rangle = a/2$ and $\langle 3, 13\rangle = -\langle 1, 33\rangle = av_2/2$. So, $k_1^1 = 2\lambda_2/a$ and $k_1^2 = k_1^3 = 0$. So, $\widehat{\varphi}$ and $\widehat{w}$ have low bias but not $\widehat{a}$. The non-zero A_{ijk}, B^{ijk} and k_2^{ijk} are $A_{122} = a/2$, $A_{133} = av_2/2$, $B^{122} = 2a^{-3}$, $B^{133} = 2a^{-3}v_2^{-1}$ and $k_2^{122} = -8\lambda_2^2 a^{-3} = v_2 k_2^{133}$. Since $\widehat{\varphi}$ and $\widehat{w}$ both have $k_1^i = k_2^{iii} = 0$, we should check whether their distributions are symmetric.

Next we show that

$$k_2^{33}/\lambda_2^2 = 4a^{-4}\left(5v_2^{-1} + v_4/v_2^3\right). \tag{5.2}$$

By (4.17), with $i_5 = i$, $i_7 = j$,

$$\begin{aligned} a_{33\cdot 2}^{13}/\lambda_2^2 &= (A^{33})^2 A^{ii}[A^{jj}c_1 + \langle 33, ii\rangle] - 2^{-1}(A^{33})^2 A^{ii}A^{jj}c_2 \\ &\quad +(A^{33})^2[-2^{-1}\langle 3, 3i\rangle B^i - \langle j, 2i\rangle B_1^{ij} + 2A_{33i}B^i + 2^{-1}A_{3ij}^2 A^{ii}A^{jj} \\ &\quad +2^{-1}\langle 3, 3ii\rangle A^{ii} + \langle i, 33i\rangle A^{ii} - 2^{-1}A_{33ii}A^{ii}], \end{aligned}$$

where $c_1 = \langle 3, 3i\rangle\,\langle j, ji\rangle + \langle 3, ij\rangle\,\langle j, 3i\rangle$ and $c_2 = \langle 3, 3i\rangle A_{ijj} + 2\langle j, 3i\rangle A_{3ij}$. So,

$$\begin{aligned} a_{33\cdot 2}^{13}\lambda_2^{-2}A_{33}^2 &= A^{ii}[A^{jj}\{c_1 - 2^{-1}c_2\} + \langle 33, ii\rangle \\ &\quad +2^{-1}A_{3ij}^2 A^{jj} + 2^{-1}\langle 3, 3ii\rangle + \langle i, 33i\rangle - 2^{-1}A_{33ii}] \\ &\quad +2A_{33i}B^i - 2^{-1}\langle 3, 3i\rangle B^i - \langle j, 2i\rangle B_1^{ij} \\ &= g_i + f_{ij} = \sum_{i=1}^{3} g_i + \sum_{i,j=1}^{3} f_{ij} \end{aligned}$$

say. Note that

$$A_{1133} = \langle 11, 33\rangle + 2\langle 13, 13\rangle + 2\langle 1, 133\rangle + 2\langle 3, 113\rangle = 0 + v_2 - v_2 + 0 = 0$$

since

$$\langle 13, 13\rangle = (2M)^{-1}\sum_{m=0}^{M-1}\nu_m^2 = 2^{-1}v_2.$$

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Note that

$$\begin{aligned}
B^1 &= A^{ii}A_{1ii} = A^{22}A_{122} + A^{33}A_{133} = 2a^{-2}a/2 + 2a^{-2}v_2^{-1}av_2/2 = 2a^{-1},\\
f_{ij} &= A^{ii}[A^{jj}\{c_1 - 2^{-1}c_2\} + 2^{-1}A_{3ij}^2A^{jj}] - \langle j, 2i\rangle B_1^{ij},\\
g_i &= A^{ii}[\langle 33, ii\rangle + 2^{-1}\langle 3, 3ii\rangle + \langle i, 33i\rangle - 2^{-1}A_{33ii}] + (2A_{33i} - 2^{-1}\langle 3, 3i\rangle)B^i,\\
g_1 &= 2[\langle 1, 331\rangle - 2^{-1}A_{3311}] + (2A_{331} - 2^{-1}\langle 3, 31\rangle)B^1\\
&= -v_2 + av_2 \cdot 2a^{-1} - (av_2/4)\cdot 2a^{-1} = v_2/2,\\
g_2 &= 2a^{-2}[\langle 22, 33\rangle + 2^{-1}\langle 3, 322\rangle + \langle 2, 332\rangle - 2^{-1}A_{3322}],
\end{aligned}$$

where

$$\langle 22, 33\rangle = (2M)^{-1}\sum_{m=0}^{M-1} a^2\nu_m^2 = 2^{-1}a^2v_2.$$

Note that $s_{m1\cdot 223} = a\nu_m \sin\alpha_m$, $s_{m1\cdot 233} = a\nu_m^2 \sin\alpha_m$, $s_{m1\cdot 333} = a\nu_m^3 \sin\alpha_m$, $s_{m2\cdot 223} = -a\nu_m \cos\alpha_m$, $s_{m2\cdot 233} = -a\nu_m^2 \cos\alpha_m$ and $s_{m1\cdot 333} = -a\nu_m^3 \cos\alpha_m$, so

$$\langle 3, 322\rangle = -(2M)^{-1}\sum_{m=0}^{M-1} (a\nu_m)^2 = -2^{-1}a^2v_2$$

and

$$\langle 2, 332\rangle = -(2M)^{-1}\sum_{m=0}^{M-1} a^2\nu_m^2 = -2^{-1}a^2v_2.$$

Note that

$$A_{2233} = \langle 22, 33\rangle + 2\langle 23, 23\rangle + 2\langle 2, 233\rangle + 2\langle 3, 223\rangle$$

and

$$\langle 23, 23\rangle = (2M)^{-1}\sum_{m=0}^{M-1} (\nu_m a)^2 = 2^{-1}a^2v_2,$$

so $A_{2233} = 2^{-1}a^2v_2(1 + 2 - 2 - 2) = -2^{-1}a^2v_2$. So, $g_2 = v_2 - 2^{-1}v_2 - v_2 + 2^{-1}v_2 = 0$. Note that

$$\begin{aligned}
g_3 &= A^{33}[\langle 33, 33\rangle + 3\langle 3, 333\rangle/2 - 2^{-1}A_{3333}] - 2^{-1}\langle 3, 33\rangle B^3,\\
A_{3333} &= 3\langle 33, 33\rangle + 4\langle 3, 333\rangle,\\
\langle 33, 33\rangle &= (2M)^{-1}\sum_{m=0}^{M-1} a^2\nu_m^4 = 2^{-1}a^2v_4,\ \langle 3, 333\rangle = -(2M)^{-1}\sum_{m=0}^{M-1} a^2\nu_m^4 = -2^{-1}a^2v_4,
\end{aligned}$$

so $g_3 = -a^{-2}v_2^{-1}[2^{-1}a^2v_4 - 2^{-1}a^2v_4] = 0$. Note that

$$f_{1j} = A^{11}[A^{jj}\{c_1 - 2^{-1}c_2\} + 2^{-1}A_{31j}^2A^{jj}] - \langle j, 21\rangle B_1^{1j}$$

at $c_1 = \langle 3, 31\rangle\langle j, j1\rangle + \langle 3, 1j\rangle\langle j, 31\rangle$ and $c_2 = \langle 3, 31\rangle A_{1jj} + 2\langle j, 31\rangle A_{31j}$. So, $f_{11} = 0$ and $f_{12} = A^{11}[A^{22}\{\langle 3, 31\rangle\langle 2, 21\rangle - 2^{-1}\langle 3, 31\rangle A_{122}\}] - \langle 2, 21\rangle B_1^{12}$. By (4.23), $B_1^{12} = 2^{-1}A_{112}A^{11}A^{22} = 0$, so

$$f_{12}/2 = 2a^{-2}\{(av_2/2)(a/2) - 2^{-1}(av_2/2)(a/2)\} = v_2/4,\ f_{13} = A^{11}[A^{33}c_3 + 2^{-1}A_{313}^2A^{33}] - 0$$

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at $c_3 = 2\langle 3,13\rangle^2 - 3.2^{-1}\langle 3,13\rangle A_{133}$. So,

$$f_{13} = A^{11}A^{33}[\langle 3,13\rangle\{2\langle 3,13\rangle - 3A_{133}/2\} + 2^{-1}A_{133}^2] = v_2,\ f_{2j} = A^{22}[A^{jj}c_4 + 0] - \langle j,22\rangle B_1^{2j}$$

at $c_4 = c_1 - c_2/2$ and $c_1 = c_2 = 0$. By (4.23), $B_1^{21} = 0$ as $B_1^{23} = A_{14}B^{234}$ is symmetric in 2, 3. So, $f_{2j} = 0$. Note that

$$f_{3j} = A^{33}[A^{jj}c_4 + 2^{-1}A_{33j}^2A^{jj}] - \langle j,23\rangle B_1^{3j},\ c_1 = 0 + \langle 3,3j\rangle\langle j,33\rangle,\ c_2 = 0 + 2\langle j,33\rangle A_{33j}.$$

So, $f_{32} = f_{33} = 0$ and $f_{31} = A^{33}A^{11}c_5$ at

$$c_5 = c_4 + 2^{-1}A_{133}^2 = \langle 3,31\rangle\langle 1,33\rangle - \langle 1,33\rangle A_{133} + 2^{-1}A_{133}^2 = a^2v_2^2/8.$$

So, $f_{31} = v_2/2$. So,

$$a_{33\cdot 2}^{13}\lambda_2^{-2}a^4v_2^2/4 = \sum_{ij} f_{ij} + \sum_i g_i = 3v_2/2.$$

So, $a_{33\cdot 2}^{13} = 6\lambda_2^2a^{-4}v_2^{-1}$. By (4.18),

$$a_{33\cdot 2}^{22}/\lambda_2^2 = \sum_i g_i + \sum_{ij} f_{ij}$$

for $g_i = (A^{33})^2\ A^{ii}\ \langle 3i,3i\rangle$ and $f_{ij} = (A^{33})^2\ A^{ii}\ A^{jj}\ \langle j,3i\rangle\langle i,3j\rangle + f_{ij}^1 + f_{ij}^2$, where $f_{ij}^1 = -2A^{33}\ B^{3ij}\ \langle j,3i\rangle$ and $f_{ij}^2 = 2B_j^{3i}\ B_i^{jj}$, giving $\sum g_i = 4a^{-4}\ v_2^{-3}\ (v_4 + 2v_2^2)$ as before. Note that $\sum_j f_{1j}^1 = -2A^{33}A^{133}\langle 3,31\rangle$, $f_{2j}^1 = -2A^{33}\cdot 0 = 0$ and $f_{3j}^1 = -2A^{33}A^{33j}\langle j,31\rangle = 0$, giving $\sum_{ij} f_{ij}^1 = -4a^{-4}v_2^{-1}$. Note that

$$\begin{aligned}
f_{1j}^2 &= (A^{33})^2A^{11}A^{jj}\langle 31,j\rangle\langle 3j,1\rangle + 2B_j^{31}B_1^{3j},\\
f_{11}^2 &= 2(B_1^{13})^2 = 0 \text{ as by (4.23) } B_1^{13} = 2^{-1}A_{113}A^{11}A^{33} = 0,\\
f_{12}^2 &= 2B_2^{31}B_1^{32} = 0 \text{ as } B_1^{23} = 0,\\
f_{13}^2 &= 2B_3^{31}B_1^{33} + (A^{33})^3 2A^{11}\langle 31,3\rangle\langle 33,1\rangle,\\
B_3^{13} &= 2^{-1}A_{133}A^{11}A^{33},\ B_1^{33} = 2^{-1}A_{133}(A^{33})^2.
\end{aligned}$$

So,

$$\begin{aligned}
f_{13}^2 &= 2^{-1}A_{133}^2A^{11}(A^{33})^3 + 2(A^{33})^3\langle 1,33\rangle\langle 3,31\rangle = 0,\\
f_{22}^2 &= (A^{33})^2(A^{22})^2\langle 32,2\rangle^2 + 2(B_2^{32})^2 = 0
\end{aligned}$$

as $B_2^{32} = 2^{-1}A_{223}^0A^{33}A^{22} = 0$. Note that

$$f_{23}^2 = (A^{33})^2A^{22}\langle 32,3\rangle\langle 33,2\rangle + 2B_3^{32}B_2^{33} = 0,\ f_{33}^2 = (A^{33})^4\langle 3,33\rangle^2 + 2(B_3^{33})^2 = 0.$$

So, $\sum f_{ij}^2 = 0$. So, $a_{33\cdot 2}^{22}/\lambda_2^2 = 4a^{-4}v_2^{-3}(v_4 + 2v_2^2)$. So, (5.2) holds: compare it with $k_1^{33}/\lambda_2 = A^{33} = 2a^{-2}v_2^{-1}$. So,

$$\operatorname{var}\widehat{w} = 2\lambda_2a^{-2}v_2^{-1}n^{-1} + 4\lambda_2^2a^{-4}(5v_2^{-1} + v_4v_2^{-3})n^{-2} + O(n^{-3})$$

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at $n = 2KM$ and $\lambda_2 = v/2$. Set SNR $= a^2/(v/KM)$ so that $SNR^{-1} = v/(a^2KM)$. So,

$$\begin{aligned} \text{var } \widehat{w} &= 2^{-1}v_2^{-1}SNR^{-1}[1 + 2^{-1}SNR^{-1}(5 + v_4/v_2^2)] + O(SNR^{-3}) \\ &= 6(1 - M^{-2})^{-1}SNR^{-1}[1 + SNR^{-1}3.4(1 + O(M^{-2}))] + O(SNR^{-3}) \end{aligned}$$

since $v_2 = 12^{-1}(1 - M^{-2})$ and $v_4 = 80^{-1} + O(M^{-2})$.

6. When linear parameters are nuisance parameters

Here we give an approach that allows us to reduce the dimension of the parameter space by eliminating nuisance parameters that enter the model linearly, when using the LSE. (For example, in Example 5.1, θ has dimension $3R$, but the parameter of interest w has dimension only R.)

The method of Section 2 does not require $\Lambda = \Lambda(\theta)$ to be the mean of random functions $\Lambda_1(\theta), \ldots, \Lambda_n(\theta)$, but only that $A_1 = 0$, $A = (A_{12})$ is bounded away from zero, $A_{1\ldots r}$ is bounded for $r \geq 1$, and

$$\Delta_{\cdot 1\ldots r} = O_p(n^{-1/2}). \tag{6.1}$$

First consider again the *real* model (4.1) $Y_N = x_N'(w)\beta + e_N$ in $\mathcal{C}$ with β in $\mathcal{R}^q$, w in $\mathcal{R}$, that is $Y = X(w)\beta + e$ in $\mathcal{R}^n$, where $X(w)' = (x_1 \cdots x_n) = X$ and $x_N = x_N(w)$. That is, we now suppose that $\theta = \binom{\beta}{w}$ and $S_N(\theta) = x_N(w)'\beta$. So, $S(\theta) = S = X(w)\beta$. Set

$$\Lambda(w) = \inf_\beta n^{-1} \sum_{N=1}^{n} (Y_N - X_N'\beta)^2 = n^{-1}|Y - X\widehat{\beta}(w)|^2 = n^{-1}Y'QY,$$

where $I - Q = I - Q(w) = P = X(X'X)^{-1}X'$ and $\widehat{\beta}(w) = (X'X)^{-1}X'Y$, assuming $X'X$ has full rank $q \leq n$. So, the LSE is

$$\widehat{\theta} = \begin{pmatrix} \widehat{\beta}(\widehat{w}) \\ \widehat{w} \end{pmatrix},$$

where $\widehat{w}$ minimises $\Lambda(w)$. Theorem 6.1 verifies that the conditions (6.1) hold with $(\theta, \Lambda(\theta))$ replaced by $(w, \Lambda(w))$.

Theorem 6.1. *Consider the model (4.1). Assume that $X'X$ has full rank $q \leq n$ and that x_N satisfies*

$$x_N(w) = y(N/n, w) = y(N/n) \tag{6.2}$$

for some function $y(t, w)$. Also assume that $e_1, \ldots, e_n$ are i.i.d. in $\mathcal{R}$ with mean zero and cumulants $\{\lambda_r\}$. Finally, assume that

$$\det(T(1:2)) \neq 0, \tag{6.3}$$

where $T(\pi_1 : \pi_2) = T(\pi_1, \pi_2) - T(\cdot, \pi_1)'T^{-1}T(\cdot, \pi_2)$, where $T(\pi_1, \pi_2) = \int_0^1 y_{\cdot\pi_1}(t)y_{\cdot\pi_2}(t)'\mathrm{d}t$, and $\cdot\pi$ denotes differentiation with respect to w, not t. Then $A_1 = 0$, $A = (A_{12})$ is bounded away from zero, $A_{1\ldots r}$ is bounded for $r \geq 1$, and (6.1) holds.

We now consider how to adapt Section 3. Since $\Lambda(w)$ is no longer a sum of independent random variables, (3.1) with (3.2) no longer applies. So, let us take (3.1) as the *definition* of $[\cdot]$. To be of use we need $[\cdot]$ to be bounded as $n \to \infty$. Note that $[\pi] = A_\pi$ is bounded. For $r > 1$,

$$[\pi_1, \ldots, \pi_r] = n^{r-1}\kappa[\Delta_{\pi_1}, \ldots, \Delta_{\pi_r}], \ [\pi_1, \pi_2] = n^{-1}\kappa[e'Q_{\cdot\pi_1}S + e'Q_{\cdot\pi_1}e, \ e'Q_{\cdot\pi_2}S + e'Q_{\cdot\pi_2}e].$$

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Clearly $\{\Delta_{\pi_1}\}$ has rth-order cumulants $O(n^{1-r})$:

$$\kappa_r(\Delta_{\pi_1 1}, \ldots, \Delta_{\pi_r r}) = n^{1-r}[\pi_1, \ldots, \pi_r]_1,$$

where

$$\begin{aligned} [\pi_1, \ldots, \pi_r]_1 &= \lambda_r \langle \gamma_{\pi_1}, \ldots, \gamma_{\pi_r} \rangle_1, \\ \langle \gamma_{\pi_1}, \ldots, \gamma_{\pi_r} \rangle_1 &= n^{-1} \sum_{N=1}^{n} \gamma_{\pi_1 N} \ldots \gamma_{\pi_r N}, \ \gamma_\pi = (\gamma_{\pi N}) = Q_{\cdot\pi} S. \end{aligned}$$

By (D.12),

$$n^2 E \Delta_{\pi_1 1} \Delta_{\pi_2 2} = \lambda_4 \sum_{N=1}^{n} Q_{NN\cdot\pi_1} Q_{NN\cdot\pi_2} + 2\lambda_2^2 \mathrm{tr} Q_{\cdot\pi_1} Q_{\cdot\pi_2} = O(1).$$

Similarly, one expects the moments and cumulants of $\{n\Delta_{\pi_2}\}$ to be $O(1)$, and so the rth-order cross-cumulants of $\{\Delta_{\pi_2}\}$ to be $O(n^{-r})$. So, one expects the rth-order cross-cumulants of $\{\Delta_{\pi_1 1}, \Delta_{\pi_2 2}\}$ to be $O(n^{1-r})$ or $O(n^{-r})$ and so the $[\cdot]$ functions of (3.2) to be bounded. So, the results of Section 3 should hold with $[\cdot]$ defined by (3.1), not (3.2).

The *complex* case is similar: observe

$$Y_N = x_N^* \beta + e_N \text{ in } \mathcal{C} \text{ with } \beta \text{ in } \mathcal{C}^q, \ x_N = x_N(w),$$

that is $Y = X\beta + e$ in $\mathcal{C}^n$, where $X^* = (x_1 \cdots x_n)$. Assume that $e_N = e_{N1} + j e_{N2}$ for $j = \sqrt{-1}$ with $\{e_{Nk}\}$ i.i.d. with mean zero and rth cumulants $\lambda_r/2$. So, $\lambda_2 = E|e_N|^2$. Set

$$\Lambda(w) = \inf_\beta n^{-1} \sum_{N=1}^{n} |Y_N - x_N^* \beta|^2 = n^{-1} |Y - X\widehat{\beta}(w)|^2 = n^{-1} Y^* Q Y,$$

where $Q = Q(w) = I - P$, $P = X(X^*X)^{-1}X^*$ and $\widehat{\beta}(w) = (X^*X)^{-1}X^*Y$, assuming X^*X has full rank $q \leq n$. So, the LSE is

$$\widehat{\theta} = \begin{pmatrix} \widehat{\beta}(\widehat{w}) \\ \widehat{w} \end{pmatrix},$$

where $\widehat{w}$ minimises $\Lambda(w)$. Again we assume (6.2) and (6.3) hold with $'$ replacing * in (D.9)–(D.11) so that $A_1 = 0$, $A_{1\ldots r}$ is bounded, $A = (A_{12})$ is bounded away from zero. Note that $A_{1\ldots r}$ is given by (D.6)–(D.9) and their extensions, with $'$ replacing *. Also

$$\Delta_\pi = 2\Delta_{1\pi} + \Delta_{2\pi} \tag{6.4}$$

for $2n\Delta_{1\pi} = 2\mathrm{Re}\, e^*\gamma_\pi = e^*\gamma_\pi + \gamma_\pi^* e$, $\gamma_\pi = Q_{\cdot\pi} S$ and $n\Delta_{2\pi} = e^* Q_{\cdot\pi} e$, where again $S = S(\theta) = X\beta$. The argument that Δ_π is $O_p(n^{-1/2})$ carries over. If w lies in $\mathcal{R}^q$ then A_π and Δ_π are still real.

Example 6.1. Observe $y_k = S + \varepsilon_k$ in $\mathcal{C}^n$ for $k = 1, \ldots, K$ with $\varepsilon_k \sim \mathcal{CN}(0, VI_n)$ for some scalar $V > 0$. So, $\{y_k\}$ have arithmetic mean $Y = S + e$ with $e \sim \mathcal{CN}(0, \nu I_n)$ and $\nu = K^{-1}V$. (So, $\lambda_2 = \nu/2$ and $\lambda_r = 0$ for $r \neq 2$.) Suppose $Y = (Y_0, \ldots, Y_{n-1})'$ and $S = (S_0, \ldots, S_{n-1})'$, where $S_N = x_N\beta$ for β in $\mathcal{C}$, $x_N = \exp(j\gamma_N)$, $j = \sqrt{-1}$, $\gamma_N = \nu_N w$, $\nu_N = (N + N_0)/n$ and $N_0 = -(n-1)/2$. So, $q = 1$, $X^*X = n$, $I - Q = P = n^{-1}XX^*$ has (a, b) element

$$n^{-1} x_a^* x_b = n^{-1} \exp\{j(\gamma_b - \gamma_a)\} = n^{-1} \exp\{jw(b-a)/n\}$$

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and $\Lambda(w) = n^{-1}|Y|^2 - n^{-2}|X^*Y|^2$ and $X^*Y = \sum_{N=0}^{n-1} x_N Y_N$. Since $q = 1$ we now replace $1 \dots r$, where it occurs by r and use the convention $f(w)_{\cdot r} = (\partial/\partial w)^r f(w)$ for any smooth function $f(w)$. For example, $x_{N\cdot r} = (\partial/\partial w)^r x_N = (j\nu_N)^r x_N$. So,

$$X_{\cdot r} = (x_{0\cdot r}, \dots, x_{n-1\cdot r})^* = (-j)^r(\nu_0^r x_0, \dots, \nu_{n-1}^r x_{n-1})^*,\ X_{\cdot r}^* X_{\cdot s} = (-1)^s j^{r+s} n v_{r+s}$$

for $v_r = n^{-1}\sum_{N=0}^{n-1} \nu_N^r$. So,

$$\begin{aligned}
v_1 &= 0,\ v_2 = (1-n^{-2})/12,\ v_4 = (1-n^{-2})(1-7n^{-2}/3)/80,\\
v_r &= 2^{-r}/(r+1) + O(n^{-2}),\\
X_r^* X/n &= j^r v_r,\ X_{\cdot 1}^* X = 0,\ X_{\cdot 1}^* X_{\cdot 1}/n = v_2,\\
Q_{\cdot 1} &= -P_{\cdot 1} = -n^{-1}(X_{\cdot 1}X^* + XX_{\cdot 1}^*),\ Q_{\cdot 2} = -n^{-1}(X_{\cdot 2}X^* + 2X_{\cdot 1}X_{\cdot 1}^* + XX_{\cdot 2}^*),\\
Q_{\cdot 3} &= -n^{-1}(X_{\cdot 3}X^* + 3X_{\cdot 2}X_{\cdot 1}^* + 3X_{\cdot 1}X_{\cdot 2}^* + XX_{\cdot 3}^*),\\
Q_{\cdot 4} &= -n^{-1}(X_{\cdot 4}X^* + 4X_{\cdot 3}X_{\cdot 1}^* + 6X_{\cdot 2}X_{\cdot 2}^* + 4X_{\cdot 1}X_{\cdot 3}^* + XX_{\cdot 4}^*),\\
Q_{\cdot 1}X &= -X_1,\ Q_{\cdot 2}X = -(X_{\cdot 2} - Xv_2),\ Q_{\cdot 3}X = -(X_{\cdot 3} - 3X_{\cdot 1}v_2 - Xjv_3),\\
Q_{\cdot 4}X &= -(X_{\cdot 4} - 6X_{\cdot 2}v_2 - 4X_{\cdot 1}jv_3 + Xv_4),\\
X^*Q_{\cdot 2}X/n &= 2v_2,\ X^*Q_{\cdot 3}X/n = 0,\ X^*Q_{\cdot 4}X/n = -2(v_4 + 3v_2^2).
\end{aligned}$$

By (D.6), $A_r = |\beta|^2 X^* Q_{\cdot r} X/n$, so $A_2 = 2|\beta|^2 v_2$, $A_3 = 0$ and $A_4 = -2|\beta|^2(v_4 + 3v_2^2)$. Also var $\widehat{w} = k_{21}n^{-1} + k_{22}n^{-2} + \dots$, where by (3.5), (3.6)–(3.11), replacing $[1,111]$ by $[1,3]$ and so on,

$$\begin{aligned}
k_{21} &= A_2^{-2}[1,1],\ k_{22} = 2a^{12} + 2a^{13} + a^{22},\\
a^{12} &= -A_2^{-3}[1,1,2],\ a^{22} = A_2^{-4}([1,1]\,[2,2] + [1,2]^2),\\
a^{13} &= A_2^{-4}(2[1,2]^2 + [1,1]\,[2,2] + 3[1,1]\,[1,3]/2) - A_2^{-5}A_4[1,1]^2/2,
\end{aligned}$$

giving

$$k_{22} = -2A_2^{-3}[1,1,2] + A_2^{-4}(5[1,2]^2 + 3[1,1]\,[2,2] + 3[1,1]\,[1,3]) - A_2^{-5}A_4[1,1]^2.$$

By (6.4), $n\Delta_r = e^*\gamma_r + \gamma_r^* e + e^* Q_{\cdot r} e$ for $\gamma_r = Q_{\cdot r}S$. By Appendix B,

$$\begin{aligned}
En^2\Delta_r\Delta_s &= E(\sum_{rs}^{2} \gamma_r^* e e^* \gamma_s + e^* Q_{\cdot r} e e^* Q_{\cdot s} e) = \nu \sum_{rs}^{2} \gamma_r^*\gamma_s + \nu^2(\text{tr } Q_{\cdot r}\text{tr } Q_{\cdot s} + \text{tr } Q_{\cdot r}Q_{\cdot s})\\
&= n\nu|\beta|^2(d_{rs} + d_{rs}^*) + \nu^2 D_{rs},
\end{aligned}$$

where $d_{rs} = (Q_{\cdot r}X)^*(Q_{\cdot s}X)/n$ and $D_{rs} = \text{tr } Q_{\cdot r}Q_{\cdot s}$ are $O(1)$. In particular, $d_{11} = v_2$, $d_{12} = X_{\cdot 1}^*(X_{\cdot 2} - Xv_2)/n = -jv_3$, $d_{13} = -v_4 - 3v_2^2$, $d_{22} = v_4 + 3v_2^2$, $D_{11} = 2v_2$, $D_{12} = 0$, $D_{13} = 2(v_4 - 3v_2^2)$ and $D_{22} = 2(v_4 + v_2^2)$. So,

$$\begin{aligned}
[1,1] &= nE\Delta_1^2 = 2\nu|\beta|^2 v_2(1+\varepsilon) \text{ for } \varepsilon = \nu n^{-1}|\beta|^{-2},\\
[1,2] &= nE\Delta_1\Delta_2 = 0,\ [1,3] = nE\Delta_1\Delta_3 = -2\nu|\beta|^2\{v_4 + 3v_2^2 - \varepsilon(v_4 - 3v_2^2)\},\\
[2,2] &= 2\nu|\beta|^2\{v_4 + 3v_2^2 + \varepsilon(v_4 + v_2^2)\}.
\end{aligned}$$

Also for $q_r = e^* Q_{\cdot r} e$ and $T_r = \text{tr } Q_{\cdot r} = 0$, by Appendix B

$$\begin{aligned}
n\,[1,1,2] &= n^3 E\Delta_1^2\Delta_2 = E(2e^*\gamma_1\gamma_1^* e + q_1^2)q_2\\
&= 2\nu^2(\gamma_1^*\gamma_1 T_2 + \gamma_1^* Q_{\cdot 2}\gamma_1) + \nu^3(T_1^2 T_2 + 2T_1\text{tr } Q_{\cdot 1}Q_{\cdot 2}T_2\text{tr } Q_{\cdot 1}^2 + 2\text{tr } Q_{\cdot 1}^2 Q_{\cdot 2})\\
&= 2n\nu^2|\beta|^2 d_{121} + 2\nu^3 D_{112},
\end{aligned}$$

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where $d_{121} = (Q_{\cdot 1}X)^* Q_{\cdot 2}(Q_{\cdot 1}X)/n = -2v_2^2$ and $D_{112} = \text{tr}\ Q_{\cdot 1}^3 Q_{\cdot 2} = 0$. So, var $\widehat{w} = 2^{-1}v_2^{-1}\varepsilon + c\varepsilon^2 + O(\varepsilon^3)$, where

$$\begin{aligned} c &= 2^{-1}v_2^{-1}\ (\text{from } k_{21})\ + v_2^{-1}\ (\text{from } 2a^{13}) + 2^{-2}v_2^{-3}(v_4 + 3v_2^2)\ (\text{from } 2a^{13} + a^{22}) \\ &= 6 \times 27/5 + O(n^{-2}). \end{aligned}$$

So, var $\widehat{w} = 6\varepsilon(1 + 5.4\varepsilon) + O(\varepsilon^3)$.

Appendix A

The following is the list of distinct summation notations used in the paper:

$$\sum_{ij}^{2} f_{ij} = f_{ij} + f_{ji},\ \sum_{ijk}^{3} f_{ijk} = f_{ijk} + f_{kij} + f_{jki},\ \sum^{3} f_{ij}g_{kl} = f_{ij}g_{kl} + f_{ik}g_{jl} + f_{il}g_{jk},$$

$$\sum_{ijkl}^{4} f_{ijkl} = f_{ijkl} + f_{lijk} + f_{klij} + f_{jkli},\ \sum_{ijklm}^{5} f_{ijklm} = f_{ijklm} + f_{mijkl} + f_{lmijk} + f_{klmij} + f_{jklmi},$$

$$\sum_{ijk}^{6} f_{ijk} = f_{ijk} + f_{ikj} + f_{jik} + f_{jki} + f_{kij} + f_{kji},$$

$$\sum_{iijj}^{6} f_{iijj} = f_{iijj} + f_{ijij} + f_{ijji} + f_{jiij} + f_{jiji} + f_{jjii},$$

$$\sum^{6} f_{ij}g_{kl} = f_{ij}g_{kl} + f_{ik}g_{jl} + f_{il}g_{jk} + f_{jk}g_{il} + f_{jl}g_{ik} + f_{kl}g_{ij},$$

$$\sum^{6} g_{lm}f_{ijk} = g_{il}f_{jkm} + g_{im}f_{jkl} + g_{jl}f_{ikm} + g_{jm}f_{ikl} + g_{kl}f_{ijm} + g_{km}f_{ijl},$$

$$\begin{aligned} \sum^{10} f_{ij}g_{klm} &= f_{ij}g_{klm} + f_{ik}g_{jlm} + f_{il}g_{jkm} + f_{im}g_{jkl} + f_{jk}g_{ilm} + f_{jl}g_{ikm} + f_{jm}g_{ikl} + f_{kl}g_{ijm} \\ &\quad + f_{km}g_{ijl} + f_{lm}g_{ijk}, \end{aligned}$$

$$\begin{aligned} \sum^{10} f_{ijk}g_{lmn} &= f_{ijk}g_{lmn} + f_{ijl}g_{kmn} + f_{ijm}g_{kln} + f_{ijn}g_{klm} + f_{ikl}g_{jmn} + f_{ikm}g_{jln} + f_{ikn}g_{jlm} \\ &\quad + f_{ilm}g_{jkn} + f_{iln}g_{jkm} + f_{imn}g_{jkl}, \end{aligned}$$

$$\begin{aligned} \sum^{12} f_{il}g_{jkmn} &= f_{il}g_{jkmn} + f_{im}g_{jkln} + f_{in}g_{jklm} + f_{jl}g_{ikmn} + f_{jm}g_{ikln} + f_{jn}g_{iklm} + f_{kl}g_{ijmn} \\ &\quad + f_{km}g_{ijln} + f_{kn}g_{ijlm} + f_{lm}g_{ijkn} + f_{ln}g_{ijkm} + f_{mn}g_{ijkl}, \end{aligned}$$

$$\begin{aligned} \sum^{15} f_{ij}g_{kl}h_{mn} &= f_{ij}g_{kl}h_{mn} + f_{ik}g_{jl}h_{mn} + f_{il}g_{jk}h_{mn} + f_{im}g_{jk}h_{ln} + f_{in}g_{jk}h_{lm} + f_{jk}g_{il}h_{mn} \\ &\quad + f_{jl}g_{ik}h_{mn} + f_{jm}g_{ik}h_{ln} + f_{jn}g_{ik}h_{lm} + f_{kl}g_{ij}h_{mn} + f_{km}g_{ij}h_{ln} + f_{kn}g_{ij}h_{lm} \\ &\quad + f_{lm}g_{ij}h_{kn} + f_{ln}g_{ij}h_{km} + f_{mn}g_{ij}h_{kl}, \end{aligned}$$

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$$\sum_{\pi_1,\pi_3}^{2} [\pi_1,\pi_2]\,[\pi_3,\pi_4] = [\pi_1,\pi_2]\,[\pi_3,\pi_4] + [\pi_3,\pi_2]\,[\pi_1,\pi_4]\,,$$

$$\sum_{\pi_2,\pi_3}^{2} [\pi_1,\pi_2]\,[\pi_3,\pi_4] = [\pi_1,\pi_2]\,[\pi_3,\pi_4] + [\pi_1,\pi_3]\,[\pi_2,\pi_4]\,,$$

$$\sum^{3} [\pi_1,\pi_2]\,[\pi_3,\pi_4] = [\pi_1,\pi_2]\,[\pi_3,\pi_4] + [\pi_1,\pi_3]\,[\pi_2,\pi_4] + [\pi_1,\pi_4]\,[\pi_2,\pi_3]\,,$$

$$\begin{aligned}\sum^{6} [\pi_4,\pi_5]\,[\pi_1,\pi_2,\pi_3] \;=\;& [\pi_1,\pi_4]\,[\pi_2,\pi_3,\pi_5] + [\pi_1,\pi_5]\,[\pi_2,\pi_3,\pi_4] + [\pi_2,\pi_4]\,[\pi_1,\pi_3,\pi_5] \\ &+ [\pi_2,\pi_5]\,[\pi_1,\pi_3,\pi_4] + [\pi_3,\pi_4]\,[\pi_1,\pi_2,\pi_5] + [\pi_3,\pi_5]\,[\pi_1,\pi_2,\pi_4]\end{aligned}$$

and

$$\begin{aligned}\sum^{15} [\pi_1,\pi_2]\,[\pi_3,\pi_4]\,[\pi_5,\pi_6] \;=\;& [\pi_1,\pi_2]\,[\pi_3,\pi_4]\,[\pi_5,\pi_6] + [\pi_1,\pi_3]\,[\pi_2,\pi_4]\,[\pi_5,\pi_6] \\ &+ [\pi_1,\pi_4]\,[\pi_2,\pi_3]\,[\pi_5,\pi_6] + [\pi_1,\pi_5]\,[\pi_2,\pi_3]\,[\pi_4,\pi_6] \\ &+ [\pi_1,\pi_6]\,[\pi_2,\pi_3]\,[\pi_4,\pi_5] + [\pi_2,\pi_3]\,[\pi_1,\pi_4]\,[\pi_5,\pi_6] \\ &+ [\pi_2,\pi_4]\,[\pi_1,\pi_3]\,[\pi_5,\pi_6] + [\pi_2,\pi_5]\,[\pi_1,\pi_3]\,[\pi_4,\pi_6] \\ &+ [\pi_2,\pi_6]\,[\pi_1,\pi_3]\,[\pi_4,\pi_5] + [\pi_3,\pi_4]\,[\pi_1,\pi_2]\,[\pi_5,\pi_6] \\ &+ [\pi_3,\pi_5]\,[\pi_1,\pi_2]\,[\pi_4,\pi_6] + [\pi_3,\pi_6]\,[\pi_1,\pi_2]\,[\pi_4,\pi_5] \\ &+ [\pi_4,\pi_5]\,[\pi_1,\pi_2]\,[\pi_3,\pi_6] + [\pi_4,\pi_6]\,[\pi_1,\pi_2]\,[\pi_3,\pi_5] \\ &+ [\pi_5,\pi_6]\,[\pi_1,\pi_2]\,[\pi_3,\pi_4]\,.\end{aligned}$$

Appendix B

Theorem B.1. *Suppose $X \sim \mathcal{CN}_p(0,V)$ so $V = EXX^*$. Its non-zero moments*

$$\mu_{1\ldots n,n+1\ldots 2n} = EX_1 \ldots X_n \bar{X}_{n+1} \ldots \bar{X}_{2n}$$

are $\mu_{1,2} = V_{12}$, $\mu_{12,34} = V_{13}V_{24} + V_{14}V_{23}$, and so on, satisfying the recurrence relation

$$\mu_{1\ldots n,n+1\ldots 2n} = \sum_{i=1}^{n} \mu_{1,n+i}\mu_{2\ldots n,(n+1\ldots 2n)_i}, \tag{B.1}$$

where $(n+1\ldots 2n)_i$ drops the ith term. Similarly, for $i_1,\ldots,i_n,j_1,\ldots,j_n$ in $1,\ldots,p$,

$$E\; X_{i_1} \ldots X_{i_n} \bar{X}_{j_1} \ldots \bar{X}_{j_n} = \sum^{n!} V_{i_1 k_1} \ldots V_{i_n k_n}, \tag{B.2}$$

where $\sum^{n!}$ sums over all $n!$ permutations $k_1 \ldots k_n$ of $j_1 \ldots j_n$.

So, in expanded form, the right hand side of (B.1) has $n!$ terms, as compared with $1 \cdot 3 \ldots (2n-1) = (2n)!2^{-n}/n! = E\mathcal{N}(0,1)^{2n}$ terms for the real case $X \sim \mathcal{N}_p(0,V)$. Reed [6] proved (B.2) for the stationary case V_{ij} a function of $i-j$, but his proof holds without this assumption.

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Theorem B.2. *Let $q_r = X^* a_r X$ for a_r in $\mathcal{C}^{p\times p}$. Set $B_r = a_r V$ and $t_{1\dots r} = \mathrm{trace}(B_1 \cdots B_r)$. Then*

$$\begin{aligned} Eq_1 &= t_1,\ Eq_1q_2 = t_1t_2 + t_{12},\ Eq_1q_2q_3 = t_1t_2t_3 + \sum_{123}^{3} t_1t_{23} + \sum_{23}^{2} t_{123}, \\ Eq_1 \dots q_4 &= t_1t_2t_3t_4 + \sum^{6} t_1t_2t_{34} + \sum^{4} t_1 \sum_{34}^{2} t_{234} + \sum^{3} t_{12}t_{34} + \sum_{234}^{6} t_{1234}, \end{aligned}$$

and in general

$$Eq_1 \dots q_r = \sum_{j=1}^{r} T_{rj} \text{ for } T_{rj} = \sum_{\pi} \{t_{\pi_1} \dots t_{\pi_j} \ :\ |\pi_1| + \dots + |\pi_j| = r\},$$

where for $\pi = (i_1 \dots i_r)$, $t_\pi = \mathrm{trace}(B_{i_1} \dots B_{i_r})$, and $\sum_\pi$ sums over all such $(\pi_1, \dots, \pi_j)$ giving different terms.

Appendix C

Theorem C.1 gives a closed expression for

$$g_{Mr}(\delta) = M^{-1} \sum_{m=0}^{M-1} \nu_m^r \exp(j\nu_m\delta),$$

where $\nu_m = m/M - (M-1)/2$.

Theorem C.1. *Let $a = (M+1)/(2M)$, $b = (1-M)/(2M)$, $c = 1/M$, $N(x) = \exp(ax) - \exp(bx)$, $D(x) = \exp(cx) - 1$, $I(x) = 1/D(x)$ and $G(x) = \sum_{m=0}^{M-1} \exp(x\nu_m) = N(x)I(x)$. We have*

$$g_{Mr}(\delta) = M^{-1} G^{(r)}(j\delta), \tag{C.1}$$

where

$$G^{(r)}(x) = \sum_{k=0}^{r} \binom{r}{k} N_{r-k} c^k P_k,\ N_i = a^i \exp(ax) - b^i \exp(bx)$$

and

$$P_k = \sum_{i=0}^{k} (-1)^i i! D^{-i-1} \exp(cix) C_{ki},\ C_{ki} = \sum_{n=0}^{i} \binom{i}{n} (-1)^{i-n} n^k.$$

In particular,

$$\begin{aligned} P_0 &= D^{-1},\ P_1 = D^{-1} - D^{-2}\exp(cx),\ P_2 = D^{-1} - D^{-2}\exp(cx) + 4D^{-3}\exp(2cx), \\ P_3 &= D^{-1} - D^{-2}\exp(cx) + 12D^{-3}\exp(2cx) - 36D^{-4}\exp(3cx), \\ P_4 &= D^{-1} - D^{-2}\exp(cx) + 28D^{-3}\exp(2cx) - 216D^{-4}\exp(3cx) + 576D^{-5}\exp(4cx), \end{aligned}$$

and so on. Also

$$g_{Mr}(0) = M^{-1} \sum_{m=0}^{M-1} \nu_m^r = g_r$$

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say, is given by $g_r = 0$ for r odd, and for $d = M^{-2}$, by

$$\begin{aligned} g_0 &= 1,\ g_2 = (1-d)/12,\ g_4 = (3-10d+7d^2)/240,\ g_6 = (3-21d+49d^2-31d^3)/1344, \\ g_8 &= (5-60d+294d^2-620d^3+381d^4)/11520, \\ g_{10} &= (3-55d+462d^2-2046d^3+4191d^4-2555d^5)/33792, \end{aligned}$$

and so on.

Appendix D

Proof of Theorem 2.1. For $\{\Lambda_N\}$ sufficiently smooth, $\widehat{\theta}$ satisfies

$$0 = \Lambda_{\cdot 1}(\widehat{\theta}) = \Lambda_{\cdot 1} + \Lambda_{\cdot 12}\delta_2/1! + \Lambda_{\cdot 123}\delta_2\delta_3/2! + \dots$$

by the multivariate Taylor expansion, using the tensor summations convention: repeated pairs of suffixes are implicitly summed over their range $1 \dots p$. For example, $\Lambda_{\cdot 12}\delta_2 = \sum_{i_2=1}^{p} \Lambda_{\cdot 12}\delta_2$. So,

$$0 = \Lambda_{\cdot 1} + (A_{12} + \Delta_{12})\delta_2 + (A_{123} + \Delta_{123})\delta_2\delta_3/2 + \dots$$

Multiply by A^{01}. Since $A^{01}A_{12} = 1$ or 0 for $i_0 = i_2$ or $i_0 \neq i_2$, this gives

$$0 = \delta_0 + A^{01}\{\Lambda_{\cdot 1} + \Delta_{12}\delta_2 + (A_{123} + \Delta_{123})\delta_2\delta_3/2 + \dots\}, \tag{D.1}$$

where the first two terms have magnitude $O_p(n^{-1/2})$, the next two are $O_p(n^{-1})$ and so on. So, an expansion of the form (2.1) is possible. Substituting (2.1) into (D.1) gives $0 = \sum_{a=1}^{\infty} T_a$, where $T_a = O_p(n^{-a/2})$:

$$\begin{aligned} T_1 &= \delta_0^1 + A^{01}\Delta_1,\ T_2 = \delta_0^2 + A^{01}(\Delta_{12}\delta_2^1 + A_{123}\delta_2^1\delta_3^1/2), \\ T_3 &= \delta_0^3 + A^{01}(\Delta_{12}\delta_2^2 + A_{123}\sum_{23}^{2}\delta_2^1\delta_3^2 + \Delta_{123}\delta_2^1\delta_3^1/2 + A_{1\dots4}\delta_2^1\delta_3^1\delta_4^1/6), \\ T_4 &= \delta_0^4 + A^{01}\{\Delta_{12}\delta_2^3 + A_{123}(\delta_2^2\delta_3^2 + \sum_{23}^{2}\delta_2^1\delta_3^3)/2 + \Delta_{123}\sum_{23}^{2}\delta_2^1\delta_3^2/2 \\ &\quad + A_{1\dots4}\sum_{234}^{3}\delta_2^1\delta_3^1\delta_4^2/6 + \Delta_{1\dots4}\delta_2^1\delta_3^1\delta_4^1/6 + A_{1\dots5}\delta_2^1\delta_3^1\delta_4^1\delta_5^1/24)\}, \end{aligned}$$

and so on. We can now obtain $\{\delta_0^a\}$ in (2.6)–(2.8) by setting $T_a \equiv 0$ and solving these recurrence relations for δ_0^a. □

Proof of Theorem 3.1. Note that $a_{12}^{11} = A^{13}A^{24}\kappa(\Delta_3, \Delta_4)$, so (3.7) follows. Note that $a_0^1 = 0$, so $a_{0.1}^1 = 0$, $a_0^2 = A^{01}A^{23}E\Delta_{12}\Delta_3 - B^{045}E\Delta_4\Delta_5$, so (3.8) follows. Note that $a_{123}^{111} = -A^{14}A^{25}A^{36}\kappa(\Delta_4, \Delta_5, \Delta_6)$, so (3.9) follows. Note that

$$a_{123}^{112} = A^{14}A^{25}\left\{A^{36}A^{78}\sum_{45}^{2}\kappa(\Delta_4, \Delta_{67})\kappa(\Delta_5, \Delta_8) - B^{367}\sum_{45}^{2}\kappa(\Delta_4, \Delta_6)\kappa(\Delta_5, \Delta_7)\right\} + O\left(n^{-3}\right)$$

since if

$$EX_i \equiv 0 \text{ and } \mu_{j_1\dots j_r} \equiv EX_{j_1}\dots X_{j_r}, \tag{D.2}$$

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then

$$\kappa(X_1, X_2, X_3X_4) = \kappa(X_1, X_2, X_3, X_4) + \sum_{12}^{2} \mu_{13}\mu_{24}.$$

So, (3.10) follows. So, $k_{\cdot 2}^{123}$ is given by (3.4), (3.9) and (3.10). We next give $k_{\cdot 2}^{12}$ of (3.5). Recall that $a_{12\cdot k}^{11} = 0$ for $k \neq 1$. Note that

$$a_{12}^{12} = -A^{13}\kappa(\Delta_3, A^{24}A^{56}\Delta_{45}\Delta_6 - B^{245}\Delta_4\Delta_5),$$

so (3.11) follows since (D.2) implies $\kappa(X_1, X_2X_3) = \mu_{123}$. Also

$$\begin{aligned} a_{12}^{13} &= A^{13}A^{24}\kappa(\Delta_3, \Delta_{45}(A^{56}A^{78}\Delta_{67}\Delta_8 - B^{567}\Delta_6\Delta_7) \\ &\quad -2^{-1}A_{456}\sum_{56}^{2} A^{57}\Delta_7(A^{68}A^{9,10}\Delta_{89}\Delta_{10} - B^{689}\Delta_8\Delta_9) \\ &\quad +2^{-1}A^{57}A^{68}\Delta_{456}\Delta_7\Delta_8 - C_4^{567}\Delta_5\Delta_6\Delta_7)) \\ &= A^{13}A^{24}\left\{A^{56}A^{78}\kappa(\Delta_3, \Delta_{45}\Delta_{67}\Delta_8) - B^{567}\kappa(\Delta_3, \Delta_{45}\Delta_6\Delta_7)\right. \\ &\quad -2^{-1}A_{456}\sum_{56}^{2} A^{57}(A^{68}A^{9,10}\kappa(\Delta_3, \Delta_7\Delta_{89}\Delta_{10}) - B^{689}\kappa(\Delta_3, \Delta_7\Delta_8\Delta_9)) \\ &\quad \left.+2^{-1}A^{57}A^{68}\kappa(\Delta_3, \Delta_{456}\Delta_7\Delta_8) - C_4^{567}\kappa(\Delta_3, \Delta_5\Delta_6\Delta_7)\right\}. \end{aligned}$$

Also (D.2) implies

$$\kappa(X_1, X_2X_3X_4) = \mu_{1\ldots4} = \kappa(X_1, \ldots, X_4) + \sum^{3} \mu_{12}\mu_{34}.$$

So, (3.12) follows. Also

$$\begin{aligned} a_{12}^{22} &= \kappa(A^{13}A^{45}\Delta_{34}\Delta_5 - B^{134}\Delta_3\Delta_4,\ A^{26}A^{78}\Delta_{67}\Delta_8 - B^{267}\Delta_6\Delta_7) \\ &= A^{13}A^{45}A^{26}A^{78}\kappa(\Delta_{34}\Delta_5, \Delta_{67}\Delta_8) \\ &\quad -\sum_{12}^{2} A^{13}A^{45}B^{267}\kappa(\Delta_{34}\Delta_5, \Delta_6\Delta_7) + B^{134}B^{267}\kappa(\Delta_3\Delta_4, \Delta_6\Delta_7). \end{aligned}$$

Also (D.2) implies

$$\kappa(X_1X_2, X_3X_4) = \mu_{1\ldots4} - \mu_{12}\mu_{34} = \kappa(X_1, \ldots, X_4) + \sum_{12}^{2} \mu_{13}\mu_{24}.$$

So, (3.13) follows. So, $k_{\cdot 2}^{12}$ is given by (3.5), (3.11), (3.12), (3.13). Finally, we give $k_{\cdot 3}^{1234}$ starting from (3.6):

$$\begin{aligned} a_{1234}^{1111} &= A^{15}A^{26}A^{37}A^{48}\kappa(\Delta_5, \Delta_6, \Delta_7, \Delta_8), \\ a_{1234}^{1112} &= -A^{15}A^{26}A^{37}\kappa(\Delta_5, \Delta_6, \Delta_7, A^{48}A^{9,10}\Delta_{89}\Delta_{10} - B^{489}\Delta_8\Delta_9), \end{aligned}$$

so (3.14) follows. Also (D.2) implies

$$\kappa(X_1, X_2, X_3, X_4X_5) = \kappa(X_1, \ldots, X_5) + \sum^{6} \mu_{45}\mu_{123}.$$

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So, (3.15) follows and

$$\begin{aligned} a_{1234}^{1113} &= A^{15}A^{26}A^{37}A^{48}\kappa(\Delta_5,\Delta_6,\Delta_7,\Delta_{89}(A^{9,10}A^{11,12}\Delta_{10,11}\Delta_{12}-B^{9,10,11}\Delta_{10}\Delta_{11}) \\ &\quad -2^{-1}A_{89,10}\sum_{9,10}^{2}A^{9,11}\Delta_{11}(A^{10,12}A^{13,14}\Delta_{12,13}\Delta_{14}-B^{10,12,13}\Delta_{12}\Delta_{13}) \\ &\quad +2^{-1}A^{9,11}A^{10,12}\Delta_{89,10}\Delta_{11}\Delta_{12}-C_8^{9,10,11}\Delta_9\Delta_{10}\Delta_{11}). \end{aligned}$$

Also setting $k_{1\ldots r}=\kappa(X_1,\ldots,X_r)$, (D.2) implies

$$\begin{aligned} \kappa(X_1,X_2,X_3,X_4X_5X_6) &= \mu_{1\ldots6}-\sum_{123}^{3}\mu_{12}\mu_{3\ldots6}=k_{1\ldots6}+\sum^{10}k_{123}k_{456}+\sum^{15}\mu_{12}\mu_{34}\mu_{56} \\ &\quad +\sum^{12}\mu_{14}k_{2356}. \end{aligned}$$

So, (3.16) follows and

$$a_{1234}^{1122}=A^{15}A^{26}\kappa(\Delta_5,\Delta_6,A^{37}A^{89}\Delta_{78}\Delta_9-B^{378}\Delta_7\Delta_8,A^{4,10}A^{11,12}\Delta_{10,11}\Delta_{12}-B^{4,10,11}\Delta_{10}\Delta_{11}).$$

Also (D.2) implies

$$\begin{aligned} \kappa(X_1,X_2,X_3X_4,X_5X_6) &= k_{1\ldots6}+\sum^{15}\mu_{12}k_{3\ldots6}+\sum^{10}\mu_{123}\mu_{456}+\sum^{15}\mu_{12}\mu_{34}\mu_{56} \\ &\quad -\mu_{134}\mu_{256}-\mu_{234}\mu_{156}. \end{aligned}$$

So, (3.17) follows. So, $k_{.3}^{1234}$ is given by (3.6), (3.14)–(3.17). □

Proof of Theorems 4.1 and 4.2. Note that $A_1=0$ as required. Also

$$A_{12} = [12]=n^{-1}\sum_{N=1}^{n}S_{N\cdot1}S_{N\cdot2}=\langle1,2\rangle, \tag{D.3}$$

$$A_{123} = [123]=\sum^{3}\langle1,23\rangle, \tag{D.4}$$

$$\begin{aligned} A_{1234} &= [1234]=\sum^{3}\langle12,34\rangle+\sum^{4}\langle1,234\rangle, \\ [\pi_1,\ldots,\pi_r] &= (-1)^r\lambda_r\langle\pi_1,\ldots,\pi_r\rangle \end{aligned} \tag{D.5}$$

for $r\geq1$. So, $[1,2]=\lambda_2\langle1,2\rangle=\lambda_2A_{12}$. So, by (3.7), the asymptotic covariance of $\widehat{\theta}$ is given by (4.4). By (3.8), the asymptotic bias of $\widehat{\theta}$ is given by (4.5). For A diagonal (4.5) reduces to (4.14). By (3.4), (3.9) and (3.10), we have (4.6) and (4.15). Note that $k_{.2}^{12}$ is given by (3.5) in terms of (4.7)–(4.9). For A diagonal, (4.7), (4.8) and (4.9) reduce to (4.16), (4.17) and (4.18), respectively. Note that $k_3^{i_1i_2i_3i_4}=k_{.3}^{1234}$ is given by (3.6) in terms of (4.10)–(4.13). For A diagonal, (4.10)–(4.13) reduce to (4.19)–(4.22). □

Proof of Theorem 6.1. Using the notational conventions of Section 2, tr $Q=n-q$ so tr $Q_{\cdot1\ldots r}=0$ for $r\geq1$. Also $\Lambda_{\cdot1\ldots r}=n^{-1}Y'Q_{\cdot1\ldots r}Y$, so

$$A_{1\ldots r}=n^{-1}\mathrm{tr}Q_{\cdot1\ldots r}(\lambda_2I_n+SS')=n^{-1}S'Q_{\cdot1\ldots r}S=n^{-1}\beta'X'Q_{\cdot1\ldots r}X\beta. \tag{D.6}$$

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Also

$$PX = X,\ QX = 0,\ Q_{\cdot 1}X + QX_{\cdot 1} = 0,\ X'Q_{\cdot 1}X = -X'QX_{\cdot 1} = 0,\ A_1 = 0,$$
$$Q_{\cdot 12}X + Q_{\cdot 1}X_{\cdot 2} + Q_{\cdot 2}X_{\cdot 1} + QX_{\cdot 12} = 0,$$
$$X'Q_{\cdot 12}X = -\sum_{12}^{2} X'Q_{\cdot 1}X_{\cdot 2} = \sum_{12}^{2} X'_{\cdot 1}QX_{\cdot 2}, \tag{D.7}$$
$$Q_{\cdot 123}X + \sum_{123}^{3}(Q_{\cdot 12}X_{\cdot 3} + Q_{\cdot 3}X_{\cdot 12}) + QX_{\cdot 123} = 0,$$
$$X'Q_{\cdot 123}X = -\sum_{123}^{3} X'(Q_{\cdot 12}X_{\cdot 3} + Q_{\cdot 3}X_{\cdot 12}) = \sum_{123}^{6} X'_{\cdot 1}Q_{\cdot 2}X_{\cdot 3} + \sum_{123}^{3} X'_{\cdot 1}QX_{\cdot 23}. \tag{D.8}$$

By (6.2),

$$n^{-1}X'X = n^{-1}\sum_{N=1}^{n} x_N x'_N = T + O(n^{-1}) \tag{D.9}$$

as $n \to \infty$, where $T = \int_0^1 y(t)y(t)'\mathrm{d}t$. (Higher order terms are given by the Euler-McLaurin expansion.) Similarly, for π_1, π_2 sequences of integers,

$$n^{-1}X'_{\cdot\pi_1}X_{\cdot\pi_2} = n^{-1}\sum_{N=1}^{n} x_{N\cdot\pi_1}x'_{N\cdot\pi_2} = T(\pi_1, \pi_2) + O(n^{-1}). \tag{D.10}$$

Also

$$n^{-1}X'_{\cdot\pi_1}PX_{\cdot\pi_2} = (n^{-1}X'X_{\cdot\pi_1})'(n^{-1}X'X)^{-1}(n^{-1}X'X_{\cdot\pi_2}) = T(\cdot, \pi_1)'T^{-1}T(\cdot, \pi_2) + O(n^{-1}),$$

where $T(\cdot, \pi_2)$ is $T(\pi_1, \pi_2)$ with π_1 empty. So,

$$n^{-1}X'_{\cdot\pi_1}QX_{\cdot\pi_2} = T(\pi_1 : \pi_2) + O(n^{-1}). \tag{D.11}$$

So, by (D.7), $n^{-1}X'Q_{\cdot 12}X$ is bounded. So, by (D.6) A_{12} is bounded. The condition that $A = (A_{12})$ be bounded away from zero holds because of (6.3). Similarly, the second term in the right hand side of (D.8) is $O(n)$. That the first term is also follows similarly from

$$Q_{\cdot 1} = -P_{\cdot 1} = -X_{\cdot 1}MX' - XM_{\cdot 1}X' - XMX'_{\cdot 1} \text{ and } M_{\cdot 1} = -ML_{\cdot 1}M,$$

where $M^{-1} = L = X'X$. So, A_{123} is bounded. Similarly, $A_{1\ldots r}$ is bounded.

We now show that $\Delta_\pi = O_p(n^{-1/2})$. Note that $\Delta_\pi = 2\Delta_{1\pi} + \Delta_{2\pi}$ for $\Delta_{1\pi} = n^{-1}e'Q_{\cdot\pi}S$ and $\Delta_{2\pi} = n^{-1}e'Q_{\cdot\pi}e$. Also $\Delta_{1\pi} \dot{\sim} \mathcal{N}(0,\ n^{-1}\lambda_2\beta'V_n\beta)$ for $V_n = n^{-1}X'Q^2_{\cdot\pi}X = O(1)$ by a similar argument to above. So, $\Delta_{1\pi} = O_p(n^{-1/2})$. Also

$$E\Delta_{2\pi} = n^{-1}\lambda_2 \operatorname{tr} Q_{\cdot\pi} = 0,\ n^2E\Delta^2_{2\pi} = \lambda_4\sum_{N=1}^{n} Q^2_{NN\cdot\pi} + 2\lambda_2^2 \operatorname{tr} Q^2_{\cdot\pi}, \tag{D.12}$$

where $Q^2_{N_1N_2}$ are the elements of Q. By a similar argument to above

$$\sum_{N=1}^{n} Q^2_{NN\cdot\pi} = O(n^{-1}) \text{ and } \operatorname{tr} Q^2_{\cdot\pi} = O(1).$$

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So, $\Delta_{2\pi} = O_p(n^{-1})$. So, the conditions needed for Section 2 hold. □

Proof of Theorem C.1. Note that (C.1) holds. By Leibniz' rule,

$$G^{(r)}(x) = \sum_{k=0}^{r} \binom{r}{k} N^{(r-k)}(x) I^{(k)}(x).$$

Also note that $N^{(i)}(x) = N_i$. By Faa di Bruno's rule, Comtet ([2], page 137), for $f(D) = D^{-1}$,

$$I^{(k)}(x) = \sum_{i=0}^{k} f_i B_{ki}(y)$$

for $f_i = f^{(i)}(D) = (-1)^i i! D^{-i-1}$, $y_i = D^{(i)}(x) = c^i \exp(cx) - \delta_{i0}$, and $B_{ki}(y)$ the partial exponential Bell polynomial tabled on page 307 of [2]. These are defined in terms of

$$S = \sum_{k=1}^{\infty} z^k y_k / k!$$

by

$$S^i / i! = \sum_{k=i}^{\infty} z^k B_{ki}(y) / k!.$$

But in our case $S = \exp(cx)\{\exp(cz) - 1\}$ giving $B_{ki}(y) = \exp(icx) c^k C_{ki} / i!$. As an aside, since $B_{kk}(y) = y_1^k$, we obtain the interesting identity $C_{ki} = 0$ for $k < i$, $C_{kk} = k!$. So, with (C.1) we obtain the results of the theorem. □

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