them. Of the eight original lines, a', b', e', f' can be taken arbitrarily; they determine c, d, while a, b are arbitrary members of the reguli conjugate to $(b'e'f')$, $(a'e'f')$. The line ϵ_a, and therefore in particular its meet with a', which is also the meet of ϕ_a and a', are defined completely by a, a', b', f'. The line ϕ_0 on the other hand meets a' in the point $a'b$, which can be selected quite independently of a, b', f'. Thus ϕ_0, ϕ_a, ϕ_b are the three required distinct self-corresponding members of the ranges.

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NOTE ON SEMI-GROUPS

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This note contains a new criterion for the complete simplicity of a semi-group which is more general than that given in a previous paper†.

LEMMA 1. *If S is a simple semi-group and e, f are two non-zero idempotents of S, then there exists a non-zero idempotent g of S under e such that*

$$fSf \cong gSg.$$

For since S is simple, there exist non-zero elements x, x' in S such that $xex' = f$. Then, if $y = fxe$ and $y' = ex'f$, we have

$$yy' = f, \quad fy = y = ye \quad \text{and} \quad ey' = y' = y'f.$$

Then we shall show that $g = y'y$ is an idempotent of S satisfying the conditions of this lemma. First, g is clearly a non-zero idempotent under e. Secondly, the correspondence $z \longleftrightarrow z^*$, where $z \in y'Sy$ and $z^* \in fSf$, which is defined by the equivalent relations

$$z = y'z^*y, \quad z^* = yzy',$$

is an isomorphism between $y'Sy$ and fSf. The verification of this is trivial. But

$$y'Sy \supset y' \,.\, ySy' \,.\, y = gSg,$$

† D. Rees, 'On semi-groups', *Proc. Cambridge Phil. Soc.* 36 (1940), 387–400. We refer the reader to this paper for the relevant definitions.

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and $$y'Sy = y'f \,.\, S \,.\, fy = y'y \,.\, y'Sy \,.\, y'y \subset gSg.$$

Thus $y'Sy = gSg$ and the lemma is proved.

COROLLARY. *If a simple semi-group S contains one primitive idempotent e, then all non-zero idempotents of S are primitive.*

For, by the above, if f is another non-zero idempotent of S, then fSf is isomorphic with eSe. But since e is primitive, eSe is either a group or a group with zero. Hence fSf is a group or group with zero, and so f is a primitive idempotent.

LEMMA 2. *If S is a simple semi-group containing at least one non-zero primitive idempotent e, then*

$$S = \Sigma e_i S e_j,$$

where the summation extends over all pairs of non-zero idempotents e_i, e_j of S.

First, we note that all idempotents of S are primitive. Let

$$S^* = \Sigma e_i S e_j, \quad L = \Sigma e_i S, \quad R = \Sigma S e_i,$$

e_i and e_j varying over all non-zero idempotents of S. Then L, R and S^* are none of them null, since they all contain the non-zero element e.

Now let $x \in L$, so that $e_\alpha x = x$ for some e_α. Since S is simple, there will exist elements y, y' such that $yxy' = e_\alpha$, and we may choose them so as to satisfy also the relations $e_\alpha y = ye_\alpha = y$, $y'e_\alpha = y'$. Then $f = xy'y$ is a non-zero idempotent of S such that $e_\alpha f = fe_\alpha = f$. Hence, since e_α is primitive, $e_\alpha = f$. Further, $y'yx = g$ is a non-zero idempotent of S such that $xg = xy'yx = e_\alpha x = x$. Hence

$$x \in e_\alpha S g \subset S^*,$$

and so $S^* = L$. Similarly $S^* = R$. Therefore $S^* = R = L$ is a non-zero ideal of the simple semi-group S, and so $S^* = S$ and the lemma is proved.

We may now prove the following

THEOREM. *If S is a simple semi-group containing the non-zero primitive idempotent e, then S is completely simple.*

For (i) S is simple; (ii) every idempotent of S is primitive by the corollary to Lemma 1; (iii) $S = \Sigma e_i S e_j$ by Lemma 2.

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