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Saddle-node bifurcations for hyperbolic sets

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Abstract. Hyperbolic sets are robust under perturbations: they persist on an open set of the parameter space. In this paper we investigate the boundary of this open set. Generalizing the theory of fixed points we define saddle-node bifurcations for hyperbolic sets K with one-dimensional unstable directions. In this bifurcation the geometrical splitting of the tangent space is preserved but the expansion in the unstable direction degenerates near a periodic orbit. The compact set K can be followed on a closed half-space bounded by a codimension-one manifold $\mathcal{O}^0$. On $\mathcal{O}^0$ the saddle-node bifurcation occurs. On one side of $\mathcal{O}^0$, K is hyperbolic and on the other side, it has disappeared.

1. *Introduction*

Let us recall an elementary property: a fixed point P of a diffeomorphism F such that all its eigenvalues are different from one persists under small perturbations of F. More generally the same result holds for any hyperbolic set: if $(F_\lambda)_{\lambda\in\Lambda}$ is a continuous family of diffeomorphisms and if K is an invariant hyperbolic set for F_{λ_0}, $\lambda_0 \in \Lambda$, the continuation of K is defined on an open subset of Λ.

The next step is to understand how a hyperbolic fixed point can disappear. One possibility involves the saddle-node bifurcation: at the time of the bifurcation, the derivative $\mathrm{D}F(P)$ at the fixed point P admits one eigenvalue equal to one. The other ones have their module strictly smaller than one. Moreover if v is any non-zero vector in the unstable direction, we require that $\mathrm{D}^2F(P)(v, v)$ is non-zero. The behaviour under perturbation is described by the following proposition (see [**NPT**]).

PROPOSITION 1.0.1. *Let $(F_\lambda)_{\lambda\in\Lambda}$ be a smooth family of diffeomorphisms. If (P, λ_0) is a transversal saddle-node bifurcation, there exists an open neighbourhood $\mathcal{O}$ of λ_0 in Λ such that:*

(i) *$\mathcal{O}$ has a one-codimensional submanifold $\mathcal{O}^0$ passing through λ_0; $\mathcal{O} \setminus \mathcal{O}^0$ has two components, $\mathcal{O}^+$ and $\mathcal{O}^-$;*

(ii) *the continuation $P(\lambda)$ of P is defined if and only if $\lambda \in \mathcal{O}^0 \cup \mathcal{O}^+$;*

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(iii) *$P(\lambda)$ is hyperbolic if and only if $\lambda \in \mathcal{O}^+$; for $\lambda \in \mathcal{O}^0$, $(P(\lambda), \lambda)$ is a transversal saddle-node bifurcation.*

Item (iii) deals with the stability of the bifurcation. Precise definitions for objects considered in this introduction will be given in §§2 and 4.1.

The goal of this paper is to generalize this result for hyperbolic sets and is motivated by the following questions:

- How can a hyperbolic set be destroyed? Describe some of its bifurcations.
- Which bifurcations are one-codimensional?
- What happens after the bifurcation has occurred?

Some recent results can be viewed as addressing these questions (see [**C1, C2, CK, R**]).

Our aim here is to describe a bifurcation associated to a lack of expansion at a particular fixed point of the hyperbolic set. It is due to the coalescence of a saddle point in the set with a sink outside the set while keeping the geometrical splitting of the tangent space into stable and unstable subspaces. Similar phenomena with more restrictive families of diffeomorphisms have already been studied for different purposes in [**C1, C2, CK**].

We will define a saddle-node hyperbolic set associated with a diffeomorphism F over a manifold to be any F-invariant compact set K containing a distinguished point P with the following properties.

(1) P is a saddle-node fixed point. All other periodic orbits in K are hyperbolic.

(2) There exists an open neighbourhood U of K and an open set Ω intersection of a neighbourhood of P with $\mathrm{int}(W^s_{\mathrm{loc}}(P))$ such that K is the maximal invariant set of $U \setminus \Omega$.

(3) There is on U a continuous stable invariant cone field $\mathcal{C}^s$ expanded by backward iteration. The dual cone field is noted $\mathcal{C}^u$. We stress that the central direction is one-dimensional and that F contracts in the other directions.

(4) Curves in U that intersect K with tangent space in $\mathcal{C}^u$ do not have self-intersection. Such a curve can always be extended so that it contains a point attracted by P or leaving U under forward iteration.

Property (1) expresses that P is the only lack of hyperbolicity. Compare (3) with the well-known property that any invariant compact set is hyperbolic if and only if it carries a stable invariant cone field such that $\mathcal{C}^s$ is expanded under backward iterations and $\mathcal{C}^u$ is expanded under forward iterations. Properties (2) and (4) are related to a property of local maximality of K and are satisfied if K is a topological horseshoe (see Example 2.4.4(2) below).

We can now state our result.

THEOREM 1.0.2. *Let $(F_\lambda)_{\lambda\in\Lambda}$ be a smooth family of diffeomorphisms on a Riemannian manifold M. If (K, P) is a saddle-node hyperbolic set with respect to F_{λ_0} and if (P, λ_0) is a non-degenerate saddle-node bifurcation then there exists an open neighbourhood $\mathcal{O}$ of λ_0 such that:*

(i) *$\mathcal{O}$ decomposes as in Proposition 1.0.1;*

(ii) *the continuation $K(\lambda)$ of K is defined if and only if $\lambda \in \mathcal{O}^0 \cup \mathcal{O}^+$; there exists a unique continuous map : $\Phi : (\mathcal{O}^0 \cup \mathcal{O}^+) \times K \to M$ which satisfies that $\Phi_{\lambda_0} = \mathrm{Id}_K$*

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and, for all λ, Φ_λ *is a homeomorphism from* K *to* $K(\lambda)$ *which conjugates the dynamics;*

(iii) $K(\lambda)$ *is hyperbolic if and only if* $\lambda \in \mathcal{O}^+$*; for* $\lambda \in \mathcal{O}^0$, $(K(\lambda), P(\lambda))$ *is a saddle-node hyperbolic set.*

A more detailed description of the dynamics is given. Note that in this situation the hyperbolic set exists on an open set and that the bifurcation describes an open set of the boundary $\partial\mathcal{O}^+ = \mathcal{O}^0$. In other words, the bifurcation is robust (i.e. one-codimensional). When $\lambda \in \mathcal{O}^-$ some parts of K are not destroyed but nothing is said about the maximal invariant set in U (see [**C1, C2**]).

For $\lambda \in \mathcal{O}^0$ every point $x \in K(\lambda)$ possesses two invariant spaces, one stable E^s_x and one unstable E^u_x, so that

$$\mathrm{T}_x M = E^s_x \oplus E^u_x.$$

It is a well-known fact that for every hyperbolic set the metric can be adapted so that each iteration of the diffeomorphism strictly contracts or expands the bundles E^s_x and E^u_x. A similar result holds in our setting.

PROPOSITION 1.0.3. *Let* (K, P) *be a saddle-node hyperbolic set for the diffeomorphism* F *on the Riemannian manifold* $(M, \|\cdot\|)$. *Then there exists a continuous Riemannian metric* $\|\cdot\|_a$ *on* M *such that*

$$\forall x \in K \setminus \{P\},\ \forall v \in E^u_x \setminus \{0\}, \quad \|\mathrm{D}F(x).v\|_a > \|v\|_a,$$

and

$$\forall x \in K,\ \forall v \in E^s_x \setminus \{0\}, \quad \|\mathrm{D}F(x).v\|_a < \|v\|_a.$$

The organization of this paper is as follows. Definitions, notation and examples are introduced in §2. The main ingredient for our result is to prove expansion for vectors of $\mathcal{C}^u$. In §3 this is done for $K(\lambda_0)$ using arguments of one-dimensional dynamics (see the proof of Mañé's Theorem in [**MS**]).

THEOREM 1.0.4. *Let* (K, P) *be a saddle-node hyperbolic set for the diffeomorphism* F. *Fix* W *to be any small open neighbourhood of* P. *Then for any* $k > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that for any finite orbit* $z_0, z_1, \ldots, z_n$ *in* K *with* $z_n \notin W$ *and* $n \geq n_0$ *and for any* $v \in \mathcal{C}^u(z_0) \setminus \{0\}$,

$$\frac{\|\mathrm{D}F^n(z_0).v\|}{\|v\|} > k.$$

Section 4 studies expansion in a neighbourhood of P for any perturbed map F_λ in $\mathcal{O}^+$. Theorem 1.0.2 and Proposition 1.0.3 are proved in §5.

2. *Preliminaries*

2.1. *Notation.* In §§2 and 3 we consider a smooth diffeomorphism F on a smooth Riemannian manifold M. d will stand for the dimension of M.

The tangent and the unitary tangent bundles over $X \subset M$ are denoted $\mathrm{T}X$ and $\mathrm{U}X$.

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$\mathrm{T}_z M$ is the tangent space at the point $z \in M$. $|\sigma|$ is the length of the Lipschitz curve σ in M.

In a metric space we denote $B(z, r)$ to be the open ball centred at z with radius r. For example, $B^k(0, 1)$ will stand for the standard open ball of $\mathbb{R}^k$. We also use the notation $B(Y, r) = \bigcup_{y \in Y} B(y, r)$ for any subset Y.

2.2. *Saddle-node fixed points.* Let P be a saddle-node fixed point as defined at the beginning of §1. The tangent space at P, $\mathrm{T}_P M$, splits into $E^s_P \oplus E^c_P$. E^s and E^c are invariant subspaces; E^s is strictly contracted by $\mathrm{D}F$ and E^c is associated to the eigenvalue one. It is well known (see [**HPS**, §4, Theorem 4.1]) that P admits a unique strong stable manifold tangent to E^s_P: W^{ss}_P. It separates M locally into two regions. One is attracted by P; the other contains a unique invariant half central manifold W^c_P tangent to E^c_P (see [**NPT**]).

Remarks 2.2.1.

(1) One can also assume in the following that P is a periodic orbit (see Remark 5.3.1).

(2) Of course by considering F^{-1} instead of F one can also consider saddle-node fixed points having one eigenvalue equal to one and other eigenvalues with module larger than one.

2.3. *Stable cone field.*

Definition 2.3.1. An invariant subset $K \subset M$ admits a (one-codimensional) continuous stable invariant cone field if:

(1) the tangent bundle splits into two continuous bundles: $\mathrm{T}K = E^1 \oplus E^2$ with $\dim(E^1) = 1$; for any $v \in \mathrm{T}K$ we note that $v = v^1 + v^2$;

(2) there exist two continuous norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E^1 and E^2, respectively, such that the cone field

$$\mathcal{C}^s = \{v \in \mathrm{T}K, \|v^1\|_1 \leq \|v^2\|_2\}$$

is backward invariant: for any $z \in K$,

$$\mathrm{D}F^{-1}(\mathcal{C}^s_z) \subset \mathrm{int}(\mathcal{C}^s_{F^{-1}(z)}) \cup \{0\};$$

as a consequence the dual cone field

$$\mathcal{C}^u = \{v \in \mathrm{T}U, \|v^1\|_1 \geq \|v^2\|_2\}$$

is forward invariant;

(3) there are some constants $0 < \beta < 1$ and $n_0 \in \mathbb{N}$ such that, for any $z \in K$,

$$\forall v \in \mathcal{C}^s_z, \quad \|\mathrm{D}F^{-n_0}(z).v\| \geq \beta^{-1}\|v\|,$$

and

$$\forall v \in \mathcal{C}^u_z,\ \forall w \in \mathcal{C}^s_{F^{n_0}(z)}, \quad \beta\|\mathrm{D}F^{n_0}(z).v\|^2\|\mathrm{D}F^{-n_0}(F^{n_0}(z)).w\| \geq \|v\|^2\|w\|.$$

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Remarks 2.3.2.

(1) Note that we do not assume that $\mathrm{D}F$ is expanding on $\mathcal{C}^u$.

(2) The existence of a continuous stable invariant cone field on K does not depend on the choice of a metric on M.

(3) The previous definition can be extended for subsets U that are not invariant. The backward invariance (property (2)) and the second part of property (3) are only required for points z in $U \cap F^{-1}(U)$ and $U \cap F^{-1}(U) \cap \cdots \cap F^{-n_0}(U)$, respectively. If K is an invariant compact set admitting a continuous stable invariant cone field a neighbourhood of K does also. If $P \in K$ is a saddle-node fixed point the splitting $E^s_P \oplus E^c_P$ is compatible with the cone field:

$$E^s_P \subset \mathcal{C}^s_P \quad \text{and} \quad E^c_P \subset \mathcal{C}^u_P.$$

(4) The usual definition of invariant stable cone field (or of partial hyperbolicity, see [**BP**]) is more general. One does not assume $\dim(E)^u = 1$ and the domination hypothesis is weaker. For any $z \in K$,

$$\forall v \in \mathcal{C}^u_z,\ \forall w \in \mathcal{C}^s_{F^{n_0}(z)}, \quad \beta \|\mathrm{D}F^{n_0}(z).v\| \|\mathrm{D}F^{-n_0}(F^{n_0}(z)).w\| \geq \|v\| \|w\|.$$

However, here we want to describe invariant sets which are close to being hyperbolic. Any condition that bounds the contraction in the central unstable direction (i.e. in $\mathcal{C}^u$) is appropriate.

Suppose that a point z is in K; we define at z its stable and unstable cones of order n, $\mathcal{C}^s_z(n)$ and $\mathcal{C}^u_z(n)$, as respectively $\mathrm{D}F^{-n}(\mathcal{C}^s_{F^n(z)})$ and $\mathrm{D}F^n(\mathcal{C}^u_{F^{-n}(z)})$. Condition (3) implies that the angle between vectors in the same stable or unstable cone decreases uniformly as n increases. Condition (2) implies $\mathcal{C}^s(n+1) \subset \mathcal{C}^s(n)$.

As a preliminary to the next paragraph we recall that an invariant compact set K is *hyperbolic* if the tangent space at each point z of K splits $\mathrm{T}_z K = E^s_z \oplus E^u_z$ such that:

(1) the splitting is invariant, $\mathrm{D}F^{-1}(E^s) = E^s$ and $\mathrm{D}F(E^u) = E^u$;

(2) there exist some $0 < \beta < 1, c > 0$ such that if $z \in K$, $v \in E^s_z$ (respectively $v \in E^u_z$), $n \in \mathbb{N}$, then

$$\|\mathrm{D}F^{-n}(z).v\| \geq c\beta^{-n}\|v\|$$
$$(\text{respectively } \|\mathrm{D}F^{n}(z).v\| \geq c\beta^{-n}\|v\|).$$

The cone field criterion asserts that any invariant compact set admitting a stable invariant cone field, and such that there exists a constant $\mu > 1$ and an integer $n \geq 1$ with

$$\forall z \in K,\ \forall v \in \mathcal{C}^u, \quad \|\mathrm{D}F^n(z).v\| \geq \mu\|v\|,$$

is hyperbolic.

Suppose that on an open set U the diffeomorphism admits a continuous stable invariant cone field. An *u-curve* (tangent to $\mathcal{C}^u$) is a rectifiable map from $[0, 1]$ in U with tangent subspaces in $\mathcal{C}^u$. (Up to a reparametrization such a curve is Lipschitz.) Unless specified we will however assume that u-curves are C^2.

An embedding of $B^{d-1}(0, 1)$ is a *s-leaf* (tangent to $\mathcal{C}^s$) if its tangent subspaces are in $\mathcal{C}^s$.

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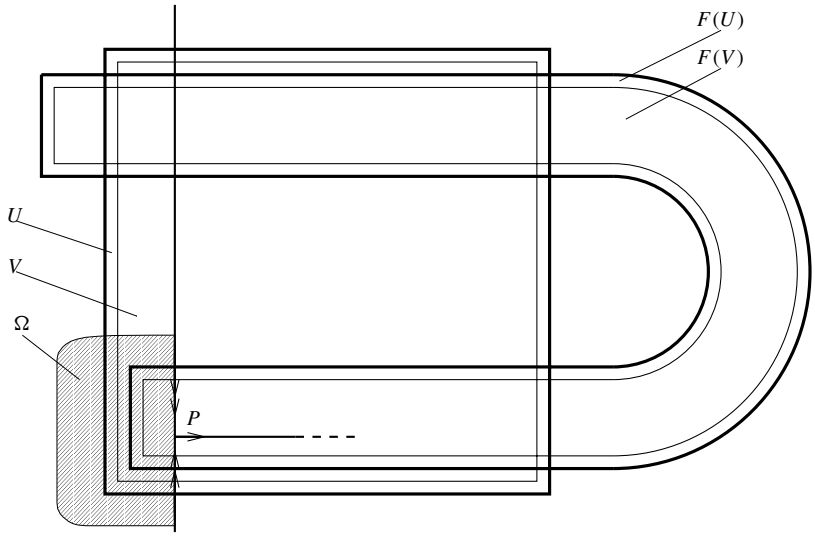


FIGURE 1. Open sets U, V, Ω associated to the saddle-node hyperbolic set (K, P).

2.4. *Saddle-node hyperbolic sets.* We first recall the following.

Definition 2.4.1. Let X be a subset of M. The *maximal invariant subset* of X, Y is

$$Y = \bigcap_{k\in\mathbb{Z}} F^k(X).$$

Let us come to the major definition of our study.

Definition 2.4.2. A compact invariant set K of the diffeomorphism F associated to the point $P \in K$ is a *saddle-node hyperbolic set* if it satisfies the following properties (see Figure 1).

(1) *Hyperbolicity of periodic orbits*: P is a saddle-node fixed point and every periodic orbit in K that is different from P is hyperbolic.

(2) *Local maximality*: There exist a small open set Ω intersection of a small neighbourhood of P with $W^s_{\mathrm{loc}}(P) \setminus W^{ss}(P)$ and an open neighbourhood U of K such that K is the maximal invariant set of $\bar{U} \setminus \Omega$.

(3) *Existence of a stable cone field*: There exists on a neighbourhood of $\bar{U}$ a continuous stable invariant cone field $\mathcal{C}^s$.
We introduce the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and a constant β as in Definition 2.3.1.

(4) *Topological expansion*: In a neighbourhood of $\bar{U}$, u-curves that intersect K do not have self-intersection. There exists an open set V with $K \subset V \subset \bar{V} \subset U$ such that for any u-curve γ intersecting K,

$$\gamma \subset \bar{U} \cap F^{-1}(\bar{U}) \setminus \Omega \Rightarrow \gamma \subset \bar{V}.$$

The open set Ω is small enough so that for any u-curve γ in $\bar{U}$ that intersects K in z, at most one component of $\gamma \setminus \{z\}$ intersects $F^{-1}(\Omega)$.

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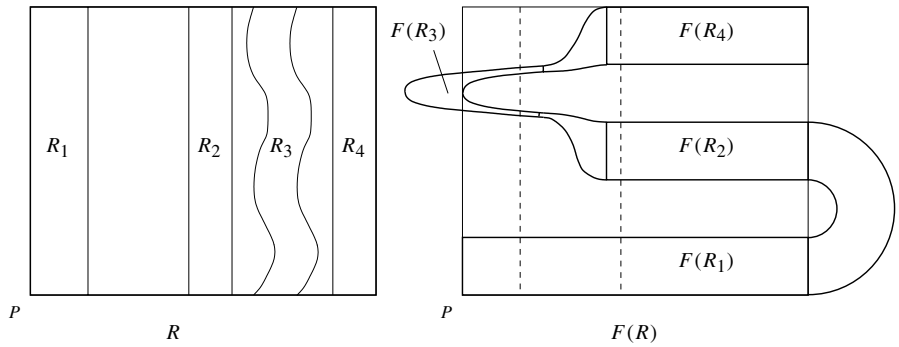


FIGURE 2. A counterexample with homoclinic tangencies.

Remarks 2.4.3.

(1) Condition (1) expresses that P is the only lack of hyperbolicity of K. Indeed it will be a corollary of the next section to see that condition (1) is equivalent to: for all the open neighbourhoods W of P, the maximal invariant set in $K \setminus W$ is hyperbolic.

(2) The first part of condition (4) implies that u-curves in $\bar{U}$ tangent to $\mathcal{C}^u(1)$ that intersect K are not closed. Such u-curves that are tangent to $\mathcal{C}^s(1)$ have their length uniformly bounded by some constant k_l: otherwise one could find a u-curve in $\bar{U}$ tangent to $\mathcal{C}^u(1)$ whose endpoints were arbitrarily close together. Hence one could close it and build a closed u-curve tangent to $\mathcal{C}^u$ in any neighbourhood of $\bar{U}$. This is a contradiction.

The second part of condition (4) asserts that for any u-curve γ in $\bar{U} \setminus \Omega$ intersecting K and $\bar{U} \setminus \bar{V}$, there exists $z \in \gamma$ such that $F(z)$ is not in $\bar{U}$. Both are important to prove that K is totally disconnected in the unstable direction. They are illustrated by some examples below. In particular, we will see that for horseshoes condition (4) is automatically satisfied.

(3) For any iterate F^n, (K, P) is again a saddle-node hyperbolic set (see Lemma 2.5.1 below).

(4) One cannot require instead of condition (2) that K is the maximal invariant subset of one of its neighbourhoods: in some examples there exists an orbit on W^c_P whose points are eventually attracted by Ω and which is arbitrarily close to a homoclinic orbit of K between W^{ss}_P and W^c_P.

(5) Condition (3) is necessary to ensure that the tangent space splits into two supplementary subspaces at each point of K. One can construct a counterexample satisfying only properties (1) and (2) (see also [**R**]). Consider in dimension two the degenerate horseshoe K in the rectangle R: the rectangles R_1, R_2, R_3 and R_4 form a Markov partition of K; F is a product on R_1, R_2 and R_4. The saddle-node fixed point P is an endpoint of R_1 and R. The image of an unstable side of R_3 is tangent to a stable side of R_1 (see Figure 2). This last property contradicts the existence of distinct stable and unstable directions at each point of K. However, any compact invariant subset of K which does not contain P is hyperbolic (similar but detailed arguments are contained in [**R**]). Indeed every 'bad point' of K, i.e. whose stable and unstable directions are tangent, is forward or backward attracted by P.

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Examples 2.4.4.

(1) Explicit classes of examples have already been considered in [**C1, C2, CK**]. These are horseshoes in dimension two built from maps on one-dimensional manifolds.

(2) More generally, one can consider horseshoes in any dimension $d \geq 2$ with one-dimensional unstable direction. We require the following.
   (a) There exists an open set U carrying a continuous stable invariant cone field $\mathcal{C}^s$.
   (b) There exists in U a finite number of rectangles $R_0, \dots, R_1$. Each of them is the image of an immersion $\phi : B^{d-1}(0,1) \times [0,1] \to U$. The stable boundary of R_i is made of s-leaves $\partial^s_j R_i = \phi_i(B^{d-1}(0,1) \times \{j\})$, $j = 0, 1$. The curves $\phi(\{z\} \times [0,1])$ are u-curves. We assume that the sets $\bar{R}_i$ are disjoint. K is the maximal invariant set in $\cup R_i$.
   (c) There exists a saddle-node fixed point $P \in \partial^s_0 R_0$, $\partial^s_0 R_0 \subset W^{ss}_P$ and $W^s_{\mathrm{loc}}(P) \cap \mathrm{int}(R_0) = \emptyset$. Other periodic orbits in $\cup R_i$ are hyperbolic.
   (d) $F(\partial^s_0 R_0) \subset \partial^s_0 R_0$. Other stable boundaries $\partial^s_j R_i$ are sent outside $\cup \bar{R}_i$ under forward iteration. Moreover, for any i, j,
   $$F(R_i) \cap \partial R_j \subset \partial^s_0 R_j \cup \partial^s_1 R_j.$$

   Assumptions (1) to (3) in Definition 2.4.2 are obviously satisfied. Considering a higher iterate of F one can shrink the cone field $\mathcal{C}^u$ and the rectangles R_i. Thus any u-curve contained in some R_i has no self-intersection. Take U small enough. We get the same property for u-curves in U.

   With slight modifications one can assume that u-curves which intersect K cannot intersect ∂R_j outside $\partial^s_0 R_j \cup \partial^s_1 R_j$. For some large $n \geq 0$ one can replace $\mathcal{C}^u$ by $\mathcal{C}^u(n)$ and $\cup R_i$ by $\bigcap_{k=0}^n F^{-k}(\cup R_i)$ which is again a finite union of rectangles. V is now a small neighbourhood of $\cup R_i$ and $\bar{\Omega}$ is a local stable manifold of P that contains $\partial^s_0 R_0$. The second part of condition (4) is a consequence of (d).

(3) Other interesting examples can be considered, in particular those modelled on hyperbolic attractors.

We have chosen to construct an example modelled on a DA-map (derived from Anosov, see [**Sm**]) using the method of [**HK**, §17.2, pp. 537–540]: we start from the linear Anosov diffeomorphism on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

There are local coordinates in a neighbourhood of $(0,0)$ where the map can be written as:

$$(x, y) \to (\sigma_1 x, \sigma_2 y)$$

with $0 < \sigma_2 < 1 < \sigma_1$. We now perturb the diffeomorphism in a small neighbourhood of $(0,0)$ so that the diffeomorphism can now be written as:

$$F : (x, y) \to (\phi(x, y), \sigma_2 y).$$

$\phi(\cdot, 0)$ satisfies $\phi(0,0) = 0$, $(\partial/\partial x)\phi(0,0) = 1$ and $(\partial^2/\partial x^2)\phi(0,0) > 1$ as it is represented in Figure 3. We set $\phi(x, y) = \eta(y)\phi(x, 0) + (1 - \eta(y))\sigma_1 x$ with some bump function η.

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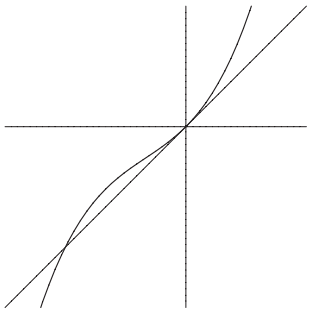


FIGURE 3. The map $x \mapsto \phi(x, 0)$.

For $\theta \ll \varepsilon$ small enough we consider, for any y, the intervals (perhaps empty) $[x(y), -\theta]$ and $(-\theta, 0)$ where $x(y)$ is the smallest real satisfying

$$\frac{\partial}{\partial x}\phi(x, y) = 1 + \varepsilon.$$

We set

$$X = \bigcup_y (x(y), 0) \times \{y\}.$$

Choosing ε and η carefully, X is contained in the basin of $(0, 0)$ and satisfies $F(X) \subset X$.

We consider the maximal invariant subset (compact) K of $\mathbb{T}^2 \setminus X$ and $P = (0, 0)$. Note that the horizontal direction is preserved: on $\mathbb{T}^2 \setminus X$, there exists an invariant cone field $\mathcal{C}^u$ arbitrarily sharp and close to the unstable direction. We introduce the open sets

$$U = \mathbb{T}^2 \setminus \left(\bigcup_y [x(y), -\theta] \times \{y\} \right), \quad \Omega = \bigcup_{x(y)<\theta} (-\theta, 0) \times \{y\}$$

and claim that (K, P) is a saddle-node hyperbolic set. Assumptions (1) to (3) in Definition 2.4.2 are obvious. The first part of property (4) is now clear: any u-curve is close to the horizontal foliation of the torus and has to intersect $\mathbb{T}^2 \setminus U$. For the second part we remark that the image of ∂U is contained in $U \cup \Omega$. We set $\bar{V} = U \setminus B(\partial U, \varepsilon')$ with $\varepsilon' > 0$ small.

2.5. *Adapted neighbourhoods.* We explain here how U, V and Ω can be modified in order to be contained in an arbitrarily small neighbourhood of K so that our definition of a saddle-node hyperbolic set is really a local property of (K, P): for any integer $n \geq 1$, the sets

$$U'_n = \left(\bigcap_{|k| \leq n} F^k(U) \right) \setminus \bigcup_{|k| \leq n-1} F^k(\Omega), \quad V'_n = \left(\bigcap_{|k| \leq n} F^k(V) \right) \setminus \bigcup_{|k| \leq n-1} F^k(\Omega)$$

are contained in a neighbourhood $B(K, \varepsilon)$ of K and by Definition 2.4.2(2), $\varepsilon > 0$ can be taken arbitrarily small if n is large enough.

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We set

$$U_n = \Big(\bigcap_{|k|\le n} F^k(U) \Big) \cap B(K, 2\varepsilon), \quad V_n = \Big(\bigcap_{|k|\le n} F^k(V) \Big) \cap B(K, \varepsilon),$$
$$\Omega_n = \Big(\bigcup_{|k|\le n+1} F^k(\Omega) \Big) \cap (B(K, 3\varepsilon) \cup F(B(K, 3\varepsilon))).$$

and prove the following.

LEMMA 2.5.1. *Let U, V and Ω be open sets associated to a saddle-node hyperbolic set (K, P) as in Definition 2.4.2. Then for any integer $n \ge 0$, the cone field $\mathcal{C}'^s = \mathcal{C}^s(-n)$ and the sets U_n, V_n and Ω_n defined above satisfy again Definition 2.4.2 and are contained in an arbitrarily small neighbourhood of K.*

Proof. Properties (1) and (2) are obvious. On U_n we consider the cone field $\mathcal{C}'^u = \mathrm{D}F^n(\mathcal{C}^u)$.

Note that $\bar{U}_n \setminus \Omega_{n-1}$ is contained in $\bar{U}'_n$ so that

$$\bar{U}_n \cap F^{-1}(\bar{U}_n) \setminus \Omega_n \subset \bar{U}'_n \cap F^{-1}(\bar{U}'_n).$$

For the last assumption we consider now any u-curve γ in U_n tangent to $\mathcal{C}'^u$ that intersects K. For any $-n \le k \le n$, $F^k(\gamma)$ is again a u-curve in U tangent to $\mathcal{C}^u$. In particular, the first part of condition (4) is realized. Let us assume now that

$$\gamma \subset \bar{U}_n \cap F^{-1}(\bar{U}_n) \setminus \Omega_n.$$

For any $-n \le k \le n$,

$$F^k(\gamma) \subset \bar{U} \cap F^{-1}(\bar{U}) \setminus \Omega.$$

Thus, $\gamma \subset F^{-k}(\bar{V})$ and $\gamma \subset \bar{V}'_n \subset \bar{V}_n$. □

Remark 2.5.1. By taking Ω larger, one can assume that Ω is stable: $F(\Omega) \subset \Omega$. From this, one gets that $F(\Omega_n) \cap U_n \subset \Omega_n$.

3. *Proof of Theorem 1.0.4*

In this section, we fix a saddle-node hyperbolic set (K, P). We show that if one waits until a point leaves a fixed neighbourhood of the saddle-node fixed point it has gained expansion in its unstable direction. Our arguments follow those by de Melo and van Strien in [**MS**, §III.5, pp. 247–257] for proving Mañé's Theorem.

Sometimes it is easier to work with a flat metric. Our result does not depend on the choice of a metric. Thus one can cover $\bar{U} \subset M$ with a finite number of charts and consider a flat metric on each of them. Our estimates will change by a uniform constant which does not depend on the number of iterations we could do.

Note that it is enough to prove Theorem 1.0.4 for an iterate of F. Thus one replaces F by some iterate $F' = F^m$, $m \ge 1$, and considers new open sets $U' = U_m$, $V' = V_m$ and $\Omega' = \Omega_m$ and a new cone field $\mathcal{C}'^s$ as in Lemma 2.5.1. One can then assume that

(1) the new unstable cones $\mathcal{C}'^u$ are arbitrarily sharp; note that the new stable cones $\mathcal{C}'^s(1)$ for F' coincide with the former cones $\mathcal{C}^s$ and are defined on $\bar{U}'$;

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(2) the new constant β' can be taken arbitrarily small; note also that property (3) of Definition 2.3.1 implies that, for any $z \in \bar{U}' \cap {F'}^{-1}(\bar{U}')$,

$$\forall v \in \mathcal{C}'^s_z(1), \quad \|\mathrm{D}{F'}^{-1}.v\| \geq {\beta'}^{-1}\|v\|,$$
$$\forall v \in \mathcal{C}'^u_z,\ \forall w \in \mathcal{C}'^s_{F'(z)}(1), \quad \beta\|\mathrm{D}F'(z).v\|^2\|\mathrm{D}{F'}^{-1}(F'(z)).w\| \geq \|v\|^2\|w\|; \quad (1)$$

(3) if α is the minimum angle between vectors in $\mathcal{C}'^s(1)$ and $\mathcal{C}'^u$ then

$$\frac{\beta'}{\sin(\alpha)} < \frac{1}{24}. \quad (2)$$

In the following we will forget the dependence in m. We will simply denote F, U, V, Ω, $\mathcal{C}^u$ and β for F', U', V', Ω', $\mathcal{C}'^u$ and β' as in Definition 2.4.2. We also consider some constant k_l as in Remark 2.4.3(2).

3.1. *Distortion.* One can control the curvature of u-curves as in [**BP**] or [**ABV**, Remark 2.5].

LEMMA 3.1.1. *There exists $\rho > 0$ such that for every u-curve γ in $\bar{U}$ with curvature bounded by ρ the image $F(\gamma)$ is a u-curve with curvature bounded by ρ.*

Proof. We will use the following estimate for $z \in \bar{U} \cap F^{-1}(\bar{U})$:

$$\forall v \in \mathcal{C}^u_z \setminus \{0\},\ \forall \zeta \in \{v\}^\perp, \quad \frac{\|\mathrm{D}F.\zeta \wedge \mathrm{D}F.v\|}{\|\mathrm{D}F.v\|^3} \leq \frac{\|\zeta\|}{3\|v\|^2}. \quad (3)$$

One obtains it by decomposing $\zeta = \zeta^1 + \zeta^2$ with $\zeta^1 \in \mathcal{C}^s(2)$ and $\zeta^1 \in \mathbb{R}.v$. One gets from (1),

$$\frac{\|\mathrm{D}F.\zeta \wedge \mathrm{D}F.v\|}{\|\mathrm{D}F.v\|^3} < \beta\frac{\|\zeta^1\|}{\|v\|^2}.$$

Then one gets from (2) that $\beta\|\zeta^1\| \leq \frac{1}{24}\|\zeta\|$.

To prove the lemma, one assumes that γ is parametrized by its length. In this case its curvature at $\gamma(t)$ is $\|\gamma''(t)\|$. The curvature at $F(\gamma(t))$ of $F \circ \gamma$ is

$$\frac{\|(F \circ \gamma)''(t) \wedge (F \circ \gamma)'(t)\|}{\|(F \circ \gamma)'(t)\|^3}.$$

We compute

$$(F \circ \gamma)''(t) = \mathrm{D}^2F(\gamma(t)).(\gamma'(t), \gamma'(t)) + \mathrm{D}F(\gamma(t)).\gamma''(t)$$

and bound the curvature by

$$\frac{\|\mathrm{D}^2F(\gamma'(t), \gamma'(t)) \wedge \mathrm{D}F.\gamma'(t)\|}{\|(F \circ \gamma)'(t)\|^3} + \frac{\|\mathrm{D}F.\gamma''(t) \wedge \mathrm{D}F.\gamma'(t)\|}{\|(F \circ \gamma)'(t)\|^3}.$$

The first term is uniformly bounded by some constant c and the second one by $\gamma''(t)/3$ (recall (3)). Thus it is sufficient to take $\rho > \frac{3}{2}c$. □

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Suppose that γ is a curve in U parametrized by its length with curvature bounded by ρ. The derivative of $\log \|DF(\gamma(t)).\gamma'(t)\|$ is

$$\frac{\langle DF(\gamma(t)).\gamma''(t) + D^2F(\gamma(t)).(\gamma'(t),\gamma'(t)), DF(\gamma(t)).\gamma'(t)\rangle}{2\|DF(\gamma(t)).\gamma'(t)\|^2}$$

which is uniformly bounded by some $c_1 > 0$.

Recall that $|\gamma|$ denotes the length of a curve γ. We need the following standard distortion lemma.

LEMMA 3.1.2. *For each constant $c > 0$ there exists $0 < \eta < 1$ such that given two smooth u-curves $\sigma \subset \gamma \subset U$ parametrized by their length with curvature bounded by ρ and assuming*

(i) *$\gamma, F(\gamma), \dots, F^{n-1}(\gamma)$ are well-defined in U for some integer n,*
(ii) *$\sum_{i=0}^{n-1} |F^i(\sigma)| < c$,*
(iii) *$|\gamma| \le (1+\eta)|\sigma|$,*

we get

(i) *for all $i \le n$, $|F^i(\gamma)| \le 2|F^i(\sigma)|$,*
(ii) *for all $x, y \in \gamma$,*

$$\exp(-2c_1c)\|DF^n(y).\gamma'(y)\| \le \|DF^n(x).\gamma'(x)\| \le \exp(2c_1c)\|DF^n(y).\gamma'(y)\|.$$

In general we will use this lemma with $\sigma = \gamma$.

Proof. Note that the curvature of the curves $\gamma, F(\gamma), \dots, F^{n-1}(\gamma)$ is bounded by ρ (Lemma 3.1.1). The proof is now similar to the one-dimensional case (see [**MS**, Ch. I.2]). □

3.2. *Rectangles.*

3.2.1. In this section we define the notion of rectangle R associated to a u-curve γ. One may think of it as a thickened u-curve. The stable sides of R which correspond to the endpoints of γ are s-leaves that are chosen in a quite arbitrary way. The relevant size for rectangles is not the thickness, which will not vary essentially, but the length which is defined in §3.3. A key tool here, which will be used implicitly, is the graph transform (see also [**Sh**]): if L is an s-leaf of radius r centred at $z \in K$, $F^{-1}(L)$ is an s-leaf of radius bigger than r/β centred at $F^{-1}(z)$ with a higher order. Take any s-leaves L_n at $F^n(z)$. A subsequence $F^{-n}(L_n)$ converges in $B(z,r)$ to an s-leaf with infinite order. As a consequence any point of K possesses a unique local stable manifold which is smooth.

Most of the properties we state here are of geometrical nature and can be proved easily by considering appropriate local charts.

3.2.2. First of all, we introduce a large enough integer n_c that will give the order of sharp unstable and stable cones and some constants $0 < r \ll r_0 < 1$ (see below).

One can extend Ω so that Ω is stable,

$$F(\bar\Omega) \cap \bar U_n \subset (\Omega \cup W^{ss}_{\mathrm{loc}}(P)) \cap \bar\Omega, \tag{4}$$

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and $\bar{\Omega}$ is compact. By Lemma 2.5.1 one can assume that $\bar{U}$ is compact. The other assumptions in Definition 2.4.2 are unchanged.

We fix an open set W containing P as stated in Theorem 1.0.4. We have to adapt U, V and Ω to the choice of W.

In the same way, one replaces U, V and Ω by U_{n_c}, V_{n_c} and Ω_{n_c} so that the cone fields $\mathcal{C}^s(n_c)$ and $\mathcal{C}^u(n_c)$ are defined on a neighbourhood $B(\bar{U}, r_0)$ of $\bar{U}$. Note that Definition 2.4.2(4) is now satisfied by u-curves tangent to $\mathcal{C}^u(n_c)$. We also require that

$$B(V, r_0) \subset U. \tag{5}$$

It will be convenient to set, for $n \geq 0$,

$$\begin{aligned} D &= \bar{U} \setminus \Omega, \\ D(0) &= D \cap F^{-1}(D) = (\bar{U} \cap F^{-1}(\bar{U})) \setminus F^{-1}(\Omega), \\ D(n) &= D \cap \cdots \cap F^{-n-1}(D) = \bar{U} \cap \cdots \cap F^{-n-1}(\bar{U}) \setminus F^{-n-1}(\Omega). \end{aligned} \tag{6}$$

3.2.3. From now on a *u-curve* is a C^2-curve that intersects K whose tangent subspaces are in $\mathcal{C}^u$ and whose curvature is bounded by ρ. Unless specified, they will be u-curves with respect to $\mathcal{C}^u(n_c)$. A u-curve is of *order* $n \geq 0$ if all its backward iterates of order smaller than n are u-curves.

With r small, for any u-curve $J \subset \bar{U}$, $z \in J$ and $0 < \varepsilon < 8r$,

$$J \cap B(z, \varepsilon) \text{ has only one connected component.} \tag{7}$$

If not one could construct a sequence of u-curves converging to a u-curve with some self-intersection and contradict Definition 2.4.2(4).

Note that $J \subset D(0)$ implies $J \subset \bar{V}$ (Definition 2.4.2(4)).

3.2.4. A u-curve is a *long u-curve* (respectively *crosses* $\bar{U}$) if both of its ends are in $\partial\bar{U} \cup \partial\bar{\Omega}$ (respectively $M \setminus \bar{U}$). Every u-curve in $\bar{U} \cup F^{-1}(\bar{U})$ can be extended in a long u-curve (respectively in a u-curve that crosses $\bar{U}$).

For $r_0 > 0$ small enough, every long u-curve T satisfies

$$r_0 < |T| < k_l. \tag{8}$$

More generally, by changing r_0 by a bounded factor one can assume that for any u-curve $J \subset D(0)$ such that $F(J)$ intersects $\partial\bar{U}$, $|J| > r_0$.

We state another important consequence of Definition 2.4.2(4). From it we deduce that, through every point of K, one can construct a long u-curve with arbitrarily high order.

LEMMA 3.2.1. *If T is a long u-curve, every component of $F(T) \cap \bar{U}$ intersecting K again contains a long u-curve.*

Proof. Take $z \in T \cap K$ and consider any component T_1 of $T \setminus \{z\}$. Either $T_1 \cap \bar{\Omega} \neq \emptyset$ (here $F(T_1)$ intersects $\partial\bar{\Omega}$) or $T_1 \subset \bar{U} \setminus \bar{\Omega}$ (as $T_1 \not\subset \bar{V}$, $T_1 \not\subset F^{-1}(\bar{U})$ and $F(T_1)$ intersects $\partial\bar{U}$). □

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LEMMA 3.2.2. *The unstable manifold of any periodic point z of K that is different from P contains a long u-curve passing through z.*

The local branch of W^c_P which is not in W^s_P is a long u-curve. Note also that any u-curve contained in $U \setminus \Omega$ and intersecting $\bar{\Omega} \cap W^{ss}_P$ becomes of length larger than r_0 by iteration.

Proof. We prove the first part; the argument is identical for the second one. The curvature remains the same after a finite number of iterations and the proof of Lemma 3.1.1 implies that it is bounded by ρ. It remains to show that the unstable manifold is long: if not there exists a branch γ of the unstable manifold of z contained in $\bar{U} \setminus \bar{\Omega}$. γ is Lipschitz so its endpoints are well-defined and belong to $\bar{U} \setminus \Omega$. One of them is z. The other one is also a periodic point z'. Note that, by Definition 2.4.2(4), $z \neq z'$. Moreover, points on γ are attracted by z'. Thus we must have $z' = P$. In this case some points of γ are in $\bar{\Omega}$ which is a contradiction. □

3.2.5. An s-leaf L will be called a *long s-leaf* centred at $z \in \bar{U}$ if the exponential map of L at z is an homeomorphism between $B^{d-1}(0, 10r) \subset \mathrm{T}_z L$ and L. Unless specified, we will only consider s-leaves with respect to $\mathcal{C}^s(n_c)$. If $r > 0$ is small enough, every point z in $\bar{U}$ is contained in a long s-leaf.

An s-leaf is of *order* $n \geq 0$ if all its forward iterates with order smaller than n are s-leaves. Every point in $\bar{U} \cap F^{-1}(\bar{U}) \cap \cdots \cap \bar{F}^{-n}(\bar{U})$ admits a long s-leaf of order larger than $n \geq 0$. When $z \in W^{ss}(P) \cap \bar{\Omega}$ we take L as a subset of $W^{ss}(P) \cap F^{-2}(\bar{\omega})$. More generally, when $z \in K$, the §3.2 implies that there exists at z a unique long leaf with infinite order defined by

$$W^s_z = \left\{ z' \in \bar{U},\ \frac{d(F^n(z), F^n(z'))}{\beta'^n} \xrightarrow[n\to\infty]{} 0 \right\} \quad \text{for some } 0 < \beta' < \beta.$$

The smallness of r and largeness of n_c imply, for any u-curve $J \subset \bar{U}$ and long s-leaf L centred in $\bar{U}$ tangent to $\mathcal{C}^s(n_c - 1)$, the following properties.

(1) $J \cap L$ is empty or a singleton.

(2) Assume that J and L intersect at z' and consider $z \in L$. Then there exists a curve $\gamma : [0, 1] \to L$ with $\gamma(0) = z$, $\gamma(1) = z'$ such that $t \mapsto d(\gamma(t), J)$ is strictly decreasing and, for any t,

$$\tfrac{2}{3} \sin(\alpha) |\gamma_{|[t,1]}| \leq d(\gamma(t), J) \leq |\gamma_{|[t,1]}|,$$

where $d(\gamma(t), J)$ stands for the distance between $\gamma(t)$ and J in M and α stands for the minimum angle between vectors in $\mathcal{C}'^s(1)$ and $\mathcal{C}'^u$.

3.2.6. In this section and in the next one, we consider the following situation: T is any u-curve T that crosses $\bar{U}$ and $J \subset T \cap D(0)$ is a smaller compact u-curve. We choose any long disjoint s-leaves at the endpoints z_1 and z_2 of J, L_1 and L_2.

LEMMA 3.2.3. *Under this setting:*

(i) *for any $z \in B(J, 2r)$ and any long s-leaf $L(z)$ centred at z and tangent to $\mathcal{C}^s(n_c - 1)$, $L(z)$ intersects T;*

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(ii) *for any* $z \in B(J, 2r) \setminus (L_1 \cup L_2)$ *there exists a long s-leaf* $L(z)$ *centred at* z *with respect to* $\mathcal{C}^s(n_c - 1)$ *that does not intersect* $L_1 \cup L_2$*;*

(iii) *if* $L(z)$ *and* $L'(z)$ *are two long s-leaves as in* (ii)*, and* $\{w\} = L(z) \cap T$, $\{w'\} = L'(z) \cap T$,

$$w \in J \Leftrightarrow w' \in J.$$

Proof. Note that the stable and unstable cones of order $n_c - 1$ at points near z are near the cones of order $n_c - 1$ at z by continuity of the cone fields. The angles between stable and unstable vectors are bounded from below. As a consequence the u-curves and s-leaves of radius ε are near lines and hyperplanes of $\mathbb{R}^d$. So any u-curve centred at z' and s-leaf centred at z intersect if z' is at a distance smaller than $\chi\varepsilon$ from z for some uniform constant $0 < \chi < 1$ that can be chosen arbitrarily close to 1. This proves the first assertion.

For (ii) we cover $\bar{U}$ by local charts with flat metric. On them we chose linear coordinates $(x_u, y_s) \in \mathbb{R} \times \mathbb{R}^{d-1}$ compatible with the cone fields $\mathcal{C}^u(n_c)$ and $\mathcal{C}^s(n_c)$. If r is small enough, the ball centred at any point in $\bar{U}$ and with radius $2r$ is contained in a chart. One can subdivide the component of $T \cap \bar{V}$ containing J into a finite number of curves $[z(0), z(1)], [z(1), z(2)], \ldots, [z(n-1), z(n)]$ such that $z_1 = z(i_1)$ and $z_2 = z(i_2)$. We require that all the segments $[z(i), z(i+1)]$ (except $[z(i_1), z(i_1+1)]$ if $z(i_1+1) = z(i_2)$) have length larger than r. Choose for each $z(i)$ a long s-leaf $L(i)$ centred at $z(i)$. ($L_1 = L(i_1)$ and $L_2 = L(i_2)$.) They are graphs of the form $\{(h_i(y_s), y_s)\}$.

If r is small enough and n_c is large enough, all the s-leaves $L(i)$ are disjoint and the graphs $\{(th_i(y_s) + (1-t)h_{i+1}(y_s), y_s)\}$ for $t \in [0, 1]$ and $0 \le i < n$ are s-leaves tangent to $\mathcal{C}^s(n_c - 1)$ that define a foliation of $B(J, 2r)$.

We prove (iii) by contradiction. Suppose that $w \in J$, $w' \in T \setminus J$. We get $w' \in B(w, 4r)$ and, by (7),

$$\text{there exists } i \in \{1, 2\}, \quad \text{such that } w, w' \in B(z_i, 5r).$$

$B(z_i, 8r) \setminus L_i$ has two connected components. Thus $L(z)$ and $L'(z)$ are contained in different components and cannot intersect at z. □

By (iii), one defines for any $0 < \varepsilon \le r$ the *rectangle* $R_\varepsilon(J, L_1, L_2)$ as the set of points $z \in B(J, \varepsilon)$ that project on J: either $z \in L_1 \cup L_2$ or $L(z) \cap T \subset J$ for any $L(z)$ as in (ii). When $\varepsilon = r$ we only write $R_\varepsilon(J, L_1, L_2)$.

We will say that a rectangle $R_\varepsilon(J, L_1, L_2)$ is of *order* $n \ge 0$ if L_1 and L_2 are of order n and $J \subset D(n)$. In order to iterate the rectangle, we first consider some long s-leaves L_1^n and L_2^n containing $F^n(z_1)$ and $F^n(z_2)$ and take inductively the long s-leaves L_i^j as a subset of $F^{-1}(L_i^{j+1})$, $i = 1, 2$, $0 \le j < n$, to construct the rectangle $R(F^j(J), L_1^j, L_2^j)$. In this case and as a consequence of §3.2.5 property (2) and Lemma 3.2.3(ii), for any $0 < \varepsilon < r$, $F(R_\varepsilon(J, L_1, L_2))$ is contained in $R_{(3\beta/2\sin(\alpha))\varepsilon}(F(J), L_1^1, L_2^1)$ and in $R_{\varepsilon/16}(F(J), L_1^1, L_2^1)$ (recall equation (2)).

3.2.7. As in §3.2.6, $R(J, L_1, L_2)$ is a rectangle with $J \subset T$. For any u-curve T' with endpoints outside $B(\bar{V}, r)$, such that $T' \subset B(T, r)$, the intersection $T' \cap R(J, L_1, L_2)$ is either empty or a segment whose endpoints are in L_1 and L_2. If it is not empty it will be called a *section* of $R(J, L_1, L_2)$.

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By compacity, if r is small and n_c large enough, we also get the following property: there exists $r/4 < r' < r/2$ such that for any u-curve T crossing $\bar{U}$ and any u-curve $\sigma \subset B(\bar{V}, r_0/2)$, if $\sigma \cap B(T, r') \neq \emptyset$, then $\sigma \subset B(T, r/2)$. Thus, for any rectangle $R(J, L_1, L_2)$, any u-curve T' which crosses $\bar{U}$ and intersects $R_{r'}(J, L_1, L_2)$ contains a section of $R(J, L_1, L_2)$.

A section S will be called a ε-section when $S \cap R_\varepsilon(J, L_1, L_2)$ is non-empty and $0 < \varepsilon \leq r'$.

Moreover, we will need to consider $r'/4 < r_2 < r'/2$ (respectively $r_2/4 < r_3 < r_2/2$) so that any r_2-section (respectively r_3-section) of any $R(J, L_1, L_2)$ is contained in $R_{r'/2}(J, L_1, L_2)$ (respectively $R_{r_2/2}(J, L_1, L_2)$).

Note also that when $R(J, L_1, L_2)$ is of order $n \geq 0$ and σ is a section of $R(J, L_1, L_2)$, $F^k(\sigma)$ is a section of $R(F^k(J), L_1^k, L_2^k)$, $0 \leq k \leq n$.

3.3. *Length of a rectangle.* A useful distance d_L between L_1 and L_2 is defined by

$$d_L(L_1, L_2) = \sup_\sigma |\sigma|,$$

where the supremum is taken over all sections of $R(J, L_1, L_2)$.

The aim of the next proposition is to prove by a standard argument that any section gives more or less the distance $d_L(L_1, L_2)$, and that this distance is independent of the choice of the manifolds L_i.

PROPOSITION 3.3.1. *There exists $\Delta > 1$ such that for any integer $n \geq 0$, for any rectangle $R(J, L_1, L_2)$ of order n such that $F^{n+1}(J)$ has one endpoint in $B(\partial\bar{U}, r)$, for any section σ,*

$$\frac{1}{\Delta}|J| < |\sigma| < \Delta|J|.$$

We state before the proof a classical λ-lemma which controls iterations of unstable cones (see for example [**BR**, Appendix A]).

LEMMA 3.3.2. *Consider any Riemannian norm on the unitary bundle $\mathrm{U}M$ inducing a distance $\tilde{d}$. There exist constants $0 < \lambda < 1$ and $c' > 0$ such that for any integer $n \geq 0$, for any two points ζ, ξ on the same long s-leaf tangent to $\mathcal{C}^s(n_s - 1)$ of order greater than n and any $v_\zeta \in \mathcal{C}^u_\zeta(n_c) \setminus \{0\}$, $v_\xi \in \mathcal{C}^u_\xi(n_c - 1) \setminus \{0\}$, we have*

$$\tilde{d}(DF^n(\zeta).v_\zeta, DF^n(\xi).v_\xi) \leq c'\lambda^n.$$

Proof of Proposition 3.3.1. Our strategy is to construct a Lipschitz map from σ to J with uniform constant Δ.

We claim that, for any rectangle $R(J, L_1, L_2)$ of order zero and any section σ, one can define a Lipschitz homeomorphism ϕ from J to σ with some uniform Lipschitz constant τ such that, for any $z \in J$, the points z and $\phi(z)$ are on the same long s-leaf L_z tangent to $\mathcal{C}^s(n_c - 1)$.

Consider now the general case: $R(J, L_1, L_2)$ is of order n. Using the claim we define $\phi_n : F^n(J) \to F^n(\sigma)$ such that, for any $z \in F^n(J)$, the points z and $\phi_n(z)$ are on the same long s-leaf L_z^n. As above we get for each $0 \leq k \leq n$ a long s-leaf L_z^k. We denote $v(z)$

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to be a unitary vector tangent at $F^k(J)$ or $F^k(\sigma)$ for $z \in F^k(J)$ or $z \in F^k(\sigma)$. The map $\phi_0 : J \to \sigma$ defined by

$$\phi_0(z) = (F^{-n} \circ \phi_n \circ F^n)(z)$$

is a homeomorphism. Its Lipschitz constant is bounded by

$$\max_{z \in J} \|\mathrm{D}F^n(\phi_0(z)).v(\phi_0(z))\|^{-1} \operatorname{Lip} \phi_n \|\mathrm{D}F^n(z).v(z)\|.$$

We decompose

$$\|\mathrm{D}F^n(\phi_0(z)).v(\phi_0(z))\|^{-1} \|\mathrm{D}F^n(z).v(z)\|$$

into a product of the following factors indexed by $0 \leq k \leq n-1$:

$$\|\mathrm{D}F(F^k \circ \phi_0(z)).v(F^k \circ \phi_0(z))\|^{-1} \|\mathrm{D}F(F^k(z)).v(F^k z)\|.$$

Using Lemma 3.3.2 we bound each of them by

$$\exp(c\tilde{d}(v(F^k \circ \phi_0(z)), v(F^k(z)))) \leq \exp(cc'\lambda^k).$$

Hence $\operatorname{Lip} \phi$ is finite and bounded by

$$\Delta = \tau \exp\left(cc' \frac{1}{1-\lambda}\right).$$

In the same way $\operatorname{Lip} \phi^{-1} \leq \Delta$ and the proposition is proved.

It remains to prove the claim. Recall that $|J| > r_0$ and, as in the proof of Lemma 3.2.3(ii), we can subdivide J and σ into a finite number of curves $[z_i, z_{i+1}]$ and $[z'_i, z'_{i+1}]$ of length larger than r and contained in the same chart. We also construct a foliation of $R(J, L_1, L_2)$ by s-leaves: z_i and z'_i are on the same leaf H_i and the other ones are obtained by linear interpolation.

The holonomy of the foliation defines a homeomorphism $\phi : J \to \sigma$. For each i, the lengths of $[z_i, z_{i+1}]$ and $[z'_i, z'_{i+1}]$ are bounded from above and below. Thus ϕ is Lipschitz. □

Remark 3.3.1. If r is small enough and n_c is large enough, c' in Lemma 3.3.2 and $|\tau - 1|$ can be chosen arbitrarily small. Thus Δ is arbitrarily close to 1.

As a corollary we get a correspondence between the length of a rectangle and its volume.

COROLLARY 3.3.3. *There exists $\theta > 1$ such that for any rectangle $R(J, L_1, L_2)$ of order n as in Proposition 3.3.1 we have*

$$\text{Volume}(R_{r_3/2}(J, L_1, L_2)) \geq \theta |J|.$$

Proof. For small values of n, the length of J is bounded from below and the result is obvious. Thus we only have to consider the case when n is large. L_1 and L_2 are close together.

Recall that the volume of the ball centred at J in L_1 of radius $r_3/2$ and the angle between vectors of $\mathcal{C}^s$ and $\mathcal{C}^u$ are uniformly bounded from below. The volume is computed by integration along the unstable direction and the previous proposition ends the proof. □

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3.4. *Combinatorics.*

3.4.1. For any long u-curve T and any $n \in \mathbb{N}$ we define

$$E_n(T) = T \cap D(n).$$

$\mathcal{E}_n(T)$ is the set of all connected components of $E_n(T)$ intersecting K. The union of those sets over all curves T is denoted as $\mathcal{E}_n$.

The following proposition is an easy consequence of Definition 2.4.2(4) and Remark 2.4.3(2).

PROPOSITION 3.4.1.

(i) *Let T be any long u-curve. Then, for any $n \geq m$, $E_n(T) \subset E_m(T)$. If $J \in \mathcal{E}_{m+1}(T)$, there exists $J' \in \mathcal{E}_m(T)$ such that $F(J) \subset J'$.*

(ii) *For all integers $n \geq 0$ and all $J \in \mathcal{E}_n$, $F^{n+1}(J)$ contains a long u-curve.*

Proof. For (i), Lemma 3.2.1 shows that the connected part of $F(J \cap \bar{U})$ that contains $F(J)$ is a long u-curve.

For (ii), we get by formula (6)

$$\partial J \subset \partial F^{-n-1}(\Omega) \cup \partial U \cup \cdots \cup \partial F^{n+1}(\bar{U}).$$

Remark 2.4.3(2) implies then that

$$\partial J \subset \partial F^{-n-1}(\Omega) \cup \partial F^{n+1}(\bar{U}). \qquad \square$$

3.4.2. Consider u-curves $J \subset D(n)$ and $J' \subset D(m)$ for some $m \leq n$ and $0 < \varepsilon \leq r'$. If there exist $\zeta \in J$ and $\xi \in J'$ with ζ and ξ on an s-leaf of diameter smaller than ε and of order m, we will say that J' *ε-covers* J. When $\varepsilon = r'$ we will only say that J' covers J.

PROPOSITION 3.4.2. *Any integer $n \in \mathbb{N}$ and any curve $J \in \mathcal{E}_n$ with endpoints z_i, $i \in \{1, 2\}$, satisfy the following properties:*

(i) *any long s-leaves L_i of order n through z_i, $i \in \{1, 2\}$, are disjoint;*

(ii) *suppose that $J' \in \mathcal{E}_m$, $1 \leq m < n$, covers J and that $F^{m+1}(J')$ does not intersect $\bar{\Omega}$; then for any choice of long s-leaves L'_j with order n at the endpoints of J', J is contained in $R(J', L'_1, L'_2)$ and the leaves L_i and L'_j are disjoint.*

Proof. We first remark that the s-leaves L_i have their first n iterates of diameter less than r. Moreover, the distance between $F^n(z_1)$ and $F^n(z_2)$ is larger than r_0 by (5). This implies the first part of the proposition.

To prove the second part we consider the long u-curve $F^{m+1}(J')$ and the u-curve $F^{m+1}(J)$. Note that $F^{m+1}(J) \subset D^0$. $F^{m+1}(L_i)$ and $F^{m+1}(L'_i)$ are s-leaves of diameter bounded by $10r$. $\bar{\Omega} \cap F^{m+1}(J') = \emptyset$. Hence the endpoints of J' are in $\partial \bar{U}$. The leaves $F^{m+1}(L'_i)$ are contained in $B(M \setminus U, r)$ and the leaves $F^{m+1}(L_i)$ are inside $B(V, r)$. Thus they are disjoint (recall (5)). This implies that L_i and L'_i are disjoint. Take any u-curve S containing J and crossing $\bar{U}$. It also contains a section σ of $R(J', L'_1, L'_2)$ and $\sigma \cap J$ is non-empty. $F^{m+1}(\sigma)$ has its endpoints outside $\bar{V}$. Thus $J \subset \sigma$. Hence J is contained in $R(J', L'_1, L'_2)$. $\square$

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PROPOSITION 3.4.3. *Under the previous assumptions, if $J' = F^j(J)$ with $j \leq n-1$ and if J' intersects $R_{r_2}(J, L_1, L_2)$, there exists a fixed point of F^j in $R_{r'}(J, L_1, L_2)$.*

Proof. By Proposition 3.4.2, J' defines a section γ of $R_{r'/2}(J, L_1, L_2)$ (recall §3.2.7). The curve $F^j(\gamma)$ is a section of $R_{16^{-j}r'/2}(J', L_1', L_2')$ (recall the end of §3.2.6) and contains a section of the rectangle $R_{(16^{-j}+1)r'/2}(J, L_1, L_2)$.

By induction any iterate l of γ by F^j contains a section γ_l of $R_{r_l}(J, L_1, L_2)$ where

$$r_l = \frac{r'}{2}(1 + 16^{-j} + (16)^{-2j} + \cdots + (16)^{-lj}) < r'.$$

Hence γ_l is a curve contained in $R_{r'}(J, L_1, L_2)$. The sequence of those iterates converges to a Lipschitz curve $\bar{\gamma}$ with $\bar{\gamma} \subset F^j(\bar{\gamma})$ and $\bar{\gamma}$ contains a fixed point. □

3.4.3. Note that if $R(J)$ and $R(J')$ are some rectangles with J and J' as in §3.4.2 and if there is no r_3-covering relation between J and J' then $R_{r_3/2}(J)$ and $R_{r_3/2}(J')$ are disjoint. As a corollary to the previous section we get the following.

COROLLARY 3.4.4. *There exists a constant $k_v > 0$ such that for any rectangle of order $n \geq 0$, $R(J, L_1, L_2)$, and with no r_3-covering relations between the iterates $0 \leq k_1 \leq k_2 \leq \cdots \leq k_m \leq n$ of J then*

$$\sum_{j=0}^{m} |F^{k_j}(J)| < k_v.$$

Proof. By assumptions the rectangles $R_{r_3/2}(F^{k_j}(J), L_1^{k_j}, L_2^{k_j})$ are all disjoint and of volume larger than $\theta|F^{k_j}(J)|$ for $0 \leq j \leq m$. We deduce that

$$\sum_{j=0}^{m} |F^{k_j}(J)| < \frac{\text{Volume}(\bar{U})}{\theta}.$$ □

3.5. *No wandering u-curve.* We set $e_n = \max_{J \in \mathcal{E}_n} |J|$.

PROPOSITION 3.5.1.

$$e_n \xrightarrow[n \to \infty]{} 0.$$

It implies that K contains no u-curve.

Proof. We prove the proposition by contradiction: suppose that the proposition is false. One can find some $\varepsilon > 0$ and some infinite sequence $(\bar{J}_{n_i})$ with (n_i) strictly increasing, $\bar{J}_{n_i} \in \mathcal{E}_{n_i}$ and $|\bar{J}_{n_i}| > \varepsilon$ for any i. Any accumulation point for $(\bar{J}_{n_i})$ is a curve $\bar{J}$ of length larger than ε. Moreover, for all $k \in \mathbb{N}$, $\bar{J}$ is included in $D(k)$. One deduces that through each point z of J there exists a unique long s-leaf W_z^s of infinite order.

LEMMA 3.5.2. *Up to performing a change on the sequence $(\bar{J}_{n_i})$, one can form a new sequence (J_n) with the following additional property: there exists a u-curve T crossing $\bar{U}$ such that $J_n \subset T$ for all n.*

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Proof. If $\bar{J}_{n_i}$ and $\bar{J}$ are close enough, $\bar{J}$ projects by holonomy with respect to the foliation $W^s_{\bar{J}}$ on a subcurve $\bar{J}' \subset \bar{J}_{n_i}$ not reduced to a point and every point in $\bar{J}'$ remains in D by forward iterations. Consider now any u-curve T containing $\bar{J}'$ and for each n the element J_n of $\mathcal{E}_n(T)$ containing $\bar{J}'$. □

The sequence (J_n) is decreasing and converges to a u-curve $J \subset T$.

Now, we claim that there is no r_2-covering between iterates k and j of J: we form the rectangles $R(F^k(J), L_1^k, L_2^k)$ where L_1^k and L_2^k are subsets of the stable manifolds of the endpoints of $F^k(J)$. Consider some u-curves $T_k \subset F^k(T)$ containing $F^k(J)$ and crossing $\bar{U}$. Let us assume that $F^k(J) \cap R_{r_2}(F^j(J)) \neq \emptyset$ with $k - j > 0$. Then T_k defines a section S_k of $R(F^j(J), L_1^j, L_2^j)$. As $F^n(J) \subset D$ for any $n \in \mathbb{N}$ and as F contracts uniformly in the unstable direction, we get $F^n(S_k) \subset D$ for any $n \in \mathbb{N}$. Among the subsegments of T which remain in D by forward iterations, $F^k(J)$ is maximal for the inclusion. Hence $S_k \subset F^k(J)$.

Symmetrically T_j contains a section S_j of $R(F^k(J), L_1^k, L_2^k)$ and $S_j \subset F^j(J)$. Recall that L_i^j and $L_{i'}^k$ are disjoint or contained in a same s-leaf, $i, i' \in \{1, 2\}$. We deduce that in fact $S_j = F^j(J)$ and $S_k = F^k(J)$.

By the same argument as in Proposition 3.4.3, $R_{r'}(F^j(J), L_1^j, L_2^j)$ must contain a fixed point Q of F^{k-j}. By §3.2.3, Q must have an unstable branch which leaves $R(F^j(J), L_1^j, L_2^j)$ and intersects L_1^j or L_2^j in a fixed point for $F^{2(k-j)}$. This contradicts the definition of the unstable manifold and proves the claim.

We finish the proof by some length estimate: we use Corollary 3.4.4 for each J_n. We get

$$\sum_{k\in\mathbb{N}} |F^k(J)| < k_v \quad \text{and} \quad |F^k(J)| \xrightarrow[k\to\infty]{} 0. \tag{9}$$

We use Lemma 3.1.2 with constant $2k_v$ and get some constant $\eta > 0$. Iterating J a little bit more we can assume that

$$\forall k \in \mathbb{N}, \quad |F^k(J)| < \frac{r}{2+\eta}.$$

Take any u-curve $S \subset T$ with $J \subsetneq S$ of length smaller than $\min(2, 1+\eta)|J|$. We prove by induction on k that

$$\forall k \in \mathbb{N}, \quad F^k(S) \subset B(\bar{V}, r) \quad \text{and} \quad |F^k(S)| < 2|F^k(J)|.$$

We suppose it is true for all integers k with $k < k_0$ and prove it for k_0. The inductive assumption and (9) imply

$$\sum_{k=0}^{k_0-1} |F^k(S)| < 2k_v.$$

We also have $|S| < (1+\eta)|J|$ and by Lemma 3.1.2, we get

$$|F^{k_0}(S)| < 2|F^{k_0}(J)| < r.$$

$F^{k_0}(S)$ contains a point of $\bar{V}$. Hence $F^{k_0}(S) \subset B(\bar{V}, r)$ and our claim is proved.

No forward iterate $F^k(S)$ can intersect Ω: if not $F^k(S)$ would contain a point in W^s_P and a point outside W^s_P. A further iterate of S would be of length larger than r_0 by §3.2.3.

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However,

$$\forall k \geq 0, \quad |F^k(S)| < \frac{2r}{2+\eta}.$$

We get a contradiction. This proves that for all k, $F^k(S) \subset D$. Thus for all n, $S \subset J_n$. This contradicts the definition of J (recall $J \subsetneq S$). □

3.6. *Growth of derivatives: the case of periodic orbits.*

PROPOSITION 3.6.1. *There exists $c > 0$ such that for any periodic orbit $O(z)$ in U of minimal period q, and not equal to $\{P\}$, and for any unstable vector v at z,*

$$\|DF^q(z).v\| \geq \frac{c}{e_q}\|v\|.$$

Proof. We first fix such a periodic orbit O and remark that it cannot intersect Ω so that it is contained in O. Hence O is a hyperbolic orbit of K. For any point z_i, $1 \leq i \leq q$, of the orbit let us take as u-curve that crosses $\bar{U}$, a part T_i of the unstable manifold of z_i containing z_i. The corresponding element of $\mathcal{E}_0$ is called $J(i)$ and we fix any rectangle $R(i) = R(J(i), L_1(i), L_2(i))$ associated to $J(i)$. $L(i)$ is the long leaf centred at z_i contained in $W^s_{z_i}$. As in the proof of Lemma 3.2.3 for any $z_i \in R_{r'}(j)$, $J(j)$ and $L(i)$ intersect at a unique point $\tilde{z}_i$. We think of it as the projection of z_i on $J(j)$. $\tilde{z}_i$ lies on the unstable manifold of a periodic orbit and must stay in $\bar{V}$ under backward iterations (Lemma 3.2.1). It must also remain at a distance less than r from K by forward iterations and thus is contained in K.

LEMMA 3.6.2. *There are some points $z_{i_0}, \ldots, z_{i_k}$ in O and a q-periodic map $\phi : \mathbb{Z} \to \{i_0, i_1, \ldots, i_k\}$ such that the following properties (C1)–(C3) are satisfied:*

$$\forall 0 \leq i < q-1, z_i \in R_{r'}(\phi(i)), \quad \forall 0 \leq j \leq k, \phi(i_j) = i_j, \tag{C1}$$

$$\forall 0 \leq j < j' \leq k, z_{i_{j'}} \notin R_{r_3}(i_j). \tag{C2}$$

For any $0 \leq j \leq k$ and any $i \in \phi^{-1}(i_j)$ one defines $J(F(\tilde{z}_i))$ to be the element of $\mathcal{E}_0$ contained in $F(T_{i_j})$ and containing $F(\tilde{z}_i)$. We consider the largest segment $\tilde{\gamma}_i \subset J(i_j)$ containing $\tilde{z}_i$ with the following property: for any $0 \leq s \leq q-1$ such that $J(F(\tilde{z}_i)) \subset B(T_s, r')$ and $F(\tilde{\gamma}_i) \cap R(s) \neq \emptyset$ then $\tilde{\gamma}_i \subset R(s)$. We will say that z_i and $z_{i'}$ are bound *if $\phi(i) = \phi(i')$ and $\tilde{\gamma}_i = \tilde{\gamma}_{i'}$:*

$$z_i \text{ and } z_{i'} \text{ are bound together } \Rightarrow \phi(i+1) = \phi(i'+1). \tag{C3}$$

Proof. We will construct a map ϕ which satisfies two additional properties:

$$\forall 0 \leq i \leq q-1, \ 0 \leq \phi(i) \leq i. \tag{C4}$$

$$\begin{aligned} &\text{If } z_i (0 \leq i \leq q-2) \text{ is not bound to any } z_{i'}, \\ &\text{with } 0 \leq i' < i, \text{ then } z_{i+1} \in R_{r_2}(\phi(i+1)). \\ &\text{More precisely if } z_{i+1} \in R_{r_2}(0) \text{ then } \phi(i+1) = 0. \end{aligned} \tag{C5}$$

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ϕ is defined inductively on i. We first set $z_{i_0} = z_0$ and $\phi(0) = 0$. Let us assume now that $\phi(i)$ is defined for $0 \le i \le n-1$ ($n < q$) with values $0 = i_0 < \cdots < i_l < q-1$. Then $\phi(z_n)$ is defined in the following way.

There are three cases:

(1) z_{n-1} is bound to another z_i, $0 \le i < n-1$; then $\phi(n) = \phi(i+1)$;

(2) z_{n-1} is bound with no z_i $0 \le i < n-1$, and for any $0 \le j \le l$, $z_n \notin R_{r_2}(i_j)$; we then set $i_{l+1} = n$, $\phi(n) = n$;

(3) z_{n-1} is bound with no z_i but $z_n \in R_{r_2}(i_j)$ for some $0 \le j \le l$; then $\phi(n) = i_j$. There may be several possible choices. However, if $z_n \in R_{r_2}(0)$ then $\phi(n) = 0$.

First one checks conditions (C1)–(C5) by induction. Note that $\phi(n)$ is well-defined in case (1): the binding is an equivalence relation and $\phi(i+1)$ does not depend on the choice of z_i. By condition (C5) one can assume $z_{i+1} \in R_{r_2}(\phi(n))$. Thus the unstable manifold of z_{i+1} defines a section of $R_{r'/2}(\phi(n))$. On the other hand, the unstable manifold of z_i defines a section of $R_{r/2}(\phi(i))$. This implies that z_{n-1} is at a distance less than $r' + (r/2)$ from this section. Hence, z_n is at a distance less than $(r' + (r/2))/16$ from the unstable manifold of z_{i+1}. Recall that $r/4 < r'$. We deduce that the distance between z_n and the unstable manifold of $z_{\phi(n)}$ is bounded by $(r'/2) + (r'/8) + (r'/4)$. Thus $z_n \in R_{r'}(\phi(n))$.

One can check easily that the other conditions (C2)–(C5) are satisfied.

One completes the definition of ϕ by periodicity. It remains to prove that if z_i and z_{q-1} ($0 \le i < q-1$) are bound then $\phi(i+1) = \phi(q)$. The index i can be taken to be minimal so that z_i is bound with no $z_{i'}$ ($0 \le i' < i$).

Note that since z_i and z_{q-1} are bound together, z_i is at a distance less than $r' + (r/2)$ from the unstable manifold of z_{q-1}. Thus z_{i+1} is at a distance less than $(r' + (r/2))/16$ from the unstable manifold of z_0. We get $z_{i+1} \in R_{r_2}(0)$ and by (C5) $\phi(i+1) = 0$. Hence $\phi(i+1) = \phi(q) = 0$. □

LEMMA 3.6.3. *The number of points $z_{i_0}, \ldots, z_{i_k}$, is bounded by some integer N which does not depend on the choice of the orbit O.*

Proof. The volume of each $R_{r_3/2}(i_j)$ is uniformly bounded from below by θr_0 (recall Corollary 3.3.3 and $|J(i)| > r_0$). Thus it is enough to bound from above $\sum \text{Volume}(R_{r_3/2}(i_j))$.

For each $0 \le j \le k$ we bound the number of rectangles $R_{r_3/2}(i_{j'})$, $j < j' \le k$, intersecting $R_{r_3/2}(i_j)$: recall that T_j is a u-curve T_j with length less than k_l which crosses $\bar{U}$ and contains $J(i_j)$. Assuming that $R_{r_3}(i_j)$ intersects $R_{r_3}(i_{j'})$, T_j contains a section $I_{j'}$ of $R(i_{j'})$. The endpoints of $I_{j'}$ are in $B(\partial\bar{U}, r) \cup \bar{\Omega}$. If $R_{r_3/2}(i_j)$ intersects another rectangle $R_{r_3/2}(i_{j''})$ with $j'' > j'$, the projection of $z_{i_{j''}}$ on T_j is not in $I_{j'}$ (condition (C2)) and must be at a distance at least r_0 from $I_{j'}$. We deduce that at most k_l/r_0 rectangles $R_{r_3/2}(i_{j'})$, $j < j' \le k$, can intersect $R_{r_3/2}(i_j)$.

Now any point in $\bar{U}$ is at most in k_l/r_0 rectangles $R_{r_3/2}(i_j)$. Hence we have bound $\sum \text{Volume}(R_{r_3/2}(i_j))$ by $(r_0 \,\text{Volume}(\bar{U}))/k_l$. □

Now we prove the proposition. $R(0) \setminus L(0)$ has two components. If r is small enough, one of them does not intersect $F^{-1}(\bar{\Omega})$ (recall Definition 2.4.2(4)). We denote Q as the corresponding endpoint of $J(0)$.

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We consider the point z_i with $\phi(i) = 0$ whose projection $\tilde{z}_i$ is as close as possible to Q. Q_i is the endpoint of $J(i)$ corresponding to Q. We define $\sigma_i \subset [z_i, Q_i]$ to be the largest segment of $J(i)$ that contains z_i with the following property: for any $0 \leq s \leq q-1$, if $\sigma_i \subset B(T_s, r)$ and $\sigma_i \cap R(s) \neq \emptyset$, then σ_i is contained in $R(s)$.

One leaf $L'(i) = L_1(s)$ or $L_2(s)$, for some s, contains one endpoint of σ_i and defines the rectangle $S(i) = R(\sigma_i, L(i), L'(i))$. Then T_0 contains a section $\tilde{\sigma}_i$ of $S(i)$. By construction $F(L'(i))$ has one point in $\partial\bar{U}$ and

$$|\sigma_i| > \frac{r_0}{\Delta}.$$

By induction one defines, for $1 \leq s \leq q$,

$$S(i-s) = F^{-1}(S(i-s+1)) \cap R(\phi(i-s)).$$

If $F^{-1}(S(i-s+1))$ intersects some rectangle $R(j)$, it cannot intersect $L_1(j) \cup L_2(j)$. Indeed this would imply either that $S(i-s+1) \cap B(\partial\bar{U}, r) \neq \emptyset$ (but $S(i-s+1) \subset B(\bar{V}, r)$) or that $F^{-1}(S(i-s+1))$ intersects $F^{-1}(\bar{\Omega})$ (but $S(i)$ does not intersect $\bar{\Omega}$). This proves that $S(i-s)$ is a subrectangle of $R(\phi(i-s))$.

It also implies that $J(\phi(i-s))$ induces a section $\tilde{\sigma}_{i-s}$ of $S(i-s)$ (one endpoint is $\tilde{z}_{i-s}$) and that $\sigma_{i-s} = F^{-s}(\sigma_i)$ is a section of $S(i-s)$ which contains z_{i-s}.

Now we prove that, for $1 \leq s \leq q$, $\tilde{\sigma}_{i-s} \subset \tilde{\gamma}_{i-s}$. Let us assume that, for some $0 \leq j \leq q-1$, $F(J(\tilde{z}_{i-s})) \subset B(T_s, r')$ and $F(\tilde{\sigma}_{i-s}) \cap R(j) \neq \emptyset$. If $s > 1$, as $S(i-s+1) \subset F^{-1}(S(i-s+2))$ does not intersect $L_1(j)$ nor $L_2(j)$ we get that $F(\tilde{\sigma}_{i-s}) \subset R(j)$. If $s = 1$ and $F(\tilde{\sigma}_{i-1}) \not\subset R_{r/2}(s)$, then σ_i would intersect $R(s)$ and its complementary and would be contained in $B(T_s, r)$. This contradicts the definition of σ_i. Hence $F(\tilde{\sigma}_{i-1}) \subset R(s)$.

Assuming that $\phi(i-s) = \phi(i-s') = j$, $1 \leq s' < s \leq q$, one shows by contradiction that the rectangles $S(i-s)$, $S(i-s') \subset R(j)$ are disjoint: if not, as $L(i-s)$, $L(i-s')$, $L'(i-s)$ and $L'(i-s')$ are disjoint $\tilde{\sigma}_{i-s}$ and $\tilde{\sigma}_{i-s'}$ intersect each other. Thus z_{i-s} and $z_{i-s'}$ are bound together. By condition (C3), $\phi(i-s+1) = \phi(i-s'+1)$ and the rectangles $S(i-s+1)$ and $S(i-s'+1)$ intersect each other. By induction one shows that $\phi(i-s+s') = 0$ and $S(i-s+s')$ intersects $S(i)$ (ϕ is q-periodic). By our choice of z_i, $z_{i-s+s'}$ is not in $S(i)$. Hence z_i is in $S(i-s+s')$; z_i and $z_{i-s+s'}$ are bound together; $S(i-s+s')$ intersects $S(i-q)$. Repeating the same argument we show that $\phi(i+s-s') = 0$ and $z_{i+s-s'}$ is in $S(0)$ which is a contradiction.

One gets for any $o \leq j \leq k$ that $|J(i_j)| \leq k_l$ and

$$\sum_{1 \leq s \leq q} |\tilde{\sigma}_{i-s}| < N k_l$$

and by Proposition 3.3.1

$$\sum_{1 \leq s \leq q} |\sigma_{i-s}| < N k_l \Delta.$$

As $|\sigma_i| > r_0$ and $F^q(\sigma_{i-q}) = \sigma_i$, and also using that $z_i \in \sigma_{i-q}$ and $\sigma_{i-q} \subset E_q$, we get by Lemma 3.1.2 for any unstable unitary vector v at z_i,

$$\|\mathrm{D}F^q(z_i).v\| \geq \exp(-2c_1 N k_l \Delta) \frac{|\sigma_i|}{|\sigma_{i-q}|} \geq \exp(-2c_1 N k_l \Delta) \frac{r_0}{e_q}. \qquad \square$$

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3.7. *Growth of derivatives: proof of Theorem 1.0.4.* Recall that we have fixed a small neighbourhood W of P.

By Proposition 3.6.1, F has only a finite number of periodic orbits in K with all their eigenvalues in $B(0,2)$. Hence there exists $1 < \lambda$ such that, for every periodic orbit of K different from $\{P\}$, the module of the unstable eigenvalue is larger than λ^3. Once one knows that the unstable eigenvalues are bounded from below one can require a new condition on the choice of the constants r and n_c. By Remark 3.3.1, Δ can be chosen to be close to 1:

$$1 < \Delta^2 < \lambda.$$

LEMMA 3.7.1. *There exists $m_1 > 0$ which satisfies the following property. Consider any q-periodic orbit $O(z) \neq \{P\}$ and T the long u-curve through z contained in $W^u(z)$. Take for some $n \geq m_1 + q$, $J \in \mathcal{E}_n(T)$ containing z. Assume also that there is no r_3-covering between $J, F(J), \dots, F^{q-1}(J)$. Then, for any $z' \in J$ and any unitary vector w' tangent to J at z',*

$$\|\mathrm{D}F^q(z').w'\| \geq \lambda^2. \tag{10}$$

Proof. We first assume only that $J \in \mathcal{E}_n(T)$ for some $n \geq q$. For any unitary vector w tangent to J at z,

$$\|\mathrm{D}F^q(z').w'\| \geq \|\mathrm{D}F^q(z).w\| \exp\left(-2c_1 \sum_{k=0}^{q-1} |F^k(J)|\right).$$

Since there is no r_3-covering between iterates of J,

$$\sum_{k=0}^{q-1} |F^k(J)| \leq k_v.$$

If the orbit satisfies

$$\|\mathrm{D}F^q(z).w\| > \exp(+2c_1 k_v)\lambda^2, \tag{11}$$

we easily get (10). Only a finite number of periodic orbits do not satisfy (11). Their periods are bounded by some q_0. We consider an integer m_1 such that

$$\exp(2c_1 q_0 e_{m_1-1}) < \lambda$$

and some orbit $O(z)$ with period bounded by q_0. Recall that

$$\|\mathrm{D}F^q(z).w\| \geq \lambda^3.$$

We deduce that if $n \geq q_0 + m_1 - 1$, $|J| + \cdots + |F^q| < q_0 e_{m_1-1}$ and this implies (10). □

We also prove the following.

LEMMA 3.7.2. *There exists some $m_2 \geq 0$ such that, for any $n \geq m_1$, $z \in K$ and $J \in \mathcal{E}_{n+m_2}$ such that $z \in J$ and $F^n(z) \notin W$, we have $J \cap \bar{\Omega} = \emptyset$.*

Proof. We consider some integer $m_0 \geq 0$ such that

$$\forall n \geq m_0, \quad F^n(\bar{\Omega}) \subset W.$$

For $m_1 \geq m_0$ large enough the lemma is now a consequence of Proposition 3.6.1. □

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We set $n_1 = \max(m_1, m_2)$. Take $z_{-n_c}, \dots, z_0, z_1, \dots, z_n$ to be an orbit of F in K with $z_n \notin W$ and $v_{-n_c} \in \mathcal{C}^u_{z_{-n_c}}$. We set

$$v_k = \mathrm{D}F^{k+n_c}(z_{-n_c}).v_{-n_c}.$$

We fix a long u-curve T containing z_0 of order greater than n_c and tangent to v_0. J denotes an element of $\mathcal{E}_{n+n_1}(T)$ containing z_0. We also consider one rectangle $R(J, L_1, L_2)$ with L_1 and L_2 of order $n + n_1$ and its iterates $R(F^k(J), L_1^k, L_2^k)$ with $0 \le k \le n$. Note that from Lemma 3.7.2 $F^n(J)$ cannot intersect Ω.

LEMMA 3.7.3.

$$\sum_{m=0}^{n-n_1} |F^m(J)| \le \frac{k_v}{1-\lambda^{-1}}.$$

Proof. First we claim that if, for some $m < s \le l \le n - n_1$, $F^l(J)$ r_3-covers $F^m(J)$ and $F^s(J)$, then

$$\lambda |F^m(J)| < |F^s(J)|. \tag{12}$$

To prove the claim it is sufficient to assume that with l, m fixed s is minimal with those properties. By Proposition 3.4.2, we introduce a section $F^{s-m}(I) \subset F^s(T)$ of $R(F^l(J), L_1^l, L_2^l)$ containing $F^s(J)$. This implies by §3.2.7 that $F^{s-m}(I)$ is in $R_{r_2/2}(F^l(J), L_1^l, L_2^l)$. By Proposition 3.4.2, I is also contained in $R_{r_2/2}(F^l(J), L_1^l, L_2^l)$. Thus $R_{r_2}(F^{s-m}(I), L_1^l, L_2^l)$ contains I. We also define the rectangle $R(I, L_1', L_2')$ such that $F^{s-m}(L_i') \subset L_i^l$, $i = 1, 2$. By Proposition 3.4.3 there exists in the rectangle $R_{r'}(I, L_1', L_2')$ a fixed point Q of the map F^{s-m}. Recall that $R(F^n(J), L_1^n, L_2^n)$ does not contain the point P so that $R(F^l(J), L_1^l, L_2^l)$ cannot contain P. Thus Q is different from P and Q is hyperbolic. $W^u(Q)$ defines a $r' + r_2$ section S of $R(I, L_1', L_2')$. We note also that there is no r_3-covering between the curves $I, F(I), \dots, F^{m-s-1}(I)$. Indeed this would imply by Proposition 3.4.2 that some $F^j(J)$, $m + 1 \le j \le s - 1$, is contained in $R(F^l(J), L_1^l, L_2^l)$ and contradict the minimality of s.

The section S is contained in a u-curve of $\mathcal{E}_{n-l}$ with $n - l \ge n_1 - 1$. Hence by (10) and the definition of λ, for any $z \in S$ and w any unitary vector tangent to I at z,

$$\|\mathrm{D}F^{s-m}(z).w\| > \lambda\Delta^2.$$

S contains a section S' of $R(F^m(J), L_1^m, L_2^m)$ and $F^{s-m}(S')$ is a section of $R(F^s(J), L_1^s, L_2^s)$. Thus

$$|F^{s-m}(S')| > |S'|\lambda\Delta^2$$

and by Proposition 3.3.1 the claim is proved.

We now prove the announced property. Consider $F^{n-n_1}(J), F^{n-n_1-1}(J), \dots, J$. By Proposition 3.4.2 there exist $n - n_1 = l_{j_0} > l_{j_0-1} > \cdots > l_1 \ge 0$ such that every $F^m(J)$, $0 \le m \le n - n_1$, is r_3-covered by some $F^{l_j}(J)$ and for $1 \le j < j' \le r$ there is no covering relation between $F^{l_j}(J)$ and $F^{l_{j'}}(J)$. The construction is done by induction on the sequence $(F^{n-n_1-k}(J))_k$.

Using the claim for any $1 \le j \le j_0$,

$$\sum_{F^m(J) \text{ covered by } F^{l_j}(J)} |F^m(J)| \le \frac{|F^{l_j}(J)|}{1-\lambda^{-1}}.$$

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Finally using Corollary 3.4.4 we get

$$\sum_{j=1}^{j_0} |F^{l_j}(J)| \leq k_v$$

and

$$\sum_{m=0}^{n-n_1} |F^m(J)| \leq \frac{k_v}{1-\lambda^{-1}}$$

which gives the announced bound. □

We now prove Theorem 1.0.4 for the map F and the open set W. We have the estimate

$$\sum_{m=0}^{n-1} |F^m(J)| \leq \frac{k_v}{1-\lambda^{-1}} + (n_1-1)k_l.$$

Thus

$$\begin{aligned}\|\mathrm{D}F^n(z_0).v_0\| &> \exp\left(-2c_1\left(\frac{k_v}{1-\lambda^{-1}} + (n_1-1)k_l\right)\right)\frac{|F^n(J)|}{|J|}\|v_0\| \\ &> \exp\left(-2c_1\left(\frac{k_v}{1-\lambda^{-1}} + (n_1-1)k_l\right)\right)\frac{r_0}{e_n}\|v_0\|.\end{aligned}$$

By Proposition 3.5.1, $\|\mathrm{D}F^n(z_0).v_0\|$ is arbitrarily large if n is large enough. The end of the proof is straightforward.

4. *Local estimates*

We saw in the previous section that at the time of the bifurcation F is expanding in the unstable direction. It is not clear, however, that a similar work can be carried out for a perturbed map. In this section we give another ingredient for proving expansion: we analyse the dynamics in a neighbourhood of P for the perturbed maps.

4.1. *Saddle-node bifurcation.* We now continue with the discussion of §2.2. Let $(F_\lambda)_{\lambda\in\Lambda}$ be a smooth family of diffeomorphisms of M. Here the parameter space Λ is an open subset of a finite-dimensional space. If P_{λ_0} is a saddle-node fixed point of F_{λ_0}, $(P_{\lambda_0}, \lambda_0)$ is a fixed point of the diffeomorphism

$$\begin{aligned}\tilde{F}: \quad M\times\Lambda &\to M\times\Lambda \\ (z,\lambda) &\mapsto (F_\lambda(z),\lambda).\end{aligned}$$

By [**HPS**], $\tilde{F}$ preserves locally a $(\dim\Lambda+1)$-dimensional central submanifold W^c tangent to $E^c_{P_0}\times T_{\lambda_0}\Lambda$ at $(P_{\lambda_0},\lambda_0)$. For $(z,\lambda)\in W^c$, W^c is transversal with respect to $M\times\{\lambda\}$ so that $W^c\cap(M\times\{\lambda\})$ is a submanifold of $M\times\{\lambda\}$.

Definition 4.1.1. $(P_{\lambda_0},\lambda_0)$ is said to be a *transversal saddle-node bifurcation* if:
(1) P_{λ_0} is a saddle-node fixed point of F_{λ_0};
(2) for some (and then any) central manifold W^c of $\tilde{F}$ at $(P_{\lambda_0},\lambda_0)$,

$$\frac{\partial}{\partial\lambda}\tilde{F}_{|W^c}(P_{\lambda_0},\lambda_0) \neq 0.$$

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By condition (2), the bifurcation persists in a neighbourhood of $(P_{\lambda_0}, \lambda_0)$ along a submanifold $\mathcal{O}^0$ of $\mathcal{O}$ of codimension one. Those bifurcations are described in [**NPT**] through their conjugacy classes. We now recall some basic facts about the local behaviour of the dynamics (see also Proposition 1.0.1).

- We can shrink $\mathcal{O}$ to a smaller neighbourhood of λ_0 so that $\mathcal{O} \setminus \mathcal{O}^0$ is the union of two connected open sets $\mathcal{O}^+$ and $\mathcal{O}^-$.
- There exists a neighbourhood W of P_{λ_0} such that:
  (1) for $\lambda \in \mathcal{O}^-$, F_λ has no fixed point in W and every point leaves W by backward and forward iterations;
  (2) for $\lambda \in (\mathcal{O}^0 \cup \mathcal{O}^+)$, F_λ has two fixed points P_λ and Q_λ in U. They are distinct for $\lambda \in \mathcal{O}^+$ and equal for $\lambda \in \mathcal{O}^0$. Moreover, P_λ and Q_λ vary continuously with λ. When $\lambda \in \mathcal{O}^0$, (P_λ, λ) is a transversal saddle-node bifurcation. When $\lambda \in \mathcal{O}^+$, P_λ and Q_λ are hyperbolic: P is a saddle point and Q is a sink.

The choice of the manifold W^c is now fixed.

4.2. *Local coordinates.* We choose local coordinates near $(P_{\lambda_0}, \lambda_0)$ in $M \times \Lambda$ so that a point (z, λ) will be represented by

$$(x, y, \lambda) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \Lambda$$

and $y = 0$ is an equation for W^c. It is possible to take $P_\lambda = (0, 0)$ for any $\lambda \in \mathcal{O}^0$:

$$x \mapsto F_\lambda(x, 0) = (f_\lambda(x), 0)$$

defines a family of local one-dimensional diffeomorphisms.

For any integer $k \geq 0$ one can C^k-conjugate (f_λ) to the following normal form†:

$$f_\lambda(x) = -c(\lambda) + x + x^2 + o(x^2).$$

Consider the map

$$(z, \lambda) \mapsto (F_\lambda(z) - (c(\lambda), 0), \lambda).$$

The submanifold $\{(0, 0)\} \times \Lambda \subset M \times \Lambda$ contains only fixed points and possesses a $(d-1)$-dimensional invariant stable bundle. By [**HPS**, Theorem 5.5], there exists for this map an invariant manifold $\hat{W}^s$ such that:

(1) $\dim \hat{W}^s = d - 1 + \dim \Lambda$;
(2) $\forall \lambda \in \mathcal{O}$, $(0, 0, \lambda) \in \hat{W}^s$;
(3) for every $(z, \lambda) \in \hat{W}^s$, $\hat{W}^s$ is transversal with respect to $M \times \{\lambda\}$ at (z, λ) so that

$$\hat{W}^s_\lambda = \hat{W}^s \cap (M \times \{\lambda\})$$

is a submanifold;
(4) $\hat{W}^s$ and W^c are transversal.

† Assuming that f_λ is C^∞ one can even deduce from [**T**] that, when $\lambda \in \mathcal{O}^0$, f_λ is conjugated to $x \mapsto x + x^2 + \delta(\lambda)x^3$, but in general f_λ is only C^k due to the fact that W^c is not C^∞ (see [**St**]).

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We change the coordinates so that $x = 0$ is an equation for $\hat{W}^s$. Near $(0,0)$, F_λ now has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -c + x(1 + x + O(1).y + o(x)) \\ B.y + O(x,y).y \end{pmatrix},$$

with C^k dependence on λ. B is an attracting linear automorphism: there exists $\beta' < 1$ with $\|B\| < \beta'$. Note also that

$$\begin{aligned} c = 0 \quad &\text{when} \quad \lambda \in \mathcal{O}^0, \\ c > 0 \quad &\text{when} \quad \lambda \in \mathcal{O}^+, \\ c < 0 \quad &\text{when} \quad \lambda \in \mathcal{O}^-. \end{aligned}$$

Locally the parameter space can be viewed as a product

$$\lambda = (c, l) \in \mathbb{R} \times \mathbb{R}^{m-1}$$

with $m = \dim \Lambda$ and $\lambda_0 = (0,0)$. Our further estimates are needed only for $\lambda \in (\mathcal{O}^0 \cup \mathcal{O}^+)$. So we prefer to work with the new smooth family $(G_{\bar\lambda})_{\bar\lambda \in \bar{\mathcal{O}}}$ defined by

$$G_{(\bar c, \bar l)} = F_{(\bar c^2, \bar l)}$$

and the fixed point

$$\bar P_{\bar\lambda} = P_{(\bar c^2, \bar l)} = (\bar p_{\bar\lambda}, 0)$$

which is defined for any $\bar\lambda$ and varies smoothly. The map

$$(z, \bar\lambda) \mapsto (G_{\bar\lambda}(z), \bar\lambda)$$

preserves the submanifold $\{(\bar p_{\bar\lambda}, 0)\} \times \bar{\mathcal{O}}$ and a $(d-1)$-dimensional stable bundle. Again one deduces that the stable manifold of $\bar P_{\bar\lambda}$ varies continuously in the C^k-topology so that, near $\bar P_{\bar\lambda}$, G_λ can be conjugated to

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x(1 + a + \delta x + \gamma y + o(x,y)) \\ B.y + O(x,y).y \end{pmatrix}$$

with $a = 0$ if and only if $\bar c = 0$ and $a > 0$ elsewhere, $\delta(\lambda_0) = 1$ and $\|B\| < \beta'$.

Every tangent vector v decomposes into $v_x + v_y$. Suppose that F_{λ_0} satisfies a stable cone field criterion in a neighbourhood of P_{λ_0} then any diffeomorphism near F_{λ_0} satisfies a cone field criterion with the same cone field. In the following we will suppose that the unstable cone at P_{λ_0} is sufficiently sharp:

$$\mathcal{C}^u_{P_{\lambda_0}} \subset \{v = v_x + v_y \in TM_{P_{\lambda_0}},\ \|v_y\| \le \eta \|v_x\|\}$$

with some sufficiently small constant η to be fixed in the following section (consider for instance some $\mathcal{C}^u(n)$ with n large enough instead of $\mathcal{C}^u$). Taking η and W smaller one can suppose that the unstable cone is defined in the whole W and satisfies a similar condition.

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4.3. *Local estimates.* For any $\lambda \in \mathcal{O}^0 \cup \mathcal{O}^+$ (or $\bar{\lambda} \in \bar{\mathcal{O}}$) and any small open set W containing P_{λ_0}, $W^{ss}_{P_\lambda}$ cuts $W \setminus W^{ss}_{P_\lambda}$ into two open subsets. One of them, $W(\lambda)$, is attracted by P_λ when $\lambda \in \mathcal{O}^0$ and contains Q_λ when $\lambda \in \mathcal{O}^+$.

We prove here the following proposition.

PROPOSITION 4.3.1. *Consider any transversal saddle-node bifurcation* $(P_{\lambda_0}, \lambda_0)$ *of the family* (F_λ) *and any small open neighbourhood* W *of* P_{λ_0} *carrying an invariant stable cone field* $\mathcal{C}^u$. *Then there exists* $0 < k < 1$ *such that for any* $\lambda \in (\mathcal{O}^0 \cup \mathcal{O}^+)$, *any finite orbit* $z_0, z_1, \ldots, z_n$ *in* $W \setminus W(\lambda)$ *and any unstable vector* $v \in \mathcal{C}^u_{z_0}$,

$$\|\mathrm{D}F^n_\lambda(z_0).v\| \geq k\|v\|.$$

If W *is small enough,* k *is arbitrarily close to* 1.

Proof. We consider any diffeomorphism G from the family $(G_{\bar{\lambda}})$ sufficiently close to F_{λ_0}. Note that the result does not depend on the choice of a norm. We modify the norm slightly in the following way: the norms of v_x and v_y are unchanged. For the new norm $\|\cdot\|'$ we set

$$\|v\|' = \begin{cases} \|v_x\|, & \text{if } \|v_y\| \leq \eta\|v_x\|, \\ \|v_y\|, & \text{otherwise.} \end{cases}$$

We deduce from the stability of the unstable cone that, for any $z \in U$ and any $v \in \mathcal{C}^u_z$,

$$\|\mathrm{D}G(z).v\|' = \left|\frac{\partial}{\partial x}G_x(z)\right| \|v\|'.$$

If one considers an admissible orbit in $W \setminus W(\lambda)$, $z_0, z_1, \ldots, z_n$, then we note first that $x_k \geq 0$ at each point of the orbit. The first derivative in $\mathrm{D}G$ at those points is then bounded from below by

$$\frac{\partial}{\partial x}G_x(z) \geq 1 + a + 2\delta' x - \gamma'\|y\| \tag{13}$$

with constants $0 < \delta' < \delta$ and $\gamma' > |\gamma|$. One bounds the first derivative uniformly from above by $1 + a'$ with some $a' > 0$. One can then deduce the following estimates on the coordinates:

$$\|y_k\| \leq \beta'^k\|y_0\|,$$
$$x_k \geq \frac{1}{(1+a')^{n-k}}x_n.$$

We iterate an unstable vector $v \in \mathcal{C}^u_{z_0}$ along the orbit. The expansion is bounded from below by

$$\begin{aligned}\|\mathrm{D}G^n(x_0, y_0).v\|' &\geq \prod_{k=0}^{n-1}(1 + a + 2\delta' x_k - \gamma'\|y_k\|)\|v\|' \\ &\geq \prod_{k=0}^{n-1}\left(1 + a + \frac{2\delta' x_n}{(1+a')^{n-k}} - \gamma'\beta'^k\|y_0\|\right)\|v\|'.\end{aligned}$$

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We only want to control the loss of expansion and the following rough lower bound is enough:

$$\|\mathrm{D}G^n(x_0, y_0).v\|' \geq \prod_{k=0}^{n-1}(1-\gamma'\beta'^k\|y_0\|)\|v\|' \geq \left(1-\gamma'\frac{\|y_0\|}{1-\beta'}\right)\|v\|'.$$

Recall that $\|y_0\|$ is arbitrarily small if W is small enough. The proof is complete. □

Remark 4.3.1. With a more precise study one gets a stronger result when $\lambda \in \mathcal{O}^0$; however, we will not use it. Consider an orbit $z_{-1}, z_0, \ldots, z_n, z_{n+1}$ with $z_{-1}, z_{n+1} \notin W$, $z_0, \ldots, z_n \in W \setminus W(\lambda)$ and $v \in \mathcal{C}^u_{z_0}$. If δ is the distance of the orbit to P_λ then $n \sim -c\log(\delta)$ and

$$\|\mathrm{D}F^n.v\| \sim c\frac{\|v\|}{\delta}.$$

See also [**C2, CK, DRV**] for a study of the transition map near a saddle-node point.

5. *The saddle-node bifurcation is robust*

Our assumptions here are stated in Theorem 1.0.2. We also suppose that the estimates of §4 near $(P_{\lambda_0}, \lambda_0)$ are valid.

5.1. *Hyperbolicity.*

Recall that, for any small neighbourhood W of P, the set $W(\lambda)$ has been defined at the beginning of §4.3. We fix the open sets U, V and Ω as in Definition 2.4.2. We consider a small open set W such that $\Omega = W(\lambda)$.

PROPOSITION 5.1.1. *For any $\lambda \in \mathcal{O}^0 \cup \mathcal{O}^+$ the maximal invariant set $K(\lambda)$ of F_λ in $U \setminus \Omega(\lambda)$ with $\Omega(\lambda) = W \setminus W(\lambda)$ satisfies the following.*

(i) *When $\lambda \in \mathcal{O}^0$, $(K(\lambda), P_\lambda)$ and the sets $\hat{U}$, $\hat{V}$ and $\Omega(\lambda)$ satisfy Definition 2.4.2 of the saddle-node hyperbolic set for F_λ. Moreover, $K(\lambda_0) = K$.*

(ii) *When $\lambda \in \mathcal{O}^+$, $K(\lambda)$ is hyperbolic for F_λ.*

Proof. For $\lambda \in \mathcal{O}^0 \cup \mathcal{O}^+$ close to λ_0, the maximal invariant set $K(\lambda)$ in $U \setminus \Omega(\lambda)$ is close to K.

- As W is a small neighbourhood of P_{λ_0}, no periodic orbit distinct from P_λ or Q_λ is contained in W for any diffeomorphism F_λ near F_{λ_0}. Theorem 1.0.4 asserts after perturbation that there exists some integer $N \geq 1$ such that for any diffeomorphism F_λ near F_{λ_0}, any orbit $z_0, z_1, \ldots, z_N$ of F_λ close to K with $z_N \notin W$ and any $v \in \mathcal{C}^u_{z_0}$,

$$\|\mathrm{D}F^n(z_0).v\| \geq 8(1+\|\mathrm{D}F^{-1}\|^{-1})\|v\|.$$

  If W has been chosen small enough the loss of expansion in $W \setminus W(\lambda)$ is bounded from below by $\frac{1}{2}$ when $\lambda \in \mathcal{O}^0 \cup \mathcal{O}^+$. Any orbit $(z_k)_{0\leq k\leq l}$ close to K in U can be decomposed into periods of the following forms.

  Type I. (Possibly empty) $z_0, \ldots, z_{k_1-1}$ with k_1 smaller than N.

  Type II. $z_{k_i}, \ldots, z_{k_{i+1}-1}$ with $z_{k_i+1}, \ldots, z_{k_{i+1}}$ in W.

  Type III. $z_{k_i}, \ldots, z_{k_{i+1}}$ with $z_{k_{i+1}-1} \notin W$ and $k_{i+1} - k_i = N$.

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The loss of expansion during periods of type I is bounded from below by some constant $0 < C^N < 1$. Note that between two periods of type II there exists at least one period of type III. If an orbit has s periods, its expansion for vectors in the unstable cone between times 0 and l is bounded from below by

$$C^N 4^{s/2}. \tag{14}$$

- Consider for $\lambda \in \mathcal{O}^0$ a periodic orbit $O(z)$ of period q in U distinct from $\{P\}$. If one considers $z_0, \ldots, z_{kq}$ for large k, the previous argument and Definition 2.4.2(3) show that $O(z)$ is hyperbolic. On U one can change the continuous stable invariant cone field $\mathcal{C}^s$ into $\mathcal{C}'^s$ so that the unstable cones of $\mathcal{C}'^u$ are strictly smaller than those of $\mathcal{C}^u$. We deduce that if $z \in U \setminus \bar{V}$, and z' is close enough to K so that it can be connected by a u-curve with respect to $\mathcal{C}'^u$, then z can be connected to a point in K by a u-curve with respect to $\mathcal{C}^u$. Shrinking U slightly if necessary, such a point z is sent by any diffeomorphism close to F_{λ_0} outside $U \setminus \Omega(\lambda)$. As the maximal invariant set $K(\lambda)$ in $U \setminus \Omega(\lambda)$ is arbitrarily close to K, $(K(\lambda), P_\lambda)$, the open sets U, V, $W(\lambda)$ and the cone field $\mathcal{C}'^s$ satisfy the definition of a saddle-node hyperbolic set when $\lambda \in \mathcal{O}^0$.
- Suppose that $\lambda \in \mathcal{O}^+$. P_λ is then hyperbolic and, in a neighbourhood of P_λ, $\mathrm{D}F_\lambda$ is expansive in the unstable direction. Long orbits in $W \setminus W(\lambda)$ must stay near P_λ for a long time so that we get a better estimate for orbits of type II:

$$\|\mathrm{D}F^{k_{i+1}-k_i}.v\| \geq \tfrac{1}{4}\mu^{k_{i+1}-k_i}\|v\|$$

for some $\mu > 1$ depending on the parameter λ. Take $\mu^N < 2$ and observe that the lower bound (14) on expansion becomes

$$C^N \mu^l.$$

Definition 2.4.2(3) remains valid and the cone field criterion now asserts that the maximal invariant set in $U \setminus W(\lambda)$ is hyperbolic.

The proposition holds with open sets U, V and W. □

5.2. *Continuation of K.*

Proof of Theorem 1.0.2. We use here the constructions of §3 for the map F_{λ_0} in U.

First we choose an integer $n_1 \gg 1$. Any point z of K possesses a rectangle $R(z) = R(J(z), L_1(z), L_2(z))$ of order n_1 such that $J(z) \in \mathcal{E}_{n_1}$ contains z. From the family $(R(z))_{z\in K}$ we can extract a finite number of rectangles (R_k) such that the rectangles $R_{r'',k}$, with thickness $0 < r'' < r$, cover K.

We define the continuation of each R_k when $\lambda \in \mathcal{O}^0 \cup \mathcal{O}^+$: we fix some long u-curves T_k crossing $\bar{U}$ containing J_k for each k. When the endpoint $z(\lambda)$ of J_k associated to $L_{1,k}$ is in $W^{ss}_P \cap F^{-1}(\Omega(\lambda))$, $L_{1,k} \subset W^{ss}_{P_\lambda}$ can be continued thanks to the continuation $W^{ss}_{P_\lambda}$: $L_{1,k}(\lambda)$ intersects T_k at some $z_1(\lambda)$. In the other case, $L_{1,k}$ and z_1 are unchanged. Doing the same for the other endpoint, it defines $R_k(\lambda) = R(J_k(\lambda), L_{1,k}(\lambda), L_{2,k}(\lambda))$. Note that when λ is close enough to λ_0 we have the inclusions

$$K(\lambda) \subset \bigcup_k R_{k,r''} \subset U.$$

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By perturbation, the results of §§3.2.7 and 3.4.2 are valid with rectangles $R_k(\lambda)$ and $F^{n_1}(R_k(\lambda))$.

LEMMA 5.2.1. *If $\mathcal{O}^0 \cup \mathcal{O}^+$ have been chosen small enough, there exist $0 < r'' < r' < r$ such that for any $\lambda, \lambda' \in \mathcal{O}^0 \cup \mathcal{O}^+$:*

(i) *if $F_\lambda^{n_1}(R_{k,r''}(\lambda)) \cap R_{l,r''}(\lambda)$ is non-empty then, for any λ', $F_{\lambda'}^{n_1}(R_{k,r}(\lambda'))$ crosses $R_{l,r'}(\lambda')$,*

$$F_{\lambda'}^{n_1}(R_{k,r}(\lambda')) \cap R_{l,r}(\lambda') \subset R_{l,r'}(\lambda'),$$

and the leaves $F_{\lambda'}^{n_1}(L_{i,k}(\lambda'))$ and $L_{i,l}(\lambda')$, $i = 1, 2$, are disjoint;

(ii) *if $F_\lambda^{n_1}(R_{k,r'}(\lambda)) \cap R_{l,r'}(\lambda)$ is non-empty then, for any λ', $F_{\lambda'}^{n_1}(R_{k,r}(\lambda'))$ crosses $R_{l,r}(\lambda')$.*

For each λ, λ' we define a map

$$\Phi_{\lambda,\lambda'} : K(\lambda) \to K(\lambda').$$

Choose for $z \in K(\lambda)$ a sequence (k_n) such that

$$\forall n \in \mathbb{Z}, \quad F_\lambda^{nn_1}(z) \in R_{k_n,r'}(\lambda).$$

By Lemma 5.2.1, for any $n \geq 0$,

$$F_{\lambda'}^{-nn_1}(R_{k_n}(\lambda')) \cap F_{\lambda'}^{-(n-1)n_1}(R_{k_{n-1}}(\lambda')) \cap \cdots \cap R_{k_0}(\lambda')$$

is a rectangle which stays in $U \setminus \Omega(\lambda')$ during nn_1 iterations. Using §3.5 for the map $F_{\lambda'}$ when $\lambda' \in \mathcal{O}^0$ or using the hyperbolicity of $K(\lambda')$ when $\lambda' \in \mathcal{O}^+$ we see that the length of this rectangle is arbitrarily small when n increases and that

$$\bigcap_{n\in\mathbb{Z}} F_{\lambda'}^{-nn_1}(R_{k_n}(\lambda'))$$

is a point of $K(\lambda')$. This point does not depend on the choice of (k_n) and will be denoted $\Phi_{\lambda,\lambda'}(z)$. Indeed if (k'_n) is another admissible sequence note that

$$\bigcap_{-1-N\leq n\leq N+1} F_{\lambda'}^{-nn_1}(R_{k_n}(\lambda')) \subset \bigcap_{-N\leq n\leq N} F_{\lambda'}^{-nn_1}(R_{k'_n}(\lambda')).$$

We claim that for any $\lambda, \lambda', \lambda''$ we get

$$\Phi_{\lambda',\lambda''} \circ \Phi_{\lambda,\lambda'} = \Phi_{\lambda,\lambda''}.$$

To prove this fact let us choose (k_n) such that

$$\forall n \in \mathbb{Z}, \quad F_\lambda^{nn_1}(z) \in R_{k_n,r''}(\lambda).$$

Since the family satisfies (by (i))

$$\forall n \in \mathbb{Z}, \quad F_{\lambda'}^{nn_1}(\Phi_{\lambda,\lambda'}(z)) \in R_{k_n,r'}(\lambda').$$

one can compute $\Phi_{\lambda',\lambda''}(\Phi_{\lambda,\lambda'}(z))$ and $\Phi_{\lambda,\lambda''}(z)$ using the same family $(R_{k_n}(\lambda''))$ and we are done.

The map Φ is continuous. Moreover for any λ, $\Phi_{\lambda,\lambda} = \mathrm{Id}_{|K(\lambda)}$. Thus each $\Phi_{\lambda,\lambda'}$ is a homeomorphism which conjugates the dynamics. □

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5.3. *Adapted norm.*

Proof of Proposition 1.0.3. The cone field property in Definition 2.3.1 asserts that $\mathrm{T}K$ admits a unique invariant splitting $\mathrm{T}K = E^s \oplus E^u$ with

- $\dim E^u = 1$ and $E^u \subset \mathcal{C}^u(n_c)$,
- $E^s \subset \mathcal{C}^s(n_c)$,

with some arbitrarily large integer n_c. How to construct an adapted norm for the uniformly contracting direction E^s is well-known (see [**Sh**]). Thus it is sufficient to find an adapted norm for the unstable direction.

- By changing the norm continuously and replacing $\|\cdot\|$ by $\|\cdot\|'$ in a neighbourhood of P (recall §4), we first prove the following local property: there exists a neighbourhood W_0 of P and a continuous $\phi_0 : W_0 \to [\frac{1}{2}, 1]$ satisfying for any $z \in W_0 \setminus \Omega$ and $v \in \mathcal{C}^u_z(n_c) \setminus \{0\}$, n_c large,

$$\phi_0(F(z))\|\mathrm{D}F(z).v\| \geq \phi_0(z)\|v\| \quad \text{with equality if and only if } z = P.$$

  Remember the proof of Proposition 4.3.1 and the estimate (13). If W_0 is small enough there are local coordinates $z = (x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$ such that

$$\|\mathrm{D}F(z).v\| \geq (1 + 2\delta x - \gamma'\|y\|)\|v\|$$

  with $\delta, \gamma' > 0$. Let us define

$$\phi_0(z) = 1 - \frac{2\gamma'}{1 - \beta'}\|y\|.$$

  If W_0 is small enough, $\phi_0(W_0) \subset [\frac{1}{2}, 1]$. Under the previous assumptions note also that

$$\begin{aligned}
&\frac{\phi_0(F(z))\|\mathrm{D}F.v\|}{\phi_0(z)\|v\|} \\
&\quad \geq (1 + 2\delta x - \gamma'\|y\|)\left(1 - \frac{2\gamma'}{1 - \beta'}\|F(y)\|\right)\left(1 - \frac{2\gamma'}{1 - \beta'}\|y\|\right)^{-1} \\
&\quad \geq \left(1 - \gamma'\|y\| - \frac{2\gamma'}{1 - \beta'}\|F(y)\|\right)\left(1 - \frac{2\gamma'}{1 - \beta'}\|y\|\right)^{-1} \geq 1
\end{aligned}$$

  with equality if and only if $x = 0$ and $\|y\| = 0$, i.e. $z = P$. (Remember that $\|F(y)\| \leq \beta'\|y\|$ and $x \geq 0$.)

- In order to solve the global problem, consider a small open neighbourhood of P, $\tilde{W} \subset W_0$ and some integer n_0 given by Theorem 1.0.4: for any orbit $z_0, \ldots, z_n$ and any $v \in E^u_{z_0} \setminus \{0\}$,

$$n \geq n_0 \quad \text{and} \quad z_n \notin \tilde{W} \text{ implies } \|\mathrm{D}F^n(z_0).v\| > 8(1 + \|\mathrm{D}F^{-1}\|^{-1})\|v\|. \tag{15}$$

  We set $W = \tilde{W} \cap F^{-1}\tilde{W} \cap \cdots \cap F^{-n_0}\tilde{W}$. One may take W such that $\mathrm{int}(\bar{W}) = W$ and $\bar{W}$ is compact. W is an open neighbourhood of P and there exists again $n_1 \geq n_0$ such that for any finite orbit $z_0, \ldots, z_n$ and any $v \in E^u_{z_0} \setminus \{0\}$,

$$n \geq n_1 \quad \text{and} \quad z_n \notin \tilde{W} \text{ implies } \|\mathrm{D}F^n(z_0).v\| > 8(1 + \|\mathrm{D}F^{-1}\|^{-1})\|v\|. \tag{16}$$

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Moreover if $z_0, \dots, z_n$ is an orbit in K with $z_0, z_n \in W$ and $z_i \notin W$, $0 < i < n$, then for any $v \in E^u_{z_0} \setminus \{0\}$,

$$\|\mathrm{D}F^n(z_0).v\| > 4\|v\|. \tag{17}$$

Indeed we proceed as in the proof of Proposition 5.1.1. First remark that $n > n_0$ and $z_{n-1} \notin \tilde{W}$. If $z_k \notin \tilde{W}$ then

$$\frac{\|\mathrm{D}F^k.v\|}{\|\mathrm{D}F^{k-n_0}.v\|}$$

is controlled by (15). The loss of expansion during periods contained in $\tilde{W}$ is bounded from below by $\frac{1}{2}$.

- Let us define the norms

$$\|v\|_i = \|v\| + \cdots + \|\mathrm{D}F^{-i+1}.v\|, \quad 1 \le i \le n_1.$$

For any $F(z) \in K \setminus W$ and $v \in E^u_z \setminus \{0\}$, by (16),

$$\|\mathrm{D}F.v\|_{n_1} - \|v\|_{n_1} = \|\mathrm{D}F.v\| - \|\mathrm{D}F^{-n_1+1}.v\| > 0. \tag{18}$$

We also have for $z \in W_0$,

$$\phi_0(z)\|v\| < \|\mathrm{D}F.v\|_1 < \cdots < \|\mathrm{D}F^{n_1}.v\|_{n_1}. \tag{19}$$

We introduce some $n_2 > n_1$ such that for any orbit $z_0, \dots, z_{n_2-1}$ in $K \setminus W$, and $v \in E^u_{z_0}$,

$$\|\mathrm{D}F^{n_2}.v\| \ge 4\|v\|_{n_1}. \tag{20}$$

Define also the norms

$$\|v\|_{-i} = 2^{-1-(i/n_2)}\|\mathrm{D}F^i.v\|, \quad 1 \le i < n_2,$$

and

$$\|v\|^l_{-i} = 2^{-1-(i/l)}\|\mathrm{D}F^i.v\|, \quad 1 \le i < l,\ 2 \le l < n_1 + n_2$$

so that for any $z \in (K \cap W) \setminus (F(W) \cup F^2(W) \cup \cdots \cup F^{n_2}(W))$,

$$\begin{aligned}\phi_0(z)\|v\| &> \|\mathrm{D}F^{-1}.v\|_{-1} > \cdots \\ &> \|\mathrm{D}F^{-n_2+1}.v\|_{-n_2+1} > \|\mathrm{D}F^{-n_2}.v\|_{n_1},\end{aligned} \tag{21}$$

(remember (20) and $\frac{1}{2} \le \phi_0(z)$) and for any $z \in (K \cap W \cap F^l(W)) \setminus (F(W) \cup F^2(W) \cup \cdots \cup F^{l-1}(W))$,

$$\begin{aligned}\phi_0(z)\|v\| &> \|\mathrm{D}F^{-1}.v\|^l_{-1} > \cdots \\ &> \|\mathrm{D}F^{-n_2+1}.v\|^l_{-l+1} > \phi_0(F^{-l}(z))\|\mathrm{D}F^{n_2}.v\|,\end{aligned} \tag{22}$$

(remember (17)).

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- We introduce a partition of the unity $(\psi, 1-\psi)$ subordinate to $R = \bar{W}$ and $E = M \setminus W$: ψ is a smooth map from M to $[0,\infty)$ with the property that ψ is equal to zero outside a small neighbourhood $B(R,\varepsilon)$ of R and equal to one inside $R \setminus B(E,\varepsilon)$. We construct a decomposition

$$M = \left(\bigcup_{j=-n_2+1}^{n_1-1} E_j \right) \cup E_\infty \cup \bigcup_{l=2}^{n_1+n_2-1} \left(\bigcup_{j=1}^{l-1} E_j^l \right)$$

associated to a partition of the unity

$$1 = \sum_j \psi_j + \psi_\infty + \sum_l \sum_j \psi_j^l : \tag{23}$$

– $E_0 = R$ and $\psi_0 = \psi$;
– for $1 \le j < n_1$,

$$E_j = F^j(R) \cap \bigcap_{i=j+1-(n_1+n_2)}^{j-1} F^i(E),$$

$$\psi_j = \psi \circ F^{-j}. \prod_{i=j+1-(n_1+n_2)}^{j-1} (1-\psi) \circ F^{-i};$$

– for $1 \le j < n_2$,

$$E_{-j} = F^{-j}(R) \cap \bigcap_{i=j+1-(n_1+n_2)}^{j-1} F^{-i}(E),$$

$$\psi_{-j} = \psi \circ F^{j}. \prod_{i=j+1-(n_1+n_2)}^{j-1} (1-\psi) \circ F^{i};$$

– $E_\infty = \bigcap_{i=-n_2+1}^{n_1-1} F^i(E)$ and $\psi_\infty = \prod_{i=-n_2+1}^{n_1-1} (1-\psi) \circ F^{-i}$;
– for $2 \le l < n_1 + n_2$, $1 \le j < l$,

$$E_j^l = F^j(R) \cap F^{-l+j}(R) \cap \bigcap_{i=-l+j+1}^{j-1} F^i(E),$$

$$\psi_j^l = \psi \circ F^{-j}.\psi \circ F^{l-j}. \prod_{i=-l+j+1}^{j-1} (1-\psi) \circ F^{-i}.$$

One checks (23) and also that for, any $z \in M$:
– for all $3 \le l < n_1 + n_2$, $1 \le j < l-1$,

$$\psi_{j+1}^l(F(z)) = \psi_j^l(z);$$

– for all $1 \le j < n_1 - 1$,

$$\psi_{j+1}(F(z)) = \psi_j(z);$$

– for all $1 \le j < n_2 - 1$,

$$\psi_{-j-1}(F^{-1}(z)) = \psi_{-j}(z);$$

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– $\psi_\infty(F(z)) + \psi_{-n_2+1}(F(z)) = \psi_{n_1-1}(z) + \psi_\infty(z)$;
– $\psi_0(z) + \psi_1^2(F(z)) + \cdots + \psi_1^{n_3+n_2-1}(F(z)) + \psi_1(F(z))$
$= \psi_0(z) + \psi_1^2(z) + \cdots + \psi_{n_3+n_2-2}^{n_3+n_2-1}(z) + \psi_{-1}(z)$.

By taking $\varepsilon > 0$ sufficiently small in the definition of ψ, one gets that the supports of the functions ψ_j and ψ_j^l are arbitrarily close to the sets E_j and E_j^l.
Defining

$$\|v\|_a = \psi_0(z)\phi_0(z)\|v\| + \sum_{j\neq 0} \psi_j(z)\|v\|_j + \psi_\infty(z)\|v\|_{n_1} + \sum_l \sum_j \psi_j^l(z)\|v\|_j^l$$

we deduce from (18), (19), (21) and (22) that, for any $z \in K$, $v \in E_z^u \setminus \{0\}$,

$$\|\mathrm{D}F.v\|_a \geq \|v\|_a$$

with equality if and only if $\psi_0(z) = 1$ and

$$\phi_0(z)\|\mathrm{D}F.v\|_a = \phi_0\|v\|_a$$

which is equivalent to $z = P$. □

Remark 5.3.1. Theorem 1.0.2 and Proposition 1.0.3 extend when F_{λ_0} admits a finite number m of saddle-node fixed points. In this case one should consider m one-codimensional manifolds intersecting at λ_0. In particular, one can also consider the case when F has a finite number of saddle-node periodic orbits by taking a proper iterate of F.

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