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A convolution-integral representation for a class of linear operators

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1. *Introduction.* Let $\mathfrak{X}$ be the complex vector space consisting of all complex-valued functions of a non-negative real variable t. For each positive number u, *the shift operator* I_u is the mapping of $\mathfrak{X}$ into itself defined by the formula

$$(I_u x)(t) = \begin{cases} 0 & (0 \leqslant t < u), \\ x(t-u) & (t \geqslant u). \end{cases}$$

A linear operator T which maps a subspace $\mathfrak{A}$ of $\mathfrak{X}$ into itself is said to be a *V-operator* (13) if:

(a) for each x in $\mathfrak{A}$, the complex-conjugate function x^* is in $\mathfrak{A}$;

(b) both $\mathfrak{A}$ and $\mathfrak{X} \setminus \mathfrak{A}$ are invariant under the shift operators;

(c) every shift operator commutes with T.

(Property (a) ensures that every function x in $\mathfrak{A}$ can be uniquely expressed as $x_1 + ix_2$, where x_1 and x_2 are real functions in $\mathfrak{A}$.)

Let $\mathscr{A}$ be a commutative algebra whose only annihilator is zero. A *multiplier* on $\mathscr{A}$ is defined to be a mapping A of $\mathscr{A}$ into itself such that $A(pq) = (Ap)q = p(Aq)$ for all p, q in $\mathscr{A}$ (9).

The problem of characterizing multipliers on certain convolution algebras of functions has attracted the attention of several authors; see, for example, Boehme (3), Edwards (4), Johnson (7, 8), and Weston (12, 13). In particular, Weston (13) has established that the multipliers on the convolution algebra C_0 of all continuous complex-valued functions x, of a non-negative real variable, for which $x(0) = 0$, are precisely the V-operators on C_0 which are continuous relative to the topology of uniform convergence on each subinterval $[0, t]$ (referred to in (13) as the 'strong' topology on C_0).

Let C_0^∞ be the subspace of $\mathfrak{X}$ which consists of infinitely differentiable functions ϕ such that, for all non-negative integers k, $\phi^{(k)}(0) = 0$. In this paper we establish a convolution-integral representation for V-operators on C_0^∞ which are continuous in a certain sense. This result leads to a characterization of the multipliers on C_0^∞; in fact, we have established that the multipliers on C_0^∞ are precisely the continuous V-operators on C_0^∞, continuity being interpreted relative to the topology of uniform convergence for all derivatives on each subinterval $[0, t]$. We have also obtained a new characterization of the perfect operators. The results obtained in this paper are similar to the ones given in (13), and the main result we prove (Theorem 1) closely resembles a theorem given by Bertz ((2), Satz 11).

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2. *The representation theorem.* We give C_0^∞ the topology of uniform convergence for all derivatives on compact subsets of $t \geqslant 0$; this topology is defined by the family of semi-norms.

$$r_{n,m}(f) = \max_{k=0,\ldots,m} \left(\max_{0 \leqslant u \leqslant n} |f^{(k)}(u)|\right) \quad (n = 1, 2, \ldots m = 0, 1, \ldots)$$

and makes C_0^∞ into a Fréchet space. In the sequel, the topology of uniform convergence for all derivatives on compact subsets of $t \geqslant 0$ will be referred to as the C^∞-topology on C_0^∞, and it is easy to see that the C^∞-topology on C_0^∞ is finer than the one induced by the strong topology of C_0. Since C_0^∞ is strongly dense in C_0 (see (13), Theorem 6), the strongly continuous operators on C_0^∞ can be represented according to (13), Theorem 4. In this section we establish a representation theorem for V-operators on C_0^∞ which are 'stable', in a sense to be defined. Throughout this paper all topological notions will be interpreted relative to the C^∞-topology.

Let g be any complex-valued continuous function of a non-negative real variable. Then, for any non-negative integer m, the formula

$$(Tf)(t) = \int_0^t f^{(m)}(t-u)\, g(u)\, du \quad (t \geqslant 0), \tag{1}$$

where f is any function in C_0^∞, defines a V-operator T on C_0^∞. Moreover, if n is any positive integer, and k any non-negative integer, then, for $0 \leqslant t \leqslant n$,

$$\begin{aligned} |(Tf)^{(k)}(t)| &\leqslant \int_0^t |f^{(m+k)}(t-u)|\,|g(u)|\,du \\ &\leqslant \left(\int_0^n |g(u)|\,du\right) r_{n,m+k}(f), \end{aligned}$$

so that, for any non-negative integer q,

$$r_{n,q}(Tf) \leqslant \left(\int_0^n |g(u)|\,du\right) r_{n,m+q}(f),$$

which implies that T is continuous.

A linear operator u, on C_0^∞ into itself, is continuous if and only if, given any positive number t, and any non-negative integer k, there exist a non-negative integer $p_{t,k}$, and positive numbers $\alpha_{t,k}$ and $\tau_{t,k}$ such that

$$r_{t,k}(u(f)) \leqslant \alpha_{t,k}\, r_{\tau_{t,k},\, p_{t,k}}(f) \tag{2}$$

for all f in C_0^∞.

Each V-operator on C_0^∞ determined according to equation (1) has the following properties:

(P_1) $p_{t,k}$ can be chosen independently of t;

(P_2) $\tau_{t,k}$ can be chosen so that $\tau_{t,k} = t$ for all $t \geqslant 0$.

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A continuous V-operator with these properties will be called a *stable operator*.†

There are continuous V-operators on C_0^∞ which do not have property (P_1), as the following example shows.‡

For each f in C_0^∞, and each $t \geqslant 0$, let

$$(Bf)(t) = \sum_{n=0}^{[t]} f^{(n)}(t-n), \tag{3}$$

where $[t]$ denotes the integer part of t. It is easy to verify that B is a V-operator on C_0^∞. Moreover, for any non-negative integer k,

$$r_{t,k}(Bf) \leqslant ([t]+1)\, r_{t,k+[t]}(f),$$

which implies that B is continuous. However, with the notation of (2), $p_{t,k} \geqslant k+[t]$, and thus depends on t, so that B does not have property (P_1), and therefore is not stable.

There are continuous linear operators on C_0^∞ which do not have property (P_2); for example, consider the operator E, defined by

$$(Ef)(t) = f(1)\,\phi(t) \quad (f \in C_0^\infty, t \geqslant 0),$$

where

$$\phi(t) = \begin{cases} 0 & (t = 0), \\ e^{-1/t} & (t > 0). \end{cases}$$

It is easy to see that E is a linear operator on C_0^∞, and for any non-negative integer k,

$$r_{t,k}(Ef) \leqslant \alpha_{t,k}\, r_{1,0}(f),$$

where $\alpha_{t,k} = \max_{0 \leqslant u \leqslant t} |\phi^{(k)}(u)|$. This implies that E is continuous; however, with the notation of (2), $\tau_{t,k} > t$ when $0 < t < 1$, so that E does not have the property (P_2). E is not a V-operator on C_0^∞ since it does not commute with the shift operators; at present we do not know whether there are any continuous V-operators on C_0^∞ which do not have property (P_2).

The main theorem we prove (Theorem 1) shows that every stable operator on a dense subspace of C_0^∞ can be represented according to (1). We shall require the following result due to Weston (13), Theorem 1.

LEMMA. *Let T be a V-operator, and let f_1 and f_2 be two of its operands such that, for some positive number t_0, $f_1(t) = f_2(t)$ whenever $0 \leqslant t \leqslant t_0$. Then $(Tf_1)(t) = (Tf_2)(t)$ whenever $0 \leqslant t \leqslant t_0$.*

THEOREM 1. *Let T be a stable operator on a dense subspace $\mathfrak{A}$ of C_0^∞. Then for each integer n greater than some positive integer n_0, there is a continuous function x_n, uniquely determined by T, such that*

$$(Tf)(t) = \int_0^t f^{(n)}(t-u)\, x_n(u)\, du$$

for all f in $\mathfrak{A}$, and for all $t \geqslant 0$.

† We have used the word 'stable' here because a V-operator Φ on $\mathfrak{A}$ which satisfies (P_1) and (P_2) is reasonably well behaved, in the following sense. For any non-negative integer k, there exists a non-negative integer p_k such that a bound for the first k derivatives of $\Phi(f)$ $(f \in \mathfrak{A})$ in any interval $[0, t]$ is some multiple (which will depend on t and k) of a bound for the first p_k derivatives of f in the same interval $[0, t]$.

‡ This example was suggested to me by Professor Weston.

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Proof. For each f in $\mathfrak{A}$, and each $t \geqslant 0$, let f_t be the restriction of f to the closed interval $[0, t]$. Then $f \to f_t$ is a linear mapping of $\mathfrak{A}$ onto a subspace $\mathfrak{A}_t$ of the complex vector space $C^\infty[0, t]$ of all infinitely differentiable functions on the interval $[0, t]$. $C^\infty[0, t]$ is a Fréchet space when its topology η is defined by the semi-norms

$$r_m(x) = \max_{k=0, \ldots, m} \left(\max_{0 \leqslant u \leqslant t} |x^{(k)}(u)| \right) \quad (x \in C^\infty[0, t],\ m = 0, 1, 2, \ldots);$$

moreover, $r_m(f_t) = r_{t,m}(f)$ for all f in $\mathfrak{A}$. By the lemma, if $f_t = 0$, then $(Tf)(t) = 0$; we can therefore define a linear functional ϕ on $\mathfrak{A}_t$ by the equation

$$\phi(f_t) = (Tf)(t).$$

Since T is a stable operator, corresponding to each $t \geqslant 0$ and $k = 0$, there exist a positive number $\alpha_{t,0}$ and a non-negative integer p_0 (independent of t) such that

$$r_{t,0}(Tf) \leqslant \left(\frac{1}{\alpha_{t,0}}\right) r_{t,p_0}(f)$$

for all f in $\mathfrak{A}$. This implies that, for all f_t in $\mathfrak{A}_t$,

$$|\phi(f_t)| \leqslant r_{t,0}(Tf) \leqslant \left(\frac{1}{\alpha_{t,0}}\right) r_{t,p_0}(f) = \left(\frac{1}{\alpha_{t,0}}\right) r_{p_0}(f_t),$$

so that ϕ is a continuous linear functional on $\mathfrak{A}_t$. In particular, $|\phi(f_t)| \leqslant 1$ whenever $r_{p_0}(f_t) \leqslant \alpha_{t,0}$. From this, we can easily prove that $|\phi(f_t)| \leqslant 1$ for all f_t in $\mathfrak{A}_t$ which satisfy the inequality

$$\int_0^t |f_t^{(m+1)}(u)|\, du \leqslant \left(\frac{\alpha_{t,0}}{l^m}\right), \quad \text{where} \quad m = p_0, \quad \text{and} \quad l = \max(1, t).$$

Let $D^{m+1}\mathfrak{A}_t$ be the subspace of $C^\infty[0, t]$ which consists of the $(m+1)$th derivatives of the functions in $\mathfrak{A}_t$, and let D^{m+1} be the mapping of $\mathfrak{A}_t$ onto $D^{m+1}\mathfrak{A}_t$ defined by the equation

$$D^{m+1}(f_t) = f_t^{(m+1)}.$$

Now $D^{m+1}\mathfrak{A}_t$ is a subspace of the Banach space $L_1[0, t]$, and, if ξ denotes the usual norm topology on $L_1[0, t]$, it is easy to see that D^{m+1} is a linear and continuous bijection from

$$(\mathfrak{A}_t, \eta | \mathfrak{A}_t) \quad \text{onto} \quad (D^{m+1}\mathfrak{A}_t, \xi | D^{m+1}\mathfrak{A}_t).$$

We define a functional ψ on $D^{m+1}\,\mathfrak{A}_t$ by the equation

$$\psi(D^{m+1}f_t) = \phi(f_t) \quad (f_t \in \mathfrak{A}_t).$$

Now $$|\psi(D^{m+1}f_t)| \leqslant 1 \quad \text{when} \quad \left(\int_0^t |D^{m+1}f_t(u)|du\right) \leqslant \frac{\alpha_{t,0}}{l^m},$$

so that ψ is a continuous linear functional on $(D^{m+1}\mathfrak{A}_t, \xi | D^{m+1}\mathfrak{A}_t)$. By a well-known theorem (Hahn–Banach–Bohnenblest–Sobczyk extension theorem (**14**), p. 192), ψ can be extended to a continuous linear functional Ψ on $L_1[0, t]$ such that

$$\Psi(D^{m+1}f_t) = \psi(D^{m+1}f_t)$$

for all f_t in $\mathfrak{A}_t$.

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An integral representation of Ψ can be found by adapting a construction used by Banach (1), pp. 61–65. A measurable function ω_t can be defined a.e. on $[0,t]$ such that $\operatorname{ess\,sup}_{0\leqslant u\leqslant t}|\omega_t(u)|$ is finite, and Ψ is given by the Lebesgue integral

$$\Psi(y_t) = \int_0^t y_t(u)\,\omega_t(t-u)\,du,$$

where y_t is any element of $L_1[0,t]$. Hence for all f_t in $\mathfrak{A}_t$,

$$\begin{aligned}\phi(f_t) &= \int_0^t f_t^{(m+1)}(u)\,\omega\,(t-u)\,du\\ &= \int_0^t f_t^{(m+1)}(t-u)\,\omega_t(u)\,du.\end{aligned}$$

On integrating by parts, we have

$$\phi(f_t) = \int_0^t f_t^{(m+2)}(t-u)\,x_t(u)\,du,$$

where

$$x_t(u) = \int_0^u \omega_t(v)\,dv \quad (0\leqslant u\leqslant t).$$

Now the function x_t is continuous on $[0,t]$, (5), p. 398, and x_t is uniquely determined by T, as we now show.

The functions f in $C^\infty[0,t]$ for which $f^{(k)}(0) = 0$ $(k = 0, 1, \ldots)$ form a vector space, $C_0^\infty[0,t]$ say. If $0 < v \leqslant t$ and $0 < \delta < v$, there is a function g_δ in $C_0^\infty[0,t]$ such that

$$g_\delta(u) = \begin{cases} 0 & (0\leqslant u\leqslant t-v),\\ 1 & (t-v+\delta\leqslant u\leqslant t),\end{cases}$$

and $|g_\delta(u)| \leqslant 1$ $(0\leqslant u\leqslant t)$; for example, the function $I_{t-v}h_\delta$, where

$$h_\delta(u) = \begin{cases} \left(\int_0^u \exp\{(v-(v-\delta))\}^{-1}\,dv\right)\Big/\left(\int_0^\delta \exp\{v(v-\delta)\}^{-1}\,dv\right) & (0\leqslant u\leqslant\delta),\\ 1 & (\delta < u\leqslant v).\end{cases}$$

Let $h_{m+2}(u) = u^{m+1}/(m+1)!$ $(0\leqslant u\leqslant t)$. Then the function f_δ, defined by

$$f_\delta(u) = (h_{m+2} * g_\delta)(u) \quad (0\leqslant u\leqslant t),$$

belongs to $C_0^\infty[0,t]$, and $f_\delta^{(m+2)}(u) = g_\delta(u)$. If $z_t(u) = (h_1 * x_t)(u)$ $(0\leqslant u\leqslant t)$, then

$$\Psi(y_t) = \int_0^t y_t^{(1)}(t-u)\,d(z(u))$$

for all y_t in $C_0^\infty[0,t]$; in particular,

$$\begin{aligned}\Psi(f_\delta^{(m+1)}) &= \int_0^t f_\delta^{(m+2)}(t-u)\,d(z_t(u))\\ &= \int_0^{v-\delta} d(z_t(u)) + \int_{v-\delta}^v g_\delta(t-u)\,d(z_t(u)),\end{aligned}$$

and therefore

$$|\Psi(f_\delta^{(m+1)}) - z_t(v-\delta)| \leqslant \int_{v-\delta}^v |x_t(u)|\,du.$$

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The function z_t is continuous, and therefore $\Psi(f_\delta^{(m+1)}) \to z_t(v)$ as $\delta \to 0$. Since $\mathfrak{A}$ is dense in C_0^∞, we can easily prove that the function $f_\delta^{(m+1)}$ belongs to the ξ-closure of $D^{m+1}\mathfrak{A}_t$ in $L_1[0,t]$. Now Ψ is a continuous linear functional on $D^{m+1}\mathfrak{A}_t$, and therefore its extension to the ξ-closure of $D^{m+1}\mathfrak{A}_t$ is unique [(6), p. 129]. Hence $\psi(f_\delta^{(m+1)})$ is uniquely determined by ψ for each value of δ. Moreover, ψ is uniquely determined by T; therefore z_t, and hence x_t, is uniquely determined by T.

Now suppose $t' > t$. By what has been proved, we have for any f in $\mathfrak{A}$,

$$(Tf)(t) = \int_0^t f^{(m+2)}(t-u)\, x_t(u)\, du.$$

But $(Tf)(t) = (I_{t'-t}Tf)(t')$, and $I_{t'-t}T = TI_{t'-t}$; hence

$$\begin{aligned}(Tf)(t) &= \int_0^{t'} (I_{t'-t}f)^{(m+2)}(t'-u)\, x_{t'}(u)\, du \\ &= \int_0^t f^{(m+2)}(t-u)\, x_{t'}(u)\, du.\end{aligned}$$

It follows that $x_t(u) = x_{t'}(u)$ whenever $0 \leqslant u \leqslant t$; in particular, $x_t(t) = x_{t'}(t)$. We now define the function x by the equation

$$x(t) = x_t(t) \quad (t \geqslant 0),$$

so that

$$(Tf)(t) = \int_0^t f^{(m+2)}(t-u)\, x(u)\, du$$

for all f in $\mathfrak{A}$. The function x is uniquely determined by T, as we now show. Let y be any continuous function, of a non-negative real variable, such that, for any non-negative number t,

$$(Tf)(t) = \int_0^t f^{(m+2)}(t-u)\, y(u)\, du$$

for all f in $\mathfrak{A}$. Then, for $0 \leqslant u \leqslant t$, $y(u) = x_t(u)$, and since $x_t(u) = x(u)$, we infer that $y = x$. This completes the proof of the theorem.

3. *Some properties of stable operators.* Let $\mathfrak{A}_0$ be the class of functions in C_0^∞ which have compact support. It is easy to see that $\mathfrak{A}_0$ is a subspace of C_0^∞ which satisfies (a) and (b). Moreover, $\mathfrak{A}_0$ is dense in C_0^∞ ((10), p. 89), so that the stable operators on $\mathfrak{A}_0$ can be represented according to Theorem 1. Let $\mathfrak{A}_p$ be the space of perfect functions. Since $\mathfrak{A}_0 \subseteq \mathfrak{A}_p \subseteq C_0^\infty$, it follows that $\mathfrak{A}_p$ is dense in C_0^∞. Therefore, the stable operators on $\mathfrak{A}_p$ can be represented according to Theorem 1, and it is clear from this theorem that every stable operator on $\mathfrak{A}_p$ is a perfect operator.†

An operator T represented by equation (1) is a perfect operator if and only if it maps $\mathfrak{A}_p$ into itself. An equivalent condition is given by

THEOREM 2. *The formula* (1), *with f in $\mathfrak{A}_p$, represents a perfect operator T if and only if there is a positive integer n such that $D^{-n}g$ is of exponential order at infinity.*

Proof. The proof is a trivial modification of (13), Theorem 10 and will not be given here.

Let $\mathfrak{A}$ be a subspace of C_0^∞ which satisfies (a) and (b). If, further, $\mathfrak{A}$ is invariant under the differential operator, then we say that $\mathfrak{A}$ is a *D-space*. $\mathfrak{A}_p$ is an example of a

† We refer the reader to (9) for the definitions of a perfect function and a perfect operator.

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D-space. However, there are subspaces of C_0^∞ which satisfy (a) and (b) but are not D-spaces; for example, the functions f in C_0^∞ which satisfy the condition

$$f(t) = O(e^{t^2}) \quad \text{as} \quad t \to \infty.$$

THEOREM 3. *If T is a continuous V-operator on a D-space $\mathfrak{A}$, then T commutes with the differential operator.*

Proof. First, we show that, for all ϕ in $\mathfrak{A}$,

$$D\phi = \lim_{n\to\infty} n(I_0 - I_{1/n})\,\phi.$$

For any positive integer n, and any non-negative integer m,

$$r_{n,m}(D\phi - n(I_0 - I_{1/n})\,\phi) = \max_{k=0,\ldots,m}\left(\max_{0\leqslant u\leqslant n}\left|\phi^{(k+1)}(u) - \frac{(\phi^{(k)}(u) - \phi^{(k)}\{u-(1/n)\})}{1/n}\right|\right).$$

Given any $\epsilon > 0$, there exists $\delta_k > 0$ such that $|\phi^{(k)}(u) - \phi^{(k)}(u')| < \epsilon$ whenever $u, u' \in [0, n]$ and $|u - u'| < \delta_k$ (by uniform continuity). Let $\delta = \min(\delta_1, \ldots, \delta_{m+1})$, and suppose that $(1/n) < \delta$. Then

$$r_{n,m}(D\phi - n(I_0 - I_{1/n})\,\phi) = \max_{k=0,\ldots,m}\left(\max_{0\leqslant u\leqslant n}\left|\phi^{(k+1)}(u) - \phi^{(k+1)}\left(u - \frac{\theta_{u,k,n}}{n}\right)\right|\right),$$

where $0 < \theta_{u,k,n} < 1$ for all u in $[0, n]$ and $k = 0, \ldots, m$, and so

$$r_{n,m}(D\phi - n(I_0 - I_{1/n})\,\phi) < \epsilon$$

whenever $1/n < \delta$. This implies that

$$D\phi = \lim_{n\to\infty} n(I_0 - I_{1/n})\,\phi. \tag{4}$$

Since T is continuous and commutes with the shift operators,

$$\begin{aligned} TD\phi &= \lim_{n\to\infty} n(I_0 - I_{1/n})\,T\phi \\ &= DT\phi, \end{aligned}$$

and so $TD = DT$ as required.

Remark. Equation (4) implies that, if $\mathfrak{A}$ is a closed subspace of C_0^∞, with the properties (a), and (b), then $\mathfrak{A}$ is a D-space. For each positive integer n, let $\mathfrak{A}_{0,n}$ consist of the functions in $\mathfrak{A}_0$ whose supports are contained in $[0, n]$. It is easy to see that each $\mathfrak{A}_{0,n}$ is a proper closed subspace of C_0^∞ with the property (a). However, $\mathfrak{A}_{0,n}$ is not invariant under the shift operators, and at present we do not know whether there are any proper closed subspaces of C_0^∞ with both the properties (a) and (b).

THEOREM 4. *If S is a stable operator on C_0^∞ and f is any element of a D-space $\mathfrak{A}$, then Sf belongs to the closure of $\mathfrak{A}$.*

Proof. (This is a modification of the proof of (13), Theorem 8.) By Theorem 1, there is a continuous function x such that, for some non-negative integer m,

$$(Sf)(t) = \int_0^t f^{(m)}(t-u)\,x(u)\,du \quad (t \geqslant 0).$$

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By applying the rule for differentiating a convolution product, the k-th derivative of Sf is given by

$$(Sf)^{(k)}(t) = \int_0^t f^{(m+k)}(t-u)\,x(u)\,du.$$

For each non-negative integer k, and each positive number t, the value $(Sf)^{(k)}(t)$ can be arbitrarily approximated by sums of the form

$$\sum_{j=0}^{n_k-1} f^{(m+k)}(t-u_j^k)\,x(u_j^k)\,(u_{j+1}^k-u_j^k),$$

where $0 = u_0^k < u_1^k < \ldots < u_{n_k}^k = t$ is a partition $\mathscr{D}_k$ of $[0,t]$. Now the sum is the value at t of the function $g_{\mathscr{D}_k}$ given by

$$g_{\mathscr{D}_k} = \sum_{j=0}^{n_k-1} \alpha_{k,j} I_{u_j^k} f^{(m+k)},$$

where $\alpha_{k,j} = x(u_j^k)\,(u_{j+1}^k - u_j^k)$. Since $\mathfrak{A}$ is a D-space, $g_{\mathscr{D}_k}$ belongs to $\mathfrak{A}$. Moreover, $g_{\mathscr{D}_k}$ approximates uniformly the function $(Sf)^{(k)}$ on the subinterval $[0,t]$, as we now show.

Let ϵ be any positive number. There exists a positive number δ_k such that, for $0 \leqslant v \leqslant t$,

$$|f^{(m+k)}(v-u)\,x(u) - f^{(m+k)}(v-u')\,x(u')| < \epsilon/t$$

when $u, u' \in [0,v]$, and $|u-u'| < \delta_k$ (by uniform continuity);

the points $u_0^k, \ldots, u_{n_k}^k$ are then chosen so that

$$\max_{j=0,\ldots,n_k-1} (u_{j+1}^k - u_j^k) < \delta_k.$$

If $0 \leqslant v \leqslant t$, then $v \in [u_l^k, u_{l+1}^k]$ for some integer l satisfying $0 \leqslant l \leqslant n_k - 1$. Now

$$\begin{aligned}
|(Sf)^{(k)}(v) - g_{\mathscr{D}_k}(v)| &= \left| \int_0^v f^{(m+k)}(v-u)\,x(u)\,du - \sum_{j=0}^{l} f^{(m+k)}(v-u_j^k)\,x(u_j^k)\,(u_{j+1}^k-u_j^k) \right| \\
&= \left| \sum_{j=0}^{l} \int_{u_j^k}^{u_{j+1}^k} f^{(m+k)}(v-u)\,x(u) - f^{(m+k)}(v-u_j^k)\,x(u_j^k)\,du \right| \\
&< (\epsilon/t)\,(u_{l+1}^k - u_0^k) \\
&< \epsilon.
\end{aligned}$$

For any non-negative integer N, let $\mathscr{I}_N$ be the subdivision of $[0,t]$ formed by all the points in $\bigcup_{k=0}^{N} J_k$, where $J_k = \{u_0^k, \ldots, u_{n_k}^k\}$. Corresponding to $\mathscr{I}_N$, we define a function $h_{\mathscr{I}_N}$ by the equation

$$h_{\mathscr{I}_N} = \sum_{j=0}^{n_N-1} \alpha_j I_{u_j} f^{(m)}, \tag{5}$$

where $\{u_0, \ldots, u_{n_N}\}$ are the points of $\mathscr{I}_N$, and $\alpha_j = x(u_j)\,(u_{j+1} - u_j)$. Since $\mathfrak{A}$ is a D-space, $h_{\mathscr{I}_N}$ belongs to $\mathfrak{A}$. Moreover, $J_k \subseteq \mathscr{I}_N$ for $k = 0, 1, \ldots, N$, so that

$$\max_{0 \leqslant u \leqslant t} |(Sf)^{(k)}(u) - h_{\mathscr{I}_N}^{(k)}(u)| \leqslant \epsilon.$$

This implies that $r_{t,N}(Sf - h_{\mathscr{I}_N}) \leqslant \epsilon$, and so Sf belongs to the closure of $\mathfrak{A}$ in C_0^∞.

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THEOREM 5. *With the notation of Theorem 4, suppose that Sf belongs to $\mathfrak{A}$ for all f in $\mathfrak{A}$. If T is a continuous V-operator on $\mathfrak{A}$ then $TS = ST$.*

Proof. The points $u_j (j \neq 0, \ldots, n_N)$ (see the proof of Theorem 4) can be chosen in such a way that, while Sf is approximated by (5), in the sense of the C^∞-topology, STf is simultaneously approximated in the same sense by

$$\sum_{j=0}^{n_N-1} \alpha_j I_{u_j}(Tf)^{(m)}. \tag{6}$$

But, since T is a continuous V-operator on a D-space, T commutes with the differential operator (Theorem 3), and so (6) is the same as

$$T\left(\sum_{j=0}^{n_N-1} \alpha_j I_{u_j} f^{(m)}\right).$$

Since T is continuous, it follows that $TSf = STf$ if Sf is an operand of T, and so $TS = ST$ as required.

COROLLARY. *If g is a function in C_0^∞ such that $f*g$ belongs to $\mathfrak{A}$ for all f in $\mathfrak{A}$, then $T(f*g) = g*Tf$.*

With products and linear combinations defined in the usual way, the V-operators on a given space $\mathfrak{A}$ constitute a linear algebra $\mathfrak{U}(\mathfrak{A})$. The continuous V-operators form a subalgrebra, $\mathfrak{N}(\mathfrak{A})$, of $\mathfrak{U}(\mathfrak{A})$, and the stable operators form a subalgebra, $\mathfrak{N}_s(\mathfrak{A})$, of $\mathfrak{N}(\mathfrak{A})$. If $\mathfrak{A}$ is a dense subspace of C_0^∞ which satisfies (a) and (b), then it follows from Theorem 1 that $\mathfrak{N}_s(\mathfrak{A})$ is an integral domain (it is commutative and has no proper divisors of zero), and that $\mathfrak{N}_s(\mathfrak{A})$ consists of those operators in $\mathfrak{N}_s(C_0^\infty)$ which leave $\mathfrak{A}$ invariant.

It is a consequence of Theorem 4 and the remark following Theorem 3 that a subspace of C_0^∞ is closed under convolution if it is C^∞-closed. On the other hand $\mathfrak{A}_p$ is closed under convolution though it is not C^∞-closed. If $\mathfrak{A}$ is any subspace of C_0^∞ which is closed under convolution (so forming a commutative algebra with no proper divisors of zero), then we can define multipliers on $\mathfrak{A}$. It is well known that multipliers are necessarily linear (12), and, for a given $\mathfrak{A}$, they form a commutative algebra $\mathfrak{M}(\mathfrak{A})$ with no proper divisors of zero. A shift operator belongs to $\mathfrak{M}(\mathfrak{A})$ if and only if it maps $\mathfrak{A}$ into itself. Hence, if $\mathfrak{A}$ satisfies (a) and (b), in addition to being closed under convolution, then all the multipliers on $\mathfrak{A}$ are V-operators, so that $\mathfrak{M}(\mathfrak{A})$ is a subalgebra of $\mathfrak{U}(\mathfrak{A})$. If, further $\mathfrak{A}$ is dense in C_0^∞, it follows from Theorem 1 that the stable operators on $\mathfrak{A}$ are multipliers, so that

$$\mathfrak{N}_s(\mathfrak{A}) \subseteq \mathfrak{M}(\mathfrak{A}) \subseteq \mathfrak{U}(\mathfrak{A}). \tag{7}$$

By definition, the multipliers on $\mathfrak{A}_p$ are the perfect operators, which can be represented according to (11), Theorem 2. It is easy to prove from (11), Theorem 2 that the perfect operators are stable, so that $\mathfrak{N}_s(\mathfrak{A}_p) = \mathfrak{M}(\mathfrak{A}_p)$.

If $\mathfrak{A}_D$ is a closed D-space, then the multipliers on $\mathfrak{A}_D$ are continuous (cf. (7), Corollary to Theorem 14). It follows that

$$\mathfrak{M}(\mathfrak{A}_D) \subseteq \mathfrak{N}(\mathfrak{A}_D) \subseteq \mathfrak{U}(\mathfrak{A}_D). \tag{8}$$

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From (7) and (8), we deduce that

$$\mathfrak{N}_s(C_0^\infty) \subseteq \mathfrak{M}(C_0^\infty) \subseteq \mathfrak{N}(C_0^\infty).$$

By the Corollary to Theorem 5, every continuous V-operator on C_0^∞ is a multiplier on C_0^∞, so that the multipliers on C_0^∞ are precisely the continuous V-operators on C_0^∞. There are multipliers on C_0^∞ which are not stable; for example, the operator B, defined by equation (3), is a multiplier on C_0^∞ which is not stable.

Let $\mathfrak{A}$ be a subspace of C_0^∞ which satisfies (a) and (b), and suppose that $\mathfrak{A}$ is closed under convolution. In general it seems difficult to decide whether all the multipliers on $\mathfrak{A}$ are continuous or not. However, below, we show that $\mathfrak{A}$ can be chosen so that

$$\mathfrak{N}_s(\mathfrak{A}) = \mathfrak{M}(\mathfrak{A}) = \mathfrak{N}(\mathfrak{A}).$$

The uniformly perfect functions (of which a definition may be found in (9)) form a subspace, $\mathfrak{E}$, of C_0^∞ which satisfies (a) and (b), so that V-operators can be defined on $\mathfrak{E}$. Since $\mathfrak{A}_0 \subseteq \mathfrak{E} \subseteq C_0^\infty$, it follows that $\mathfrak{E}$ is dense in C_0^∞, so that the stable operators on $\mathfrak{E}$ can be represented according to Theorem 1. The multipliers on $\mathfrak{E}$ can be represented according to (9), Theorem 3, and it is easy to see that the stable operators on $\mathfrak{E}$ are precisely the multipliers on $\mathfrak{E}$; that is, $\mathfrak{N}_s(\mathfrak{E}) = \mathfrak{M}(\mathfrak{E})$. Now $\mathfrak{E}$ is a D-space which is closed under convolution, so that, by the Corollary to Theorem 5, every continuous V-operator on $\mathfrak{E}$ is a perfect operator; that is, $\mathfrak{N}(\mathfrak{E}) \subseteq \mathfrak{M}(\mathfrak{E})$. Consequently,

$$\mathfrak{N}_s(\mathfrak{E}) = \mathfrak{M}(\mathfrak{E}) = \mathfrak{N}(\mathfrak{E}),$$

as required.

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