CrossMark

Proceedings of the Royal Society of Edinburgh, **151**, 2013–2029, *2021*
DOI:10.1017/prm.2020.92

Topological complexity of real Grassmannians[1]

Petar Pavešić
Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Jadranska 21, 1000, Slovenija
(petar.pavesic@fmf.uni-lj.si)

(Received 15 July 2020; accepted 28 November 2020)

We use some detailed knowledge of the cohomology ring of real Grassmann manifolds $G_k(\mathbb{R}^n)$ to compute zero-divisor cup-length and estimate topological complexity of motion planning for k-linear subspaces in $\mathbb{R}^n$. In addition, we obtain results about monotonicity of Lusternik–Schnirelmann category and topological complexity of $G_k(\mathbb{R}^n)$ as a function of n.

Keywords: Topological complexity; Lusternik–Schnirelmann category; Stiefel–Whitney classes; Grassmann manifold; zero divisor cup-length

2020 *Mathematics Subject Classification:* 55M30; 55S40

1. Introduction

Topological complexity is a homotopy invariant of a space that was introduced by Farber [5] in his study of motion planning in robotics. Roughly speaking, motion planning problem requires to find a continuous motion that transforms a mechanical system from some given initial position to a desired final position. To give a precise mathematical statement one considers paths in a suitable *configuration space*, i.e. the topological space that describes all possible states of the mechanical system. In particular, Grassmann manifolds of linear subspaces of $\mathbb{R}^n$ arise naturally in the motion planning for geometric objects like lines or planes in Euclidean spaces.

Given a path-connected configuration space X of a mechanical system, we denote by X^I the space of all continuous paths $\alpha\colon I \to X$, and by $\mathrm{ev}\colon X^I \to X \times X$ the evaluation map that to a path α assigns its end-points, $\mathrm{ev}(\alpha) := (\alpha(0), \alpha(1))$. A *motion plan* in X is a function that takes as input a pair of points $x, y \in X$, and returns as output a path $\alpha = \alpha(x, y)$ in X starting at x and ending at y. That is to say, a motion plan is a section $\alpha\colon X \times X \to X^I$ of the evaluation map. It is important to note that while the motion through the configuration space is always assumed to be continuous, motion plans will be generally discontinuous. In fact, one can easily show that a continuous motion plan exists if and only if X is contractible.

[1]Supported by the Slovenian Research Agency program P1-0292 and grants N1-0083, N1-0064.

© The Author(s), 2021. Published by Cambridge University Press on behalf of The Royal Society of Edinburgh

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

In applications discontinuities appear as instabilities of the motion plan because small variations or errors in the input data can result in a completely different motion path of a robot. One is thus naturally led to consider motion plans whose degree of instability is as small as possible.

Farber [**5**] defined the topological complexity $\mathrm{TC}(X)$ of a path-connected space X as the minimal integer n for which there exists an open cover $U_1, \dots, U_n$ of $X \times X$ and a motion plan $\alpha\colon X \times X \to X^I$, such that the restrictions of α to each U_i is continuous. We will refer to Farber's monograph [**8**, chapter 4] for the relevant definitions and results on $\mathrm{TC}(X)$. In particular, see [**8**, § 4.2] for several alternative descriptions of topological complexity and for its relation with the instability of motion plans.

The Grassmann manifold of 1-dimensional subspaces of $\mathbb{R}^n$ is the familiar projective space $\mathbb{R}P^{n-1}$. Topological complexity of projective spaces was studied by Farber, Tabachnikov and Yuzvinsky [**6**]. They found a remarkable relation between $\mathrm{TC}(\mathbb{R}P^n)$ and the classical immersion dimension of $\mathbb{R}P^n$, which gave precise values for the topological complexity of real projective spaces in a range of dimensions. Soon afterwards Pearson and Zhang [**12**] published a short note in which they produced some estimates for the topological complexity of higher Grassmannians. Unfortunately, their work was marred by a trivial error that led them to assume that $\mathrm{TC}(X) = \mathrm{cat}(X \times X)$ (while the correct relation is $\mathrm{cat}(X) \leqslant \mathrm{TC}(X) \leqslant \mathrm{cat}(X \times X)$, see below) which compromised all of their results. Note that even after correcting the error in [**12**], the methods used in that paper would give an estimate for $\mathrm{TC}(G_k(\mathbb{R}^n))$ that is only slightly better than the standard category estimate. Through a careful analysis of the cohomology of Grassmannians we were able to find a different set of non-trivial products and derive new lower bounds $\mathrm{TC}(G_k(\mathbb{R}^n))$ that are from two to three times better than those that were previously known.

The next two sections contain preparatory material. In the first we briefly review the main concepts of Lusternik–Schnirelmann category and topological complexity, while in the second we summarize some technical results on Grassmann manifolds and their cohomology rings, and prove the non-triviality of certain cohomology classes which will serve as a base for our computations. Section 4 is the central part of the article and is dedicated to the estimates of topological complexity of Grassmannians. There are four main results: in theorems 4.5 and 4.8 we compute zero-divisor cup-length (and the corresponding estimate of topological complexity) of $G_2(\mathbb{R}^n)$ and $G_3(\mathbb{R}^n)$, respectively; in theorem 4.10 we give a lower bound for the zero-divisor cup-length of $G_k(\mathbb{R}^n)$ for $k > 3$ and finally, in theorem 4.12 we compute zero-divisor cup-length of $G_4(\mathbb{R}^n)$ for about half of the values of n. In the last section we use the above estimates to show that under suitable assumptions Lusternik–Schnirelmann category and topological complexity of $G_k(\mathbb{R}^n)$ are increasing functions of n.

2. Review of category and topological complexity

Here we recall main properties of Lusternik–Schnirelmann category and topological complexity. Note that in both cases we follow the non-normalized convention, i.e., category and complexity of contractible spaces is equal to 1 (as opposed to 0 in the normalized case). More details can be found in the monographs [**2**] and [**8**].

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

The *Lusternik–Schnirelmann category* of a space X, denoted $\mathrm{cat}(X)$, is the minimal integer n for which there exists an open cover $U_1, \ldots, U_n$ of X such that each U_i can be deformed in X to a point. Main properties of $\mathrm{cat}(X)$ are listed in the following proposition.

Proposition 2.1.

(1) $\mathrm{cat}(X) = 1$ *if, and only if* X *is contractible;*

(2) Homotopy invariance: $X \simeq Y \Rightarrow \mathrm{cat}(X) = \mathrm{cat}(Y)$*;*

(3) Dimension-connectivity estimate: if X *is* d*-dimensional and* $(c-1)$*-connected, then* $\mathrm{cat}(X) \leqslant d/c + 1$*;*

(4) Cohomological estimate: $\mathrm{cat}(X) \geqslant \mathrm{nil}\,\widetilde{H}^*(X)$*, where* $\widetilde{H}^*(X)$ *is the ideal of positive-dimensional cohomology classes in* $H^*(X)$*;*

(5) Product formula: $\mathrm{cat}(X \times Y) \leqslant \mathrm{cat}(X) + \mathrm{cat}(Y) - 1$.

In (4) $H^*(X)$ denotes cohomology with any ring coefficients, and the nilpotency $\mathrm{nil}\,\widetilde{H}^*(X)$ of the ideal $\widetilde{H}^*(X)$ is the minimal integer n for which $\widetilde{H}^*(X)^n = 0$ (i.e., all products of n elements of $\widetilde{H}^*(X)$ are equal to 0). The invariant $\mathrm{cat}(X)$ was extensively studied and it was computed exactly or to a good approximation for a wide variety of spaces (see [2]). However, the knowledge of the LS-category of real Grassmannians, which is of interest for this paper, is still very scant.

We have already mentioned Farber's topological complexity and its applications to robotics. The following definition displays a strong analogy with Lusternik–Schnirelmann category (and is easily seen to be equivalent to the one we gave earlier in terms of continuous motion plans). The *topological complexity* of a path-connected space X, denoted $\mathrm{TC}(X)$, is the minimal integer n for which there exists an open cover $U_1, \ldots, U_n$ of $X \times X$ such that each U_i can be deformed in $X \times X$ to the diagonal ΔX. Again we collect the main properties of this concept in a proposition.

Proposition 2.2.

(1) $\mathrm{TC}(X) = 1$ *if, and only if* X *is contractible;*

(2) Homotopy invariance: $X \simeq Y \Rightarrow \mathrm{TC}(X) = \mathrm{TC}(Y)$*;*

(3) Category estimate: $\mathrm{cat}(X) \leqslant \mathrm{TC}(X) \leqslant \mathrm{cat}(X \times X)$*;*

(4) If X *is a topological group, then* $\mathrm{TC}(X) = \mathrm{cat}(X)$*;*

(5) Cohomological estimate: $\mathrm{TC}(X) \geqslant \mathrm{nil}(\mathrm{Ker}\,\Delta^*)$*, where*

$$\Delta^* \colon H^*(X \times X) \to H^*(X)$$

is induced by the diagonal $\Delta \colon X \to X \times X$*;*

(6) Product formula: $\mathrm{TC}(X \times Y) \leqslant \mathrm{TC}(X) + \mathrm{TC}(Y) - 1$.

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

We will also need two special results.

Proposition 2.3. *(see [2, theorem 8.23] and [7, lemma 28.1]) Let X be a simply-connected symplectic manifold. Then* $\mathrm{cat}(X) = \dim(X)/2 + 1$ *and* $\mathrm{TC}(X) = \dim(X) + 1$.

Proposition 2.4. *(see [3, theorem 1]) If X is a CW-complex with $\pi_1(X) = \mathbb{Z}_2$, then $\mathrm{TC}(X) \leqslant 2\dim X$. Furthermore, if X is a closed n-manifold, and the generator $w \in H^1(X; \mathbb{Z}_2) = \mathbb{Z}_2$ satisfies $w^n = 0$, then $\mathrm{TC}(X) \leqslant 2\dim X - 1$.*

3. Review of finite Grassmannians and their cohomology

Let $G_k(\mathbb{R}^n)$ denote the space of all k-dimensional linear subspaces of $\mathbb{R}^n$. This space is a manifold (actually a non-singular algebraic variety) of dimension $k(n-k)$. The correspondence between a subspace and its orthogonal complement determines a homeomorphism between $G_k(\mathbb{R}^n)$ and $G_{n-k}(\mathbb{R}^n)$, so we will always tacitly assume that $k \leqslant n/2$.

The space $G_k(\mathbb{R}^n)$ has a standard CW-decomposition which we briefly recall following [10, § 6]. For each element (i.e., k-subspace of $\mathbb{R}^n$) $V \in G_k(\mathbb{R}^n)$ consider the increasing sequence of integers

$$0 \leqslant \dim(V \cap \mathbb{R}) \leqslant \dim(V \cap \mathbb{R}^2) \leqslant \cdots \leqslant \dim(V \cap \mathbb{R}^n) = k,$$

where two consecutive terms are either equal or differ by 1, so that there are exactly k 'jumps' of the form $\dim(V \cap \mathbb{R}^{d_i}) = \dim(V \cap \mathbb{R}^{d_i - 1}) + 1$ for $i = 1, 2, \ldots, k$. The *Schubert symbol* of V is defined as a sequence of integers

$$\sigma(V) = (d_1 - 1, d_2 - 2, \ldots, d_k - k).$$

Note that for every Schubert symbol $\sigma(V) = (\sigma_1, \ldots, \sigma_k)$ we have $0 \leqslant \sigma_1 \leqslant \ldots \leqslant \sigma_k \leqslant n - k$, and that every such sequence can appear for some $V \in G_k(\mathbb{R}^n)$. Thus, for every non-increasing sequence σ of integers between 0 and $n - k$ let

$$e_\sigma := \{V \in G_k(\mathbb{R}^n) \mid \sigma(V) = \sigma\}.$$

It turns out (see [10, theorem 6.4]) that e_σ is homeomorphic to an open cell of dimension $\sigma_1 + \ldots + \sigma_k$, and that these cells form a CW-decomposition of $G_k(\mathbb{R}^n)$. There is one 0-cell corresponding to the symbol $(0, \ldots, 0)$ and one top-dimensional $k(n-k)$-cell with symbol $(n-k, \ldots, n-k)$. In general the number of d-dimensional cells is equal to the number of partitions of d as a sum of k integers between 0 and $n - k$, so that there are $\binom{n}{k}$ cells in total.

There is a natural embedding $G_k(\mathbb{R}^n) \hookrightarrow G_k(\mathbb{R}^{n+1})$ obtained by viewing every k-dimensional subspace of $\mathbb{R}^n$ as a subspace of $\mathbb{R}^n \oplus \mathbb{R} = \mathbb{R}^{n+1}$. It is easy to see that $G_k(\mathbb{R}^n)$ is actually a subcomplex of $G_k(\mathbb{R}^{n+1})$ with respect to the Schubert decomposition and that cells up to dimension $(n-k)$ coincide, i.e.

$$G_k(\mathbb{R}^n)^{(n-k)} = G_k(\mathbb{R}^{n+1})^{(n-k)}.$$

We will base our computations on the $\mathbb{Z}_2$-cohomology ring of $G_k(\mathbb{R}^n)$ which can be described as follows (cf. [13]). Let $w_1, \ldots, w_k \in H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ denote the Stiefel-Whitney classes of the canonical k-dimensional vector bundle over $G_k(\mathbb{R}^n)$, and

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

let $\overline{w}_1, \ldots, \overline{w}_{n-k} \in H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ denote the dual Stiefel–Whitney classes (i.e., the Stiefel–Whitney classes of the orthogonal complement of the canonical bundle). The Stiefel–Whitney classes and their duals are related by the formula

$$w \cdot \overline{w} = (1 + w_1 + \ldots + w_k) \cdot (1 + \overline{w}_1 + \ldots + \overline{w}_{n-k}) = 1,$$

so the dual classes can be recursively expressed as polynomials in the variables $w_1, \ldots, w_k$. Then $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ can be described as the quotient of the $\mathbb{Z}_2$-polynomial ring, generated by the Stiefel–Whitney classes and their duals, modulo the relation $w \cdot \overline{w} = 1$:

$$H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \ldots, w_k, \overline{w}_1, \ldots, \overline{w}_{n-k}]/(w \cdot \overline{w} = 1).$$

EXAMPLE 3.1. In the cohomology of $G_2(\mathbb{R}^6)$ we have the relation

$$(1 + w_1 + w_2) \cdot (1 + \overline{w}_1 + \overline{w}_2 + \overline{w}_3 + \overline{w}_4) = 1,$$

from which we obtain $\overline{w}_1 = w_1$, $\overline{w}_2 = w_1^2 + w_2$, $\overline{w}_3 = w_1^3$, $\overline{w}_4 = w_1^4 + w_1^2 w_2 + w_2^2$, $w_1^5 + w_1 w_2^2 = 0$ and $w_1^4 w_2 + w_1^2 w_2^2 + w_2^3 = 0$. Therefore

$$H^*(G_2(\mathbb{R}^6); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2]/(w_1^5 + w_1 w_2^2,\ w_1^4 w_2 + w_1^2 w_2^2 + w_2^3).$$

In spite of a very explicit description, computations in $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ are all but straightforward, because it is often extremely difficult to determine whether a given polynomial in $w_1, \ldots, w_k$ is contained in the ideal. Stong [**13**, p. 103-104] (following earlier work by Hiller) used a different approach to check if a given product of Stiefel–Whitney classes is non-trivial in $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. He considered the natural map

$$\pi \colon \mathrm{Flag}(\mathbb{R}^n) \to G_k(\mathbb{R}^n),$$

where $\mathrm{Flag}(\mathbb{R}^n)$ is the space of all ordered n-tuples $(V_1, \ldots, V_n)$ ('flags') of mutually orthogonal 1-dimensional subspaces of $\mathbb{R}^n$ and $\pi(V_1, \ldots, V_n) = V_1 \oplus \ldots \oplus V_k$. The main facts about the cohomology of $\mathrm{Flag}(\mathbb{R}^n)$ and the homomorphism π^* are collected in the following proposition.

PROPOSITION 3.2 see Stong [**13**, pp. 104–106].

(1) $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2) \cong \mathbb{Z}_2[e_1, \ldots, e_n]/(\prod_{i=1}^n (1 + e_i) = 1)$, *where* e_i *is the first Stiefel–Whitney class of the* i*-th canonical line bundle over* $\mathrm{Flag}(\mathbb{R}^n)$.

(2) $e_i^n = 0$ *in* $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$ *for* $i = 1, \ldots, n$.

(3) a monomial $e_1^{i_1} \ldots e_n^{i_n}$ *in the top dimension* $H^{\binom{n}{2}}(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$ *is non-zero if, and only if, all exponents are different, i.e., if* $(i_1, \ldots i_n)$ *is a permutation of* $\{0, \ldots, n-1\}$.

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

(4) $\pi^*\colon H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2) \to H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$ *is injective and is given explicitly as*

$$\pi^*(1+w_1+\ldots+w_k) = \prod_{i=1}^{k}(1+e_i), \quad \textit{and} \quad \pi^*(1+\overline{w}_1+\ldots+\overline{w}_{n-k}) = \prod_{i=k+1}^{n}(1+e_i);$$

In other words, $\pi^*(w_i)$ *is the* i*-th elementary symmetric polynomial in the variables* $e_1,\ldots,e_k$ *and* $\pi^*(\overline{w}_i)$ *is the* i*-th elementary symmetric polynomial in the variables* $e_{k+1},\ldots,e_n$*.*

(5) An element $x \in H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ *is non-zero if, and only if, the product*

$$\pi^*(x)\cdot(e_1^{k-1}e_2^{k-2}\ldots e_{k-1})\cdot(e_{k+1}^{n-k-1}\ldots e_{n-1}) \in H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$$

is non-zero.

Based on these properties Stong [**13**, pp. 103–104] proved a series of results about non-triviality of products in $H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$. He used them to give what are still the best known estimates of the LS-category of real Grassmannians. The same products were also used in [**9**] to estimate the minimal size of triangulations of real Grassmannians. For the convenience of the reader we first summarize Stong's results and then derive further non-trivial cohomology products that will lead to new and much stronger estimates of topological complexity.

PROPOSITION 3.3. *(Stong, [**13**, p. 103]) If* $k \geqslant 2$ *and* $2^s < n \leqslant 2^{s+1}$ *then the height of* w_1 *(i.e., the maximal* m*, such that* $w_1^m \neq 0$ *in* $H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$*) is*

$$\mathrm{height}(w_1) = \begin{cases} 2^{s+1}-2 & k=2 \text{ or } k=3, n=2^s+1 \\ 2^{s+1}-1 & \text{otherwise.} \end{cases}$$

In particular, since $H^*(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ is generated by the Stiefel–Whitney classes w_1 and w_2, the above implies that

$$w_1^{2^{s+1}-2}\cdot w_2^{n-2^s-1} \in H^{2(n-2)}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$$

is a non-trivial cup product of maximal length in $H^*(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ (cf. note after proposition 3.6).

PROPOSITION 3.4. *(Stong, [**13**, p. 104]) Let* $2^s < n \leqslant 2^{s+1}$*. Then the following classes in* $H^{3(n-3)}(G_3(\mathbb{R}^n);\mathbb{Z}_2)$ $(1 \leqslant p,\ 0 < t < 2^{p-1}$ *are non-trivial products of maximal length in* $H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$*:*

$$\begin{aligned} &\text{if } n = 2^{s+1}-2^p+1, && \text{then } w_1^{2^{s+1}-2}\cdot w_2^{2^{s+1}-3\cdot 2^{p-1}-2} \neq 0; \\ &\text{if } n = 2^{s+1}-2^p+1+t, && \text{then } w_1^{2^{s+1}-1}\cdot w_2^{2^{s+1}-3\cdot 2^{p-1}-1}\cdot w_3^{t-1} \neq 0; \\ &\text{if } n = 2^{s+1}, && \text{then } w_1^{2^{s+1}-1}\cdot w_2^{2^{s+1}-4} \neq 0. \end{aligned}$$

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

(Note that there is a small typo in case (2) of proposition [**13**, p. 104], where it is stated that the longest non-trivial product is $w_1^{2^{s+1}-1} \cdot w_2^{2^{s+1}-3\cdot 2^{p-1}} \cdot w_3^{t-1}$. The value for the cup-length in the same proposition is correct, and the correct non-trivial product can be found in [**13**, p. 112].)

PROPOSITION 3.5. *(Stong, [**13**, p. 104]) Let $2^s < n \leqslant 2^{s+1}$. Then the following classes in $H^{4(n-4)}(G_4(\mathbb{R}^n);\mathbb{Z}_2)$ $(0 \leqslant r < s,\ 0 \leqslant t < 2^r)$ are non-trivial products of maximal length in $H^*(G_4(\mathbb{R}^n);\mathbb{Z}_2)$:*

$$\begin{array}{ll} \text{if } n = 2^s + 1, & \text{then } \ w_1^{2^{s+1}-2} \cdot w_2^{2^s-5} \neq 0,\ w_1^{2^{s+1}-1} \cdot w_2^{2^s-7} \cdot w_3 \neq 0; \\ \text{if } n = 2^s + 2^r + 1 + t, & \text{then } \ w_1^{2^{s+1}-2} \cdot w_2^{2^s+2^{r+1}-5} \cdot w_4^t \neq 0, \\ & \text{also } w_1^{2^{s+1}-1} \cdot w_2^{2^s+2^{r+1}-7} \cdot w_3 \cdot w_4^t \neq 0 \text{ if } r > 0; \end{array}$$

While the products listed in the above propositions give the best cohomological lower bounds for the Lusternik–Schnirelmann category of $G_k(\mathbb{R}^n)$ for $k \leqslant 4$, it turns out that there are other non-trivial products that lead to better estimates of topological complexity. We prove the relevant results in the rest of this section. Note that whenever we state that certain product of Stiefel–Whitney classes is non-trivial in the cohomology of $G_k(\mathbb{R}^n)$, then the same product is also non-trivial in the cohomology of $G_k(\mathbb{R}^m)$ for every $m \geqslant n$ (because the corresponding ideal of relations is smaller).

PROPOSITION 3.6. *If $2^s < n \leqslant 2^{s+1}$, then*

$$w_1^{2^s} \cdot w_2^{n-2^s-1} \neq 0 \quad \textit{in} \quad H^{2(n-2)-2^s+2}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$$

and

$$w_1^{2^s} \cdot w_2^{n-2^s} = 0 \quad \textit{in} \quad H^{2(n-2)-2^s+4}(G_2(\mathbb{R}^n);\mathbb{Z}_2).$$

Proof. By proposition 3.2(5), in order to check the first claim we must show that

$$\begin{aligned} \pi^*(w_1^{2^s} \cdot w_2^{n-2^s-1}) \cdot e_1 \cdot e_3^{n-3} \cdots e_{n-1} &= (e_1^{2^s} + e_2^{2^s})(e_1 \cdot e_2)^{n-2^s-1} \cdot e_1 \cdot e_3^{n-3} \cdots e_{n-1} \\ &= e_1^{n-2^s} \cdot e_2^{n-1} \cdot e_3^{n-3} \cdots e_{n-1} \\ &\neq 0 \ \text{ in } \ H^{2(n-2)-2^s+2}(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2). \end{aligned}$$

By multiplying the result by $e_1^{2^s-2}$ we get $e_1^{n-2} \cdot e_2^{n-1} \cdot e_3^{n-3} \cdots e_{n-1}$, which is non-trivial by proposition 3.2(3).

Similarly, for the second claim we compute

$$\pi^*(w_1^{2^s} \cdot w_2^{n-2^s}) \cdot e_1 \cdot e_3^{n-3} \cdots e_{n-1} = (e_1^{2^s} + e_2^{2^s})(e_1 \cdot e_2)^{n-2^s} \cdot e_1 \cdot e_3^{n-3} \cdots e_{n-1} = 0,$$

because $e_1^n = e_2^n = 0$ by proposition 3.2(2). □

Note that a similar computation shows that $w_1^{2^{s+1}-2} \cdot w_2^{n-2^s-1} \neq 0$ in $H^{2(n-2)}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$, and that the second claim implies that the exponent of w_1 in every other non-trivial product in $H^{2(n-2)}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ must be smaller than 2^s.

Next we turn to cohomology of $G_3(\mathbb{R}^n)$ in the range $2^s < n \leqslant 2^{s+1}$.

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

PROPOSITION 3.7.

(1) $w_1^{2^s} \cdot w_2^{2^{s-1}} \neq 0$ *and* $w_1^{2^s} \cdot w_2^{2^{s-1}} \cdot w_3 = 0$ *in* $H^*(G_3(\mathbb{R}^{2^s+1}); \mathbb{Z}_2)$;

(2) $w_1^{2^s} \cdot w_2^{2^{s-1}} \cdot w_3 \neq 0$ *and* $w_1^{2^s} \cdot w_2^{2^{s-1}} \cdot w_3^2 = 0$ *in* $H^*(G_3(\mathbb{R}^{2^s+2}); \mathbb{Z}_2)$;

(3) if $3 \leqslant t \leqslant 2^s$, *then* $w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{t-3} \neq 0$ *in* $H^{3(2^s+t-3)}(G_3(\mathbb{R}^{2^s+t}); \mathbb{Z}_2)$.

Proof. Towards the proof of (1), by proposition 3.2(5), we must show that the product

$$\pi^*(w_1^{2^s} \cdot w_2^{2^{s-1}}) \cdot e_1^2 \cdot e_2 \cdot e_4^{2^s-3} \cdots e_{2^s} \neq 0.$$

To obtain a top-dimensional class we multiply the above product by $e_1 \cdot e_2$ and take into account that $e_1^{2^s+1} = e_2^{2^s+1} = 0$ to obtain

$$\begin{aligned} &e_1 \cdot e_2 \cdot \pi^*(w_1^{2^s} \cdot w_2^{2^{s-1}}) \cdot e_1^2 \cdot e_2 \cdot e_4^{2^s-3} \cdots e_{2^s} \\ &\quad = e_1 \cdot e_2 \cdot (e_1^{2^s} + e_2^{2^s} + e_3^{2^s}) \cdot \big((e_1 \cdot e_2)^{2^{s-1}} + (e_1 \cdot e_3)^{2^{s-1}} \\ &\qquad + (e_2 \cdot e_3)^{2^{s-1}}\big) \cdot e_1^2 \cdot e_2 \cdot e_4^{2^s-3} \cdots e_{2^s} \\ &\quad = e_1^{2^s-2} \cdot e_2^{2s-3} \cdot e_3^{2^s-1} \cdot e_4^{2^s-3} \cdots e_{2^s}, \end{aligned}$$

which is non-trivial by proposition 3.2(3), therefore $w_1^{2^s} \cdot w_2^{2^{s-1}} \neq 0$.

On the other hand, it is easy to check that all terms in the expansion of the product

$$\begin{aligned} &\pi^*(w_1^{2^s} \cdot w_2^{2^{s-1}} \cdot w_3) \cdot e_1^2 \cdot e_2 \cdot e_4^{2^s-3} \cdots e_{2^s} \\ &\quad = (e_1^{2^s} + e_2^{2^s} + e_3^{2^s}) \cdot \big((e_1 \cdot e_2)^{2^{s-1}} + (e_1 \cdot e_3)^{2^{s-1}} \\ &\qquad + (e_2 \cdot e_3)^{2^{s-1}}\big) \cdot (e_1 \cdot e_2 \cdot e_3) \cdot e_1^2 \cdot e_2 \cdot e_4^{2^s-3} \cdots e_{2^s} \end{aligned}$$

have at least one factor whose exponent is bigger or equal to $2^s + 1$, so the entire expression is trivial by proposition 3.2(2).

Case (2) is entirely analogous and is left to the reader.

For case (3) we compute

$$\begin{aligned} \pi^*(w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{t-3}) \cdot e_1^2 \cdot e_2 \cdot e_4^{2^s-3} \cdots e_{2^s} &= (e_1^{2^s} + e_2^{2^s} + e_3^{2^s}) \cdot \big((e_1 \cdot e_2)^{2^s} + (e_1 \cdot e_3)^{2^s} \\ &\quad + (e_2 \cdot e_3)^{2^s}\big) \cdot (e_1 \cdot e_2 \cdot e_3)^{t-3} \cdot e_1^2 \cdot e_2 \cdot e_4^{2^s-3} \cdots e_{2^s} \\ &= e_1^{2^s+t-1} \cdot e_2^{2^s+t-2} \cdot e_3^{2^s+t-3} \cdot e_4^{2^s-3} \cdots e_{2^s}, \end{aligned}$$

which is non-trivial by proposition 3.2(5). Note that $w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{t-3}$ is a top-dimensional class in the cohomology of $G_3(\mathbb{R}^{2^s+t})$. □

For $k > 3$ the computations become increasingly complicated but we are still able to obtain some general results.

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

PROPOSITION 3.8.

(1) If $3 \leqslant k \leqslant 2^s$, *then* $w_1^{2^s} \cdot w_2^{2^s} = w_3^{2^s} \neq 0$ *in* $H^{3\cdot 2^s}(G_k(\mathbb{R}^{2^s+k}); \mathbb{Z}_2)$.

(2) If $3 \leqslant k \leqslant 2^{s-1}$, *then* $w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{2^{s-1}} \neq 0$ *in* $H^{3(2^s+2^{s-1})}(G_k(\mathbb{R}^{2^s+2^{s-1}+k}); \mathbb{Z}_2)$.

(3) If $3 \leqslant k \leqslant 2^s$, *then* $w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{2^s} = 0$ *in* $H^{6\cdot 2^s}(G_k(\mathbb{R}^{2^s+k}); \mathbb{Z}_2)$.

Proof. First note that

$$\begin{aligned}\pi^*(w_1^{2^s} \cdot w_2^{2^s}) &= \left(\sum_{1\leqslant a\leqslant k} e_a^{2^s}\right) \cdot \left(\sum_{1\leqslant b<c\leqslant k} e_b^{2^s} e_c^{2^s}\right) \\ &= \left(\sum_{1\leqslant a<b<c\leqslant k} e_a^{2^s} e_b^{2^s} e_c^{2^s}\right) = \pi^*(w_3^{2^s})\end{aligned}$$

because $e_i^{2^{s+1}} = 0$ by proposition 3.2(2), therefore all products where a equals b or c disappear. Injectivity of π^* implies that $w_1^{2^s} \cdot w_2^{2^s} = w_3^{2^s}$.

To prove that the product is non-trivial we must show that

$$\pi^*(w_3^{2^s}) \cdot (e_1^{k-1} \cdots e_{k-1}) \cdot (e_{k+1}^{2^s-1} \cdots e_{2^s+k-1})$$

is a non-zero element of $H^*(\mathrm{Flag}(\mathbb{R}^{2^s+k}); \mathbb{Z}_2)$. To obtain a top-dimensional element we multiply the above product by $e_4^{2^s} \cdots e_k^{2^s}$:

$$\begin{aligned}&\pi^*(w_3^{2^s}) \cdot (e_1^{k-1} \cdots e_{k-1}) \cdot (e_{k+1}^{2^s-1} \cdots e_{2^s+k-1}) \cdot (e_4^{2^s} \cdots e_k^{2^s}) \\ &= \left(\sum_{1\leqslant a<b<c\leqslant k} e_a^{2^s} e_b^{2^s} e_c^{2^s}\right) \cdot (e_1^{k-1} \cdots e_{k-1}) \cdot \\ &\quad (e_{k+1}^{2^s-1} \cdots e_{2^s+k-1}) \cdot (e_4^{2^s} \cdots e_k^{2^s}) = e_1^{2^s+k-1} e_2^{2^s+k-2} \cdots e_k^{2^s} e_{k+1}^{2^s-1} \cdots e_{2^s+k-1} \neq 0.\end{aligned}$$

Note that multiplication by $e_4^{2^s} \cdots e_k^{2^s}$ annihilates all summands in line two except $e_1^{2^s} e_2^{2^s} e_3^{2^s}$.

The computation in case (2) is analogous so we leave the details to the reader:

$$\begin{aligned}&\pi^*(w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{2^{s-1}}) \cdot (e_1^{k-1} \cdots e_{k-1}) \cdot (e_{k+1}^{2^s+2^{s-1}-1} \cdots e_{2^s+2^{s-1}+k-1}) \\ &\quad \cdot (e_4^{2^s+2^{s-1}} \cdots e_k^{2^s+2^{s-1}}) \\ &= e_1^{2^s+2^{s-1}+k-1} e_2^{2^s+2^{s-1}+k-2} \cdots e_k^{2^s+2^{s-1}} e_{k+1}^{2^s+2^{s-1}-1} \cdots e_{2^s+2^{s-1}+k-1} \neq 0.\end{aligned}$$

To prove (3) note that

$$\pi^*(w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{2^s}) = \pi^*(w_3^{2^{s+1}}) = \left(\sum_{1\leqslant a<b<c\leqslant k} e_a^{2^{s+1}} e_b^{2^{s+1}} e_c^{2^{s+1}}\right) = 0$$

because $e_i^{2^{s+1}} = 0$. □

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

We conclude with a result that for $G_4(\mathbb{R}^n)$ improves proposition 3.8 in a range of dimensions.

PROPOSITION 3.9. *If* $3 \leqslant t \leqslant 2^{s-1}$, *then* $w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{2^{s-1}} \cdot w_4^{t-3} \neq 0$ *in* $H^*(G_4(\mathbb{R}^{2^s+2^{s-1}+t}); \mathbb{Z}_2)$.

Proof. To simplify the notation let us denote $d := 2^s + 2^{s-1}$. Then we compute the value of the following top-dimensional element:

$$\begin{aligned}
&(e_1)^{d-4} \cdot \pi^*(w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{2^{s-1}} \cdot w_4^{t-3}) \cdot e_1^3 \cdot e_2^2 \cdot e_3 \cdot e_5^{d+t-5} \cdots e_{d+t-1} \\
&\quad = \pi^*(w_3^d) \cdot (e_1 \cdot e_2 \cdot e_3 \cdot e_4)^{t-3} \cdot e_1^{d+t-4} \cdot e_2^{t-1} \cdot e_3^{t-2} \cdot e_4^{t-3} \cdot e_5^{d+t-5} \cdots e_{d+t-1} \\
&\quad = e_1^{d+t-4} \cdot e_2^{d+t-1} \cdot e_3^{d+t-2} \cdot e_4^{d+t-3} \cdot e_5^{d+t-5} \cdots e_{d+t-1} \neq 0
\end{aligned}$$

(the other summands vanish because the exponent of e_1 is at least $d+t$). □

4. Topological complexity of Grassmann manifolds

Most of this section is devoted to the computation of lower bounds for the topological complexity of real Grassmannians but let us first mention the complex case which is much simpler: complex Grassmann manifolds $G_k(\mathcal{C}^n)$ are simply-connected and symplectic, so by proposition 2.3 we have the precise value

PROPOSITION 4.1. *Topological complexity of complex Grassmann manifold is* $\mathrm{TC}(G_k(\mathcal{C}^n)) = 2k(n-k)+1$.

The fundamental group of real Grassmannians is $\mathbb{Z}_2$, so we may apply proposition 2.4 to obtain upper bounds for topological complexity.

THEOREM 4.2. *Topological complexity of real Grassmannians is bounded above by*

$$\mathrm{TC}(G_k(\mathbb{R}^n)) \leqslant 2k(n-k).$$

In fact, unless $k = 1, n = 2^d$ *or* $k = 2, n = 2^d + 1$ *we have a better estimate*

$$\mathrm{TC}(G_k(\mathbb{R}^n)) \leqslant 2k(n-k) - 1.$$

Proof. By [**1**] the height of the class w_1 that generate $H^1(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ is strictly smaller than the dimension, except if $k = 1$ or if $k = 2$ and $n = 2^d + 1$. Moreover, if $k = 1$, then by [**6**] $\mathrm{TC}(\mathbb{R}P^n) = 2n$ if, and only if, n is a power of 2, otherwise $\mathrm{TC}(\mathbb{R}P^n) \leqslant 2n - 1$. We do not know whether $\mathrm{TC}(G_2(\mathbb{R}^{2^d+1}))$ can be equal to the stated upper bound $4(2^d - 1)$. We will see below that cohomological lower estimates for the topological complexity are much smaller. □

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

By proposition 2.2(5), in order to estimate from below topological complexity of X we need a supply of elements in

$$\mathrm{Ker}(\Delta^*\colon H^*(X\times X;\mathbb{Z}_2)\to H^*(X;\mathbb{Z}_2)).$$

For every $w\in H^*(X;\mathbb{Z}_2)$ we have

$$\Delta^*(w\times 1+1\times w)=w\smile 1+1\smile w=w+w=0,$$

therefore

$$z(w):=w\times 1+1\times w\in \mathrm{Ker}\,\Delta^*.$$

Farber [5] called elements of $\mathrm{Ker}\,\Delta^*$ *zero-divisors* and defined *zero-divisor cup-length* of X, denoted $\mathrm{zcl}(X)$, to be the maximal number of elements of $\mathrm{Ker}\,\Delta^*$ whose product is non-trivial (by analogy with the classical *cup-length* from LS- category theory, cf. [2]). In other words, $\mathrm{zcl}(X)=\mathrm{nil}(\mathrm{Ker}\,\Delta^*)-1$. Note that in general one can define $\mathrm{zcl}(X)$ with respect to any multiplicative cohomology theory, but in this paper we consider only ordinary cohomology with $\mathbb{Z}_2$-coefficients.

To determine the zero-divisor cup-length of Grassmannians we will need information about the height of basic zero-divisors of the form $z(w)$ and this can be easily expressed in terms of the height of w. For an integer $n\geqslant 0$ let $\rho(n)$ denote the minimal integral power of 2 that is strictly bigger than n (e.g., $\rho(8)=\rho(11)=16$). In particular $(\rho(n)/2)\leqslant n<\rho(n)$.

Lemma 4.3. *For every* $w\in H^*(X;\mathbb{Z}_2)$

$$\mathrm{height}(z(w))=\rho(\mathrm{height}(w))-1.$$

Proof. Let $n=\mathrm{height}(w)$, i.e. $w^n\neq 0$ and $w^{n+1}=0$. It is well-known that all binomial coefficients of the form $\binom{2^s-1}{k}$, $k=0,1,\ldots,2^s-1$ are odd and that all binomial coefficients of the form $\binom{2^s}{k}$, $k=1,\ldots,2^s-1$ are even. As a consequence

$$z(w)^{2^s-1}=(w\times 1+1\times w)^{2^s-1}$$

has a non-zero coefficient at the term $w^n\times w^{2^s-1-n}\neq 0$. Therefore if, $\rho(n)=2^s$, then $z(w)^{2^s-1}\neq 0$. On the other hand

$$z(w)^{2^s}=(w\times 1+1\times w)^{2^s}=(w^{2^s}\times 1+1\times w^{2^s})=0.$$

□

Lemma 4.4. *If for some* $u_1,\ldots,u_n\in H^*(X;\mathbb{Z}_2)$ *and positive integers* $k_1,\ldots,k_n$

$$u_1^{k_1}\cdot\ldots\cdot u_n^{k_n}\neq 0,$$

then

$$z(u_1)^{\rho(k_1)-1}\cdot\ldots\cdot z(u_n)^{\rho(k_n)-1}\neq 0.$$

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

Proof. It is sufficient to observe that the term

$$\left(u_1^{k_1}\cdot\ldots\cdot u_n^{k_n}\right)\times\left(u_1^{\rho(k_1)-k_1-1}\cdot\ldots\cdot u_n^{\rho(k_n)-k_n-1}\right)$$

appears with a non-zero coefficient in the product $z(u_1)^{\rho(k_1)-1}\cdot\ldots\cdot z(u_n)^{\rho(k_n)-1}$, and that it is non-trivial, because $\rho(k_i)-k_i-1\leqslant k_i$ for $i=1,\ldots,n$. □

The question of the topological complexity of $G_1(\mathbb{R}^n)=\mathbb{R}P^{n-1}$ is considered to be solved by being reduced to the classical immersion problem for projective spaces (see [**6**]), so we pass to the next case and derive a lower bound for the topological complexity of $G_2(\mathbb{R}^n)$ (recall our standing assumption that $k\leqslant n/2$).

THEOREM 4.5. *Assume* $2^s<n\leqslant 2^{s+1}$. *Then*

$$z(w_1)^{2^{s+1}-1}\cdot z(w_2)^{\rho(n-1-2^s)-1}\neq 0$$

is a non-trivial product of zero-divisors of maximal length, so that

$$\operatorname{zcl}(G_2(\mathbb{R}^n))=2^{s+1}+\rho(n-1-2^s)-2$$

and

$$\operatorname{TC}(G_2(\mathbb{R}^n))\geqslant 2^{s+1}+\rho(n-1-2^s)-1.$$

Proof. We proved in proposition 3.6 that $w_1^{2^s}\cdot w_2^{n-2^s-1}\neq 0$ and that $w_1^{2^s}\cdot w_2^{n-2^s}=0$ in the cohomology of $G_2(\mathbb{R}^n)$. Non-triviality of the first product implies that zero-divisor cup-length is at least

$$\operatorname{zcl}(G_2(\mathbb{R}^n))\geqslant\rho(2^s)-1+\rho(n-2^s-1)-1=2^{s+1}+\rho(n-2^s-1)-2.$$

On the other hand, triviality of the second product implies that if a non-trivial product $w_1^a\cdot w_2^b\neq 0$ has $b\geqslant n-2^s$, then $a<2^s$, which yields a shorter product of zero-divisors (indeed, $\operatorname{height}(w_2)=n-2$ by [**4**] and so $\rho(w_2)\leqslant 2^{s+1}$). Therefore $z(w_1)^{2^{s+1}-1}\cdot z(w_2)^{\rho(n-1-2^s)-1}$ is the longest non-trivial product of zero divisors.

Estimate for topological complexity follows from proposition 2.2(4), as $\operatorname{TC}(X)\geqslant\operatorname{zcl}(X)+1$. □

Observe that, by the above argument, in order to achieve maximal zero-divisor cup-length one must look for products of the form $w_1^a\cdot w_2^b\cdot w_3^c\cdots$ where $a,b,c,\ldots$ are powers of two (and as big as possible). This is why products with maximal cup-length as in [**13**] often do not give products with maximal zero-divisor cup-length.

EXAMPLE 4.6. The first non-trivial case to consider is $G_2(\mathbb{R}^4)$. It is 4-dimensional and since $w_1^4=0$ (proposition 3.3), proposition 2.4 implies that $\operatorname{TC}(G_2(\mathbb{R}^4))\leqslant 7$. A lower estimate can be computed from the above theorem as $\operatorname{TC}(G_2(\mathbb{R}^4))\geqslant\rho(3)+\rho(1)-1=5$, therefore $5\leqslant\operatorname{TC}(G_2(\mathbb{R}^4))\leqslant 7$.

EXAMPLE 4.7. Note that the gap between the upper bound of theorem 4.2 and the lower bound of theorem 4.5 grows with n. For example, by applying mentioned

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

results we obtain

$$23 \leqslant \mathrm{TC}(G_2(\mathbb{R}^{13})) \leqslant 43.$$

This is perhaps not surprising in view of the irregularities and gaps between the lower estimates and the dimensional upper bounds for the topological complexity of projective spaces. As we mentioned before, computation of $\mathrm{TC}(\mathbb{R}P^n)$ was reduced to the famous immersion problem for projective spaces—see tables in https://www.lehigh.edu/ dmd1/imms.html, whose computation has required an immense body of work. Nevertheless, estimates of theorem 4.5 represent a notable improvement with respect to those implied by propositions 2.2(3) and 2.1(4) as they would give only $\mathrm{TC}(G_2(\mathbb{R}^{13})) \geqslant 19$.

To estimate topological complexity of $G_3(\mathbb{R}^n)$ we use proposition 3.7 and consider three cases. We remind the reader that if a product of Stiefel–Whitney classes is non-zero in $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ then the same product is non-zero in $H^*(G_k(\mathbb{R}^m); \mathbb{Z}_2)$ for all $m \geqslant n$.

THEOREM 4.8. *Assume $2^s < n \leqslant 2^{s+1}$ and let $t := n - 2^s$. The results on zero-divisor cup-length and topological complexity of $G_3(\mathbb{R}^n)$ are summarized in the following table:*

t	$\mathrm{zcl}(G_3(\mathbb{R}^n))$	*realized by*	$\mathrm{TC}(G_3(\mathbb{R}^n)) \geqslant$
1	$3 \cdot 2^s - 2$	$z(w_1)^{2^{s+1}-1} \cdot z(w_2)^{2^s-1}$	$3 \cdot 2^s - 1$
2	$3 \cdot 2^s - 1$	$z(w_1)^{2^{s+1}-1} \cdot z(w_2)^{2^s-1} \cdot z(w_3)$	$3 \cdot 2^s$
$3, \ldots, 2^s$	$4 \cdot 2^s + \rho(t-3) - 3$	$z(w_1)^{2^{s+1}-1} \cdot z(w_2)^{2^{s+1}-1} \cdot z(w_3)^{\rho(t-3)-1}$	$4 \cdot 2^s + \rho(t-3) - 2$

Note that for about half of the range ($n \geqslant 2^s + 2^{s-1} + 3$) the exact value is $\mathrm{zcl}(G_3(\mathbb{R}^n)) = 5 \cdot 2^s - 3$.

Proof. The three cases correspond to the points (1),(2) and (3) of proposition 3.7. In each of those we proved that certain products in the cohomology of $G_3(\mathbb{R}^n)$ are non-trivial, and that products that would give longer products of zero-divisors are trivial. The values in the table follow by application of lemma 4.4 and proposition 2.2(5).

In particular, if $n \geqslant 2^s + 2^{s-1} + 3$, then $\rho(n - 2^s - 3) = 2^s$, therefore $\mathrm{zcl}(G_3(\mathbb{R}^n)) = 5 \cdot 2^s - 3$. □

EXAMPLE 4.9. For $G_3(\mathbb{R}^{11})$ the above theorem yields the estimate $\mathrm{TC}(G_3(\mathbb{R}^{11})) \geqslant 31$, which is again a considerable improvement from the category lower bound $\mathrm{TC}(G_3(\mathbb{R}^{11})) \geqslant 20$.

To describe the estimates of $\mathrm{TC}(G_k(\mathbb{R}^n))$ for $k > 3$ we will consider for simplicity only 'generic' pairs (k, n).

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

THEOREM 4.10. *Let* $4 \leqslant k \leqslant 2^{s-1}$ *and* $2^s + k \leqslant n \leqslant 2^{s+1}$. *Then*

$$\mathrm{TC}(G_k(\mathbb{R}^n)) \geqslant 4 \cdot 2^s - 1 \quad \textit{for} \quad n \leqslant 2^s + 2^{s-1} + 2$$

and

$$\mathrm{TC}(G_k(\mathbb{R}^n)) \geqslant 5 \cdot 2^s - 2 \quad \textit{for} \quad n \geqslant 2^s + 2^{s-1} + 3.$$

Proof. In the first case $w_1^{2^s} \cdot w_2^{2^s} \neq 0$ by proposition 3.8(1), while in the second case $w_1^{2^s} \cdot w_2^{2^s} \cdot w_3^{2^{s-1}} \neq 0$ by proposition 3.8(2). Then the stated estimates follow by proposition 2.2(5) and lemma 4.4. Note that we do not claim that the mentioned products give products of zero-divisors of maximal length. □

EXAMPLE 4.11. By the above theorem we have $\mathrm{TC}(G_5(\mathbb{R}^{13})) \geqslant 31$, compared to the category estimate $\mathrm{TC}(G_5(\mathbb{R}^{13})) \geqslant 16$, and $\mathrm{TC}(G_6(\mathbb{R}^{27})) \geqslant 78$, compared to the category estimate $\mathrm{TC}(G_6(\mathbb{R}^{27})) \geqslant 32$.

The final result of this section is a direct consequence of proposition 3.9 and gives the zero-divisor cup-length of $G_4(\mathbb{R}^n)$ for approximately half of dimensions. The proof follows the lines of the previous theorems and is thus left to the reader.

THEOREM 4.12. *Assume* $2^s + 2^{s-1} + 3 \leqslant n \leqslant 2^{s+1}$ *and let* $t := n - 2^s - 2^{s-1}$. *Then*

$$\mathrm{zcl}(G_4(\mathbb{R}^n)) = 5 \cdot 2^s + \rho(t-3) - 4,$$

realized by the product

$$z(w_1)^{2^{s+1}-1} \cdot z(w_2)^{2^{s+1}-1} \cdot z(w_3)^{2^s-1} \cdot z(w_4)^{\rho(t-3)-1} \neq 0.$$

5. Monotonicity of category and topological complexity for increasing sequences of Grassmannians

We already mentioned that $\mathrm{cat}(G_1(\mathbb{R}^n)) = \mathrm{cat}(\mathbb{R}P^{n-1}) = n$, so clearly $m \leqslant n$ implies that $\mathrm{cat}(G_1(\mathbb{R}^m)) \leqslant \mathrm{cat}(G_1(\mathbb{R}^n))$. This is not a coincidence, in fact $\mathbb{R}P^m$ is the m-skeleton of $\mathbb{R}P^n$ with respect to the standard CW-decomposition of projective spaces and it is a well-known fact (see [**2**, theorem 1.66]) that $\mathrm{cat}(X^{(m)}) \leqslant \mathrm{cat}(X)$ for every non-contractible CW-complex X. The analogous comparison of topological complexities of projective spaces is less obvious. Farber, Tabachnikov and Yuzvinsky [**6**] proved that for $n \neq 1, 3, 7$ $\mathrm{TC}(\mathbb{R}P^n)$ equals the minimal k such that $\mathbb{R}P^n$ admits an immersion in $\mathbb{R}^{k-1}$. Therefore, $\mathrm{TC}(G_1(\mathbb{R}^n))$ is clearly an increasing function of n (see example 5.7 for a more direct proof).

Intuitively, the spaces $G_k(\mathbb{R}^n)$ become more complicated by increasing the dimension, so one should naturally expect that their category and topological complexity increase as well. But this is not always the case, as we can see from the sequence of spheres $S^1 \subseteq S^2 \subseteq S^3 \subseteq \ldots \subseteq S^\infty$ whose topological complexities form the alternating sequence $2, 3, 2, 3, \ldots$ and $\mathrm{TC}(S^\infty) = 1$. The case of Grassmannians is even more complicated because (for $k > 1$) $G_k(\mathbb{R}^m)$ is not even close to being a skeleton of $G_k(\mathbb{R}^n)$. In spite of that we will show that for $k = 2, 3$ the category of $G_k(\mathbb{R}^n)$ is

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

indeed increasing with n. The case of topological complexity is more complicated but we will still manage to prove some partial results.

Throughout this section we will view $G_k(\mathbb{R}^m)$ as a subcomplex of $G_k(\mathbb{R}^n)$ $(2k \leqslant m \leqslant n)$ with respect to the standard decomposition described in § 3. Our first observation is that the cells of the respective decompositions coincide up to dimension $m-k$, that is to say

$$G_k(\mathbb{R}^n)^{(m-k)} = G_k(\mathbb{R}^m)^{(m-k)}.$$

In [**11**] we proved the following result:

THEOREM 5.1. *([**11**, theorem 3.6]) Let X be a connected CW-complex X, and let A be a subcomplex of X containing X^d. If* $\dim(A) < d + \operatorname{cat}(X) - 1$, *then* $\operatorname{cat}(A) \leqslant \operatorname{cat}(X)$.

In our case we obtain that $\operatorname{cat}(G_k(\mathbb{R}^m)) \leqslant \operatorname{cat}(G_k(\mathbb{R}^n))$ provided that $k(m-k) < m-k+\operatorname{cat}(G_k(\mathbb{R}^n)) - 1$, or equivalently, if $\operatorname{cat}(G_k(\mathbb{R}^n)) > (k-1)(m-k)+1$.

THEOREM 5.2. *If $k \leqslant 3$ and $2k \leqslant m \leqslant n$, then* $\operatorname{cat}(G_k(\mathbb{R}^m)) \leqslant \operatorname{cat}(G_k(\mathbb{R}^n))$.

Proof. The case $k=1$ is clear, so let us consider $k=2$. By the above discussion it is sufficient to show that $\operatorname{cat}(G_2(\mathbb{R}^n)) > n-2$ for all $n \geqslant 4$. By proposition 3.3, if $2^s < n \leqslant 2^{s+1}$, then $w_1^{2^{s+1}-2} \cdot w_2^{n-2^s-1} \neq 0$, therefore

$$\operatorname{cat}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} - 2 + n - 2^s = n + 2^s - 2 > n - 2.$$

If $k=3$, we must check that $\operatorname{cat}(G_3(\mathbb{R}^n)) > 2n-5$ for all $n \geqslant 6$. To do this, we consider the three cases of proposition 3.4. If $n = 2^{s+1} - 2^p + 1 + t$ $(1 \leqslant p < s,\ 0 < t < 2^{p-1})$, then

$$\begin{aligned}\operatorname{cat}(G_3(\mathbb{R}^n)) - (2n-5) &\geqslant (2^{s+1} - 1 + 2^{s+1} - 3 \cdot 2^{p-1} - 1 + t) \\ &\quad - 2(2^{s+1} - 2^p + 1 + t) + 5 = 2^{p-1} + 1 - t > 0.\end{aligned}$$

The remaining two cases are verified similarly. □

For $k=4$ our approach works only under some additional assumptions.

THEOREM 5.3. *If $m \leqslant 2^s + 1$, then* $\operatorname{cat}(G_4(\mathbb{R}^m)) \leqslant \operatorname{cat}(G_4(\mathbb{R}^{2^s+1}))$.

Otherwise, if $n = 2^s + 2^p + 1 + t$ $(0 \leqslant p < s,\ 0 \leqslant t < 2^p)$, then $\operatorname{cat}(G_4(\mathbb{R}^m)) \leqslant \operatorname{cat}(G_4(\mathbb{R}^n))$, *provided that* $n - m > (2^p + 2t - 2/3)$.

Proof. To prove the claim we must show that $\operatorname{cat}(G_4(\mathbb{R}^m)) > 3m - 11$. In the first case we have by proposition 3.5 that $w_1^{2^{s+1}-2} \cdot w_2^{2^s-5} \neq 0$, therefore

$$\operatorname{cat}(G_4(\mathbb{R}^{2^s+1})) \geqslant 2^{s+1} + 2^s - 6 = 3 \cdot (2^s + 1) - 9 > 3m - 11$$

because $m \leqslant 2^s + 1$.

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

Similarly, if $n = 2^s + 2^p + 1 + t$, then $w_1^{2^{s+1}-2} \cdot w_2^{2^s+2^{r+1}-5} \cdot w_4^t \neq 0$, therefore

$$\operatorname{cat}(G_4(\mathbb{R}^r)) \geqslant 2^{s+1} + 2^s + 2^{p+1} + t - 6 = 3\big(2^s + 2^p + 1 + t - (2^p + 2t - 2/3)\big) - 11 > 3m - 11.$$

□

Example 5.4. By comparing estimates for $\operatorname{cat}(G_4(\mathbb{R}^n))$ we find that

$$\operatorname{cat}(G_4(\mathbb{R}^8)) \leqslant \operatorname{cat}(G_4(\mathbb{R}^9)) \leqslant \ldots \leqslant \operatorname{cat}(G_4(\mathbb{R}^{13}))$$

but we cannot conclude that $\operatorname{cat}(G_4(\mathbb{R}^{13})) \leqslant \operatorname{cat}(G_4(\mathbb{R}^{14}))$. In fact, we can only claim a weaker estimate $\operatorname{cat}(G_4(\mathbb{R}^{12})) \leqslant \operatorname{cat}(G_4(\mathbb{R}^{14}))$.

In order to derive analogous results for topological complexity we rely on the following result from [**11**].

Theorem 5.5. *([**11**, theorem 3.1]). Let X be a connected CW-complex X, and let A be a subcomplex of X containing X^d. If $2\dim(A) < d + \operatorname{TC}(X) - 1$, then $\operatorname{TC}(A) \leqslant \operatorname{TC}(X)$.*

In the case of the inclusion $G_k(\mathbb{R}^m) \subseteq G_k(\mathbb{R}^n)$ the condition becomes

$$2k(m-k) < (m-k) + \operatorname{TC}(G_k(\mathbb{R}^n)) - 1,$$

so we have

Corollary 5.6. *Let $2k \leqslant m \leqslant n$. If $\operatorname{TC}(G_k(\mathbb{R}^n)) > (2k-1)(m-k)+1$, then $\operatorname{TC}(G_k(\mathbb{R}^m)) \leqslant \operatorname{TC}(G_k(\mathbb{R}^n))$.*

Example 5.7. Let $k = 1$ and $m < n$. Then we have

$$\operatorname{TC}(G_1(\mathbb{R}^m)) \geqslant \operatorname{cat}(G_1(\mathbb{R}^n)) = n > (2 \cdot 1 - 1)(m-1) + 1,$$

therefore $\operatorname{TC}(G_1(\mathbb{R}^m)) \leqslant \operatorname{TC}(G_1(\mathbb{R}^n))$ whenever $m \leqslant n$, as we already mentioned at the beginning of this section.

Example 5.8. For $k = 2$ the condition in the above corollary becomes $\operatorname{TC}(G_2(\mathbb{R}^n)) > 3m - 5$. For instance, we proved in theorem 4.5 that $\operatorname{TC}(G_2(\mathbb{R}^{2^s+1})) \geqslant 2^{s+1}$, therefore $\operatorname{TC}(G_2(\mathbb{R}^{2^s+1})) \geqslant \operatorname{TC}(G_2(\mathbb{R}^m))$ whenever $m < (2^{s+1} + 5/3)$. Hence, we have $\operatorname{TC}(G_2(\mathbb{R}^9)) \geqslant \operatorname{TC}(G_2(\mathbb{R}^6))$, but we do not know whether $\operatorname{TC}(G_2(\mathbb{R}^9)) \geqslant \operatorname{TC}(G_2(\mathbb{R}^7))$.

Acknowledgements

We are very grateful to the anonymous referee for his or her careful reading of the paper and for many valuable suggestions and comments which helped us to correct and greatly improve the article.

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press

References

1 I. Berstein. On the Lusternik-Schnirelmann category of Grassmannians. *Proc. Cambridge Philos. Soc.* **79** (1976), 129–134.

2 O. Cornea, G. Lupton, J. Oprea and D. Tanré. *Lusternik-Schnirelmann Category*, AMS, Mathematical Surveys and Monographs, vol. 103 (2003).

3 A. Costa and M. Farber. Motion planning in spaces with small fundamental groups. *Commun. Contemp. Math.,* **12** (2010), 107–119.

4 S. Dutta and S. S. Khare. On second Stiefel-Whitney class of Grassmann manifolds and cuplength. *J. Indian Math. Soc.* **69** (2002), 237–251.

5 M. Farber. Topological complexity of motion planning. *Discrete Comput. Geom.* **29** (2003), 211–221.

6 M. Farber, S. Tabachnikov and Y. Sergey. Topological robotics: motion planning in projective spaces. *Int. Math. Res. Not.* **34** (2003), 1853–1870.

7 M. Farber. Topology of robot motion planning. In *Morse theoretic methods in nonlinear analysis and in symplectic topology*. (ed. P. Biran, O. Cornea and F. Lalonde). pp. 185–230 (Springer, 2006).

8 M. Farber. *Invitation to topological robotics* (Zurich: EMS Publishing House, 2008).

9 D. Govc, W. Marzantowicz and P. Pavešić. *How many simplices are needed to triangulate a Grassmannian?*, to appear in Topological Methods in Nonlinear Analysis (2020).

10 J. Milnor and J. D. Stasheff. *Characteristic classes* (Princeton, N. J.: Princeton University Press; Tokyo: University of Tokyo Press, 1974) .

11 P. Pavešić. Monotonicity of the Schwarz genus. *Proc. AMS* **148** (2020), 1339–1349.

12 K. J. Pearson and T. Zhang. Topological complexity and motion planning in certain real Grassmannians. *Appl. Math. Lett.* **17** (2004), 499–502.

13 R. Stong. Cup products in Grassmannians. *Top. Appl.* **13** (1982), 103–113.

https://doi.org/10.1017/prm.2020.92 Published online by Cambridge University Press