$a \neq 0$ and $b \neq 0$. Thus with

$$(a, b) = \left(w(w-2)\left(w^2 - w + 1\right), -w^2(w-1)\left(w^2 - 2w + 2\right)\right)$$

where w is an integer different from 0, 1 and 2, we obtain infinitely many cubics $x^3 + ax + b$ and $x^3 - ax + b$, both of which can be resolved into factors.

The cubics considered here have a special form, in that the coefficient of x^2 is zero. It may appear that such a choice was motivated by the desire to subject the quadratics $x^2 \pm ax + b$ to a minimal modification but the reader may note that any cubic of the form $x^3 + px^2 + qx + r$ may be reduced to a cubic of the form $y^3 + ky + l$ where k and l depend on p, q and r, by changing the variable from x to $y - \frac{1}{3}p$.

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105.41 Topology haiku matrix

	Searching for holes and voids	Metrics	Fine and coarse
Sometimes metrisable sometimes not	In homotopy we can quotient with great ease, as I search for voids.	In the Kansas plains we do move like an eagle, straight lines are enough.	There are but two sets: one HUGE and the other small The coarsest of all.
Spaces and points	Square, disk, triangle each its own unique space but basically a point.	and, in Manhattan we move block by block driving our geodesic.	The discrete is the topology of open points: all sets clopen.
Counter-intuitive but true!	A wedding ring is essentially a doughnut, But NOT to my wife!	Changing our metric allows us to make circles that look just like squares.	Topologies can be extreme, but excellent counterexamples!

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105.42 Generalised binomial theorem via Laplace transform technique

Introduction

The binomial theorem is an important result which appears frequently in different areas of mathematics. It states that

$$(1 + t)^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, t^k,$$

where t is any real number and n is a non-negative integer.

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The above result has been extended for real exponents. This extension is known as the *Generalised Binomial Theorem* (GBT).

Theorem (GBT): Let α and t be any real numbers such that $|t| < 1$. Then

$$(1 + t)^{\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} t^k, \tag{1}$$

where $(\alpha)_k = \alpha(\alpha - 1)(\alpha - 2)\ldots(\alpha - k + 1)$ denotes the *falling factorial* with $(\alpha)_0 = 1$, and 0^0 is interpreted as 1.

For non-negative integer values of α, the GBT reduces to the binomial theorem.

The history of GBT goes back to the 17th century mathematician and physicist Newton followed by the 18th-19th centuries mathematician Euler, Abel and Gauss. Newton's treatment of the GBT lacked the issue of series convergence which was later discussed by Euler, Abel and Gauss. In fact, Gauss was the first mathematician to treat the convergence issue satisfactorily. For more details, the readers are referred to Coolidge [1], Smoryński [2] and Goss [3].

Apart from the inductive proof there are several other proofs available for the binomial theorem. One can refer to Fulton [4] for a combinatorial proof, Rosalsky [5] for a probabilistic proof, Hwang [6] and Kataria [7] for proofs based on the theory of differential calculus, and Kataria [8] for a proof based on the Laplace transform technique. In contrast to the binomial theorem, only a few proofs are available for the GBT. In our opinion, this is probably due to the involvement of the infinite series in the GBT. One of them is based on the theory of differential equations (see Rudin [9, p. 201]).

The purpose of this Note is to provide a proof of the GBT using the Laplace transform. We remark that the binomial theorem has been proved using Laplace transform by Kataria [8]. But that proof cannot be imitated to prove the GBT because of the convergence issues of infinite series involved. This issue is tackled here using the Laplace transform of the convolution of two appropriate functions. For the sake of completeness, we first discuss below the proof of GBT outlined in Rudin [9].

Consider the series

$$y(t) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} t^k,$$

which converges for $|t| < 1$. Now,

$$\frac{dy}{dt} = \sum_{k=1}^{\infty} (\alpha - (k - 1)) \frac{(\alpha)_{k-1}}{(k - 1)!} t^{k-1}.$$

Thus,

$$\begin{aligned}(1 + t)\frac{dy}{dt} &= \sum_{k=1}^{\infty} (\alpha - (k - 1)) \frac{(\alpha)_{k-1}}{(k-1)!} t^{k-1} + \sum_{k=1}^{\infty} (\alpha - (k - 1)) \frac{(\alpha)_{k-1}}{(k-1)!} t^k \\ &= \alpha + \sum_{k=1}^{\infty} \alpha \frac{(\alpha)_k}{k!} t^k = \alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} t^k = \alpha y(t).\end{aligned}$$

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Therefore,

$$\frac{dy}{dt} = \frac{\alpha y}{1+t}.$$

The solution of the above first order differential equation with the initial condition $y(0) = 1$ is $y(t) = (1+t)^{\alpha}$, and so it follows from the uniqueness of solution that

$$(1+t)^{\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} t^k.$$

An alternative proof of the GBT

Let α and t be real numbers with $|t| < 1$. For $t = 0$, the result is obvious. First assume that $0 < t < 1$. Then, the Laplace transform of $(1+t)^{\alpha}$ is given by

$$\mathcal{L}\left\{(1+t)^{\alpha}\right\}(s) = \int_0^{\infty} (1+t)^{\alpha} e^{-st} dt, \qquad s > 0.$$

By repeated application of integration by parts, we have

$$\begin{aligned}
\mathcal{L}\left\{(1+t)^{\alpha}\right\}(s) &= \frac{1}{s} + \frac{\alpha}{s}\mathcal{L}\left\{(1+t)^{\alpha-1}\right\}(s) \\
&= \frac{1}{s} + \frac{\alpha}{s^2} + \frac{\alpha(\alpha-1)}{s^2}\mathcal{L}\left\{(1+t)^{\alpha-2}\right\}(s) \\
&\vdots \\
&= \sum_{k=0}^{n} \frac{(\alpha)_k}{s^{k+1}} + \frac{(\alpha)_{n+1}}{s^{n+1}}\mathcal{L}\left\{(1+t)^{\alpha-n-1}\right\}(s) \\
&= \sum_{k=0}^{n} \frac{(\alpha)_k}{s^{k+1}} + \frac{(\alpha)_{n+1}}{n!}\mathcal{L}\left\{(1+t)^{\alpha-n-1}\right\}(s)\,\mathcal{L}\left\{t^n\right\}(s) \\
&= \sum_{k=0}^{n} \frac{(\alpha)_k}{s^{k+1}} + \frac{(\alpha)_{n+1}}{n!}\mathcal{L}\left\{(1+t)^{\alpha-n-1} * t^n\right\}(s),
\end{aligned}$$

where the last equality follows from the convolution property of the Laplace transform. Therefore,

$$\mathcal{L}\left\{(1+t)^{\alpha} - \frac{(\alpha)_{n+1}}{n!}(1+t)^{\alpha-n-1} * t^n\right\}(s) = \sum_{k=0}^{n} \frac{(\alpha)_k}{s^{k+1}}.$$

By applying the inverse Laplace transform, we get

$$(1+t)^{\alpha} - \frac{(\alpha)_{n+1}}{n!}(1+t)^{\alpha-n-1} * t^n = \sum_{k=0}^{n} \mathcal{L}^{-1}\left\{\frac{(\alpha)_k}{s^{k+1}}\right\}(t),$$

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and so

$$(1 + t)^{\alpha} - \int_0^t \frac{(\alpha)_{n+1}}{n!}(1 + \tau)^{\alpha - n - 1}(t - \tau)^n d\tau = \sum_{k=0}^{n} \frac{(\alpha)_k}{k!} t^k.$$

Thus,

$$(1 + t)^{\alpha} - \sum_{k=0}^{n} \frac{(\alpha)_k}{k!} t^k = \int_0^t f_n(\tau)\, d\tau, \tag{2}$$

where

$$f_n(\tau) = \frac{(\alpha)_{n+1}}{n!}(1 + \tau)^{\alpha - n - 1}(t - \tau)^n = (1 + \tau)^{\alpha - 1} \frac{(\alpha)_{n+1}}{n!} \left(\frac{t - \tau}{1 + \tau}\right)^n.$$

To prove the result it suffices to show that the right-hand side of (2) tends to zero as $n \to \infty$.

Let

$$g_n(\tau) = \frac{(\alpha)_{n+1}}{n!} \left(\frac{t - \tau}{1 + \tau}\right)^n.$$

An application of the ratio test shows that the series $\sum_{n=1}^{\infty} g_n(\tau)$ converges absolutely. This implies that $g_n(\tau) \to 0$ pointwise for $\tau \in [0, t]$ as $n \to \infty$. Consider

$$M_n = \sup_{\tau \in [0,t]} | g_n(\tau) | = \left| \frac{(\alpha)_{n+1}}{n!} \right| t^n.$$

A similar argument shows that $M_n \to 0$ as $n \to \infty$ which implies that $g_n(\tau) \to 0$ uniformly on $[0, t]$ as $n \to \infty$. Since $0 < t < 1$ and $0 \leqslant \tau \leqslant t$, it follows that $(1 + \tau)^{\alpha - 1}$ is bounded, and so $f_n(\tau) \to 0$ uniformly on $[0, t]$ as $n \to \infty$. Therefore,

$$\lim_{n \to \infty} \int_0^t f_n(\tau)\, d\tau = \int_0^t \lim_{n \to \infty} f_n(\tau)\, d\tau = 0.$$

Finally, letting $n \to \infty$ in (2), we get

$$(1 + t)^{\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha) k}{k!} t^k,$$

which establishes the result (1) for $0 < t < 1$.

Now, assume that $-1 < t < 0$. By similar arguments for the function $(1 - t')^{\alpha}$, where $t' = -t$, we get

$$(1 - t')^{\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha) k}{k!} (-t')^k,$$

and thus (1) holds for $-1 < t < 0$ also. This completes the proof.

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References

1. J. L. Coolidge, The story of the binomial theorem, *Amer. Math. Monthly* **56** (1949) pp. 147-157.
2. C. Smoryński, *A treatise on the binomial theorem*, Texts in mathematics, College Publications, London (2012).
3. D. Goss, The ongoing binomial revolution, In: F. Bars, I. Longhi, F. Trihan, (eds) *Arithmetic geometry over global function fields*, Advanced Courses in Mathematics, CRM Barcelona, Birkhauser, Basel, (2014).
4. C. M. Fulton, A simple proof of the binomial theorem, *Amer. Math. Monthly* **59** (1952) pp. 243-244.
5. A. Rosalsky, A simple and probabilistic proof of the binomial theorem, *Amer. Statist*. **61** (2007) pp. 161-162.
6. L. C. Hwang, A simple proof of the binomial theorem using differential calculus, *Amer. Statist*. **63** (2009) pp. 43-44.
7. K. K. Kataria, The binomial theorem procured from the solution of an ODE, *Math. Magazine* **90** (2017) pp. 375-377.
8. K. K. Kataria, An alternate proof of the binomial theorem, *Amer. Math. Monthly* **123** (2016) p. 940.
9. W. Rudin, *Principles of mathematical analysis* (3rd edn.), McGraw-Hill, New York (1976).

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105.43 $c^2 = a^2 + bd$, a visual extension of the Pythagorean theorem

In [1] the author provides an interesting proof of the cosine rule, starting from a Pythagorean-like formula proved in [2] for obtuse triangles. We propose a visual proof which extends this formula, generalising to any triangle the typical rearrangement proof of the Pythagorean theorem.

The sides of the large quadrilateral in the right-hand diagram are all made up of two sides of a light grey triangle and the shortest side of a dark grey triangle. Therefore the two diagrams are of equal area. Moreover, the rectangle located in the upper part of the second diagram has the same dimensions as the dotted rectangle, then can be moved from the top and superimposed on it by rotating through a right angle.

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