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Itô-Taylor Schemes for Solving Mean-Field Stochastic Differential Equations

Yabing Sun, Jie Yang and Weidong Zhao*

School of Mathematics & Finance Institute, Shandong University, Jinan 250100, China

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Abstract. This paper is devoted to numerical methods for mean-field stochastic differential equations (MSDEs). We first develop the mean-field Itô formula and mean-field Itô-Taylor expansion. Then based on the new formula and expansion, we propose the Itô-Taylor schemes of strong order γ and weak order η for MSDEs, and theoretically obtain the convergence rate γ of the strong Itô-Taylor scheme, which can be seen as an extension of the well-known fundamental strong convergence theorem to the mean-field SDE setting. Finally some numerical examples are given to verify our theoretical results.

AMS subject classifications: 60H35, 65C30, 60H10

Key words: Itô-Taylor scheme, mean-field stochastic differential equation, mean-field Itô-Taylor formula, error estimate.

1. Introduction

Let $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\}_{0\le t\le T})$ be a complete filtered probability space where $\{\mathscr{F}_t\}_{0\le t\le T}$ represents the natural filtration of a standard m-dimensional Brownian motion $W = \{W_t\}_{0\le t\le T}$. In this paper, we are interested in the numerical solutions of the following *mean-field stochastic differential equations* (MSDEs) on $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\}_{0\le t\le T})$

$$dX_t = \mathbb{E}[b(t, X_t, \mu)]\big|_{\mu=X_t} ds + \mathbb{E}[\sigma(t, X_t, \mu)]\big|_{\mu=X_t} dW_t \tag{1.1}$$

with initial value X_{t_0}, for $0 \le t_0 < t \le T$, where t_0 and T are, respectively, the deterministic initial time and terminal time; the initial condition X_{t_0} is $\mathscr{F}_{t_0}$ measurable; $b : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are the coefficient functions. It is worth noting that $b(\cdot, x', x)$ and $\sigma(\cdot, x', x)$ are $\mathscr{F}_t$-adapted for any fixed number x', x, and that the stochastic integrals with respect to W_s are of Itô-type.

Mean-field theory is developed to study the collective behaviors resulting from mutual interactions of individuals in various physical and sociological dynamical systems. Mean-field stochastic differential equations (MSDEs) and related problems have been extensively

*Corresponding author. *Email addresses:* sunybly@163.com (Y. B. Sun), yangjie218@mail.sdu.edu.cn (J. Yang), wdzhao@sdu.edu.cn (W. D. Zhao)

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studied since 1960s, and have found many important applications in diverse areas such as kinetic gas theory [3], quantum mechanics [19], quantum chemistry [23], economics and finance [9], stochastic optimal control problems [21], McKean-Vlasov type partial differential equations (PDEs) [15, 17, 24, 26], and mean-field backward stochastic differential equations (MBSDEs) [4, 5]. As an important tool to understand properties or dynamics of the mean-field SDE systems, numerical solutions deserve special attention. Nevertheless, there are still few works on the numerical topic for MSDEs, in contrast to well-developed theory of numerical methods for standard SDEs (see, e.g., [11])

In this paper, after introducing the mean-field Itô formula and the mean-field Itô-Taylor expansion, we propose our Itô-Taylor schemes of strong order γ and weak order η for solving MSDEs. With $\gamma = 0.5, 1.0, 1.5$ and $\eta = 2.0$, the schemes become the Euler scheme, the Milstein scheme, the strong order 1.5, and the weak order 2.0 schemes, respectively. Error estimate of the strong order γ Itô-Taylor scheme is rigorously obtained, and our numerical results show that the schemes are effective, accurate and stable for solving MSDEs, and are consistent with our theoretical conclusions. This work is the basis of the study on numerical methods for mean-field *forward backward stochastic differential equations* (FBSDEs). In our near future, based on this work and our previous works [7, 8, 12, 27, 30–38] on solving FBSDEs, we will study numerical methods for solving mean-field FBSDEs with applications.

The rest of this paper is organized as follows. After introducing the mean-field Itô formula and the mean-field Itô-Taylor expansion in Section 2, we propose the strong and weak Itô-Taylor schemes for solving MSDEs in Section 3. In Section 4, the error estimates of the strong Itô-Taylor scheme are obtained. Numerical examples are carried out in Section 5, and some conclusions are given in Section 6.

2. The mean-field Itô formula and Itô-Taylor expansion

In this section, we will present the mean-field Itô formula and the mean-field Itô-Taylor expansion, which play fundamental roles in proposing our Itô-Taylor schemes for solving MSDEs in Section 3.

2.1. The solvability of MSDEs

We rewrite the MSDE (1.1) with initial value X_{t_0} in the following integral form

$$X_t = X_{t_0} + \int_{t_0}^{t} \mathbb{E}\big[b(s, X_s, x)\big]\big|_{x=X_s} ds + \int_{t_0}^{t} \mathbb{E}\big[\sigma(s, X_s, x)\big]\big|_{x=X_s} dW_s, \tag{2.1}$$

for $t \in [t_0, T]$. The stochastic process X_t is called an L^2-adapted solution of the MSDE (2.1), if it is $\mathscr{F}_t$-adapted, square integrable, and satisfies (2.1). Here

$$L^p = \{X : \mathbb{E}[|X|^p] < \infty\}, \quad p \geq 1.$$

We set the following assumptions on the coefficients b and σ.

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(A1) the data $b(\cdot,0,0)$ and $\sigma(\cdot,0,0)$ are L^2-measurable continuous processes and there exists some constant $K \geq 0$ such that

$$\begin{aligned}&\left|b(t,x',x)\right| + |\sigma(t,x',x)| \leq K(1+|x'|+|x|),\\ &\quad \text{a.s. for all } 0 \leq \mathrm{t} \leq \mathrm{T}, \quad \mathrm{x},\mathrm{x}' \in \mathbb{R}^{\mathrm{d}}.\end{aligned}$$

(A2) the data b and σ are Lipschitz continuous with respect to x' and x, i.e., there exists a positive constant L such that

$$\begin{aligned}&|b(t,x',x)-b(t,y',y)| + |\sigma(t,x',x)-\sigma(t,y',y)|\\ \leq& L(|x'-y'|+|x-y|), \quad \text{a.s. for all } 0 \leq \mathrm{t} \leq \mathrm{T} \text{ and } \mathrm{x},\mathrm{x}',\mathrm{y},\mathrm{y}' \in \mathbb{R}^{\mathrm{d}}.\end{aligned}$$

Now we state the existence and uniqueness of the solution of the MSDE (2.1) and some useful properties of the solution in the following theorem [5].

Theorem 2.1. *If Assumptions* (A1) *and* (A2) *hold, then the MSDE* (2.1) *has a unique strong* L^2*-adapted solution* X_t *on* $[t_0,T]$ *with*

$$\sup_{t_0 \leq t \leq T} \mathbb{E}[|X_t|^2] < \infty,$$

for any $\mathscr{F}_{t_0}$ *measurable* $X_{t_0} \in L^2$.

In addition, for any $p \geq 2$, *there exists a* $C_p \in \mathbb{R}^+$ *such that for any initial time* $t \in [0,T]$ *and* $\mathscr{F}_t$ *measurable* $\zeta, \zeta' \in L^p$,

$$\mathbb{E}\Big[\sup_{t \leq s \leq T} |X_{t,\zeta}(s)|^p \Big| \mathscr{F}_t\Big] \leq C_p\big(1+|\zeta|^p\big), \tag{2.2a}$$

$$\mathbb{E}\Big[\sup_{t \leq s \leq T} |X_{t,\zeta}(s)-X_{t,\zeta'}(s)|^p \Big| \mathscr{F}_t\Big] \leq C_p(|\zeta-\zeta'|^p), \tag{2.2b}$$

$$\mathbb{E}\Big[\sup_{t \leq s \leq t+\delta} |X_{t,\zeta}(s)-\zeta|^p \Big| \mathscr{F}_t\Big] \leq C_p\big(1+|\zeta|^p\big)\delta^{\frac{p}{2}}, \tag{2.2c}$$

a.s. for all $\delta > 0$ *with* $t+\delta \leq T$, *where* $X_{t,\zeta}(s)$ *and* $X_{t,\zeta'}(s)$ *are the solutions of* (2.1) *with initial conditions* $X_t = \zeta$ *and* $X_t = \zeta'$, *respectively.*

We emphasize that the constant C_p *in* (2.2) *only depends on the* K *and* L.

2.2. Itô's formula

In this subsection, we first give the Itô formula for standard *stochastic differential equations* (SDEs) and then extend it to MSDEs.

Let X_t be a d-dimensional Itô process defined as

$$X_t = X_0 + \int_0^t \psi_s ds + \int_0^t \varphi_s dW_s, \quad t \geq 0, \tag{2.3}$$

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where X_0 is a d-dimensional $\mathscr{F}_0$ measurable random variable, ψ_s is a d-dimensional progressively measurable process satisfying $\int_0^T |\psi_s| ds < +\infty$, and φ_s is a $d \times m$-dimensional progressively measurable process satisfying

$$\int_0^T \mathrm{Tr}(\varphi_s \varphi_s^\top) ds < +\infty.$$

Here for a matrix $A = (a_{ij})_{d\times d}$, we use Tr(A) to denote the trace of A, i.e., $\mathrm{Tr(A)} = \sum_{i=1}^d a_{ii}$.

We state the Itô formula for standard SDEs as follows [22].

Theorem 2.2 (Standard Itô formula). *Let X_t be a d-dimensional Itô process defined by* (2.3)*, and let Φ be a $C^{1,2}$ function mapping from $[0,\infty)\times\mathbb{R}^d$ to $\mathbb{R}$. Then $\Phi(t,X_t)$ is again an Itô process, and*

$$\Phi(t,X_t) = \Phi(0,X_0) + \int_0^t \bar{L}_0\Phi(s,X_s)ds + \int_0^t \bar{L}_1\Phi(s,X_s)dW_s, \quad t>0, \tag{2.4}$$

where the operators $\bar{L}_1$ and $\bar{L}_0$ are defined by

$$\bar{L}_1\Phi(s,X_s) = \nabla_x\Phi(s,X_s)\varphi_s,$$
$$\bar{L}_0\Phi(s,X_s) = \frac{\partial\Phi}{\partial s}(s,X_s) + \nabla_x\Phi(s,X_s)\psi_s + \frac{1}{2}\mathrm{Tr}\big[\Phi_{xx}(s,X_s)\varphi_s\varphi_s^\top\big]$$

with $\nabla_x\Phi = \left(\frac{\partial\Phi}{\partial x_1},\ldots,\frac{\partial\Phi}{\partial x_d}\right)$ being a d-dimensional row vector, $\Phi_{xx} = \left(\frac{\partial^2\Phi}{\partial x_i\partial x_j}\right)_{d\times d}$ a $d\times d$ matrix.

Let X_t be a d-dimensional Itô process defined by the MSDE

$$dX_t = b^\beta(t,X_t)dt + \sigma^\beta(t,X_t)dW_t, \quad t\ge 0, \tag{2.5}$$

where

$$b^\beta(t,X_t) = \mathbb{E}\big[b(t,\beta_t,x)\big]\big|_{x=X_t} \text{ and } \sigma^\beta(t,X_t) = \mathbb{E}\big[\sigma(t,\beta_t,x)\big]\big|_{x=X_t} \tag{2.6}$$

with β_t a d-dimensional Itô process defined by

$$d\beta_t = \psi_t dt + \varphi_t dW_t, \tag{2.7}$$

where ψ_t and φ_t are progressively measurable processes and satisfy $\int_0^T |\psi_t|dt < +\infty$ and $\int_0^T \mathrm{Tr}[\varphi_s\varphi_s^T]dt < +\infty$.

Note that the SDE (2.7) has a unique strong L^2-adapted solution [22] when the Itô process β_t satisfies $\mathbb{E}\big[\int_0^T |\beta_t|^2 dt\big] < \infty$, and under conditions (A1) and (A2), with $\beta_t = X_t$, the MSDE (2.5) also has a unique strong L^2-adapted solution [5]. In the sequel, we define $f^\beta(t,x)$ as

$$f^\beta(t,x) = \mathbb{E}\left[f(t,\beta_t,x)\right] \tag{2.8}$$

for a function $f:\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ with β_t defined by (2.7).

Now we state our mean-field Itô's formula in the following theorem.

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Theorem 2.3 (Mean-field Itô's formula). *Let X_t and β_t be d-dimensional Itô processes satisfying* (2.5) *and* (2.7)*, respectively, and f be a $C^{1,2,2}$ function mapping from $[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ to $\mathbb{R}$. Then it holds that*

$$f^\beta(t,X_t)=f^\beta(0,X_0)+\int_0^t L^0 f^\beta(s,X_s)ds+\int_0^t \overrightarrow{L}^1 f^\beta(s,X_s)dW_s, \tag{2.9}$$

where the differential operators L^0 and $\overrightarrow{L}^1$ are defined by

$$\begin{aligned} L^0 f^\beta(s,X_s)=&\ \frac{\partial f^\beta}{\partial s}(s,X_s)+\nabla_x f^\beta(s,X_s)b^\beta(s,X_s)\\ &+\frac{1}{2}\mathrm{Tr}\left[\nabla_{xx}f^\beta(s,X_s)(\sigma^\beta(s,X_s))(\sigma^\beta(s,X_s))^\top\right],\\ \overrightarrow{L}^1 f^\beta(s,X_s)=&\ \nabla_x f^\beta(s,X_s)\sigma^\beta(s,X_s)=\left(L^1 f^\beta(s,X_s),\cdots,L^m f^\beta(s,X_s)\right) \end{aligned}$$

with

$$L^j f^\beta(t,x)=\sum_{k=1}^d \frac{\partial f^\beta}{\partial x^k}(t,x)\sigma^\beta_{kj}(t,x),\quad j=1,\cdots,m, \tag{2.10a}$$

$$\frac{\partial f^\beta}{\partial s}(s,x)=\mathbb{E}\left[\frac{\partial f}{\partial s}(s,\beta_s,x)+\nabla_{x'}f(s,\beta_s,x)\psi_s+\frac{1}{2}\mathrm{Tr}\left[f_{x'x'}(s,\beta_s,x)\varphi_s\varphi_s^\top\right]\right], \tag{2.10b}$$

$$\nabla_x f^\beta(s,x)=\mathbb{E}\left[\nabla_x f(s,\beta_s,x)\right],\qquad f^\beta_{xx}(s,x)=\mathbb{E}\left[f_{xx}(s,\beta_s,x)\right], \tag{2.10c}$$

where $\nabla_x f$ (or $\nabla_{x'}f$) and f_{xx} (or $f_{x'x'}$) are defined as $\nabla_x\Phi$ and Φ_{xx} in (2.4)*, respectively.*

Proof. We just give the proof for the case $d=m=1$. The general case can be obtained similarly. For $s\in\mathbb{R}^+$ and $x\in\mathbb{R}^d$, under the conditions of the theorem, by the definition of $f^\beta(s,x)$, we easily get

$$\frac{\partial f^\beta(s,x)}{\partial x}=\mathbb{E}\left[\frac{\partial f}{\partial x}(s,\beta_s,x)\right],\qquad \frac{\partial^2 f^\beta(s,x)}{\partial x^2}=\mathbb{E}\left[\frac{\partial^2 f}{\partial x^2}(s,\beta_s,x)\right], \tag{2.11}$$

which proves the formulas in (2.10c).

Now we turn to prove the Eq. (2.10b). By the standard Itô formula, it holds that

$$\begin{aligned} &\mathbb{E}\left[f(s+\tau,\beta_{s+\tau},x)-f(s,\beta_s,x)\right]\\ =&\mathbb{E}\left[\int_s^{s+\tau}\frac{\partial f}{\partial r}(r,\beta_r,x)+\frac{\partial f}{\partial x'}(r,\beta_r,x)\psi_r+\frac{1}{2}\frac{\partial^2 f}{\partial (x')^2}(r,\beta_r,x)\varphi_r^2 dr\right]. \end{aligned}$$

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Then under the conditions of the theorem, we have

$$\begin{aligned}\frac{\partial f^\beta(s,x)}{\partial s} &= \lim_{\tau\to 0}\frac{f^\beta(s+\tau,x)-f^\beta(s,x)}{\tau} = \lim_{\tau\to 0}\frac{\mathbb{E}\big[f(s+\tau,\beta_{s+\tau},x)-f(s,\beta_s,x)\big]}{\tau}\\ &= \lim_{\tau\to 0}\frac{\mathbb{E}\Big[\int_s^{s+\tau}\frac{\partial f}{\partial r}(r,\beta_r,x)+\frac{\partial f}{\partial x'}(r,\beta_r,x)\psi_r+\frac{1}{2}\frac{\partial^2 f}{\partial (x')^2}(r,\beta_r,x)\varphi_r^2 dr\Big]}{\tau}\\ &= \mathbb{E}\left[\lim_{\tau\to 0}\frac{\int_s^{s+\tau}\frac{\partial f}{\partial r}(r,\beta_r,x)+\frac{\partial f}{\partial x'}(r,\beta_r,x)\psi_r+\frac{1}{2}\frac{\partial^2 f}{\partial (x')^2}(r,\beta_r,x)\varphi_r^2 dr}{\tau}\right]\\ &= \mathbb{E}\left[\frac{\partial f}{\partial s}(s,\beta_s,x)+\frac{\partial f}{\partial x'}(s,\beta_s,x)\psi_s+\frac{1}{2}\frac{\partial^2 f}{\partial (x')^2}(s,\beta_s,x)\varphi_s^2\right]. \qquad (2.12)\end{aligned}$$

Now by the standard Itô formula (2.4) and the formulas in (2.11) and (2.12), we deduce

$$f^\beta(t,X_t)=f^\beta(0,X_0)+\int_0^t L^0 f^\beta(s,X_s)ds+\int_0^t \overrightarrow{L}^1 f^\beta(s,X_s)dW_s, \qquad (2.13)$$

where

$$\begin{aligned}\overrightarrow{L}^1 f^\beta(s,X_s) &= \frac{\partial f^\beta}{\partial x}(s,X_s)\sigma^\beta(s,X_s),\\ L^0 f^\beta(s,X_s) &= \frac{\partial f^\beta}{\partial s}(s,X_s)+\frac{\partial f^\beta}{\partial x}(s,X_s)b^\beta(s,X_s)+\frac{1}{2}\frac{\partial^2 f^\beta}{\partial x^2}(s,X_s)\left(\sigma^\beta(s,X_s)\right)^2.\end{aligned}$$

The proof is completed. □

Remark 2.1. Let f and g be $C^{1,2,2}$ functions mapping from $[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ to $\mathbb{R}$, $f^\beta(t,x)$ and $g^\beta(t,x)$ be defined as in (2.8), and β_t be defined by (2.7). From the proof of Theorem 2.3, we easily have the following formulas.

$$\begin{aligned}\frac{\partial(f^\beta g^\beta)}{\partial t}(t,x) &= \left(g^\beta\frac{\partial f^\beta}{\partial t}\right)(t,x)+\left(f^\beta\frac{\partial g^\beta}{\partial t}\right)(t,x),\\ \nabla_x(f^\beta g^\beta)(t,x) &= \left(g^\beta\nabla_x f^\beta\right)(t,x)+\left(f^\beta\nabla_x g^\beta\right)(t,x).\end{aligned}$$

Remark 2.2. When f, b and σ are independent of x', we have $L^0=\bar{L}_0$ and $\overrightarrow{L}^1=\bar{L}_1$. Therefore the Itô formula (2.9) for MSDEs is a generalization of the Itô formula (2.4) for SDEs.

Compared with the standard Itô formula, the mean-field Itô formula is much more complicated. We show this by the following example.

Example 2.1. *Consider the MSDE*

$$dX_t=\mathbb{E}[\cos(X_t x)]\big|_{x=X_t}dt+\mathbb{E}[\sin(X_t x)]\big|_{x=X_t}dW_t.$$

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That is, $b(t,x',x)=\cos(x'x)$, $\sigma(t,x',x)=\sin(x'x)$ *and* $\beta_t=X_t$. *Then*

$$b^\beta=\psi_t=\mathbb{E}[\cos(X_tx)]\big|_{x=X_t},\quad \sigma^\beta=\varphi_t=\mathbb{E}[\sin(X_tx)]\big|_{x=X_t}$$

for all $0\le t\le T$. *Let* $F(t,x',x)=\cos(x'x)$, *and so* $F^\beta(t,X_t)=\mathbb{E}[\cos(X_tx)]\big|_{x=X_t}$. *By Theorem 2.3,* $F^\beta(t,X_t)$ *is an Itô process satisfying*

$$F^\beta(t,X_t)=F^\beta(0,X_0)+\int_0^t L^0F^\beta(s,X_s)ds+\int_0^t \overrightarrow{L}^1F^\beta(s,X_s)dW_s,$$

where the differential operators L^0F^β *and* $\overrightarrow{L}^1F^\beta$ *are in the form of*

$$\begin{aligned}L^0F^\beta(t,X_t)=\mathbb{E}\Big[&-x\sin(X_tx)\big(\mathbb{E}[\cos(X_tx)]\big|_{x=X_t}\big)\\&-\frac{1}{2}x^2\cos(X_tx)\big(\mathbb{E}[\sin(X_tx)]\big|_{x=X_t}\big)^2\Big]\Big|_{x=X_t}\\&+\mathbb{E}\big[-X_t\sin(X_tx)\big]\Big|_{x=X_t}\mathbb{E}\big[\cos(X_tx)\big]\Big|_{x=X_t}\\&+\frac{1}{2}\mathbb{E}\big[-X_t^2\cos(X_tx)\big]\Big|_{x=X_t}\Big(\mathbb{E}\big[\sin(X_tx)\big]\Big|_{x=X_t}\Big)^2,\\ \overrightarrow{L}^1F^\beta(t,X_t)=\mathbb{E}\big[&-X_t\sin(X_tx)\big]\Big|_{x=X_t}\mathbb{E}\big[\sin(X_tx)\big]\Big|_{x=X_t}.\end{aligned}$$

2.3. The mean-field Itô-Taylor expansion

2.3.1. Multiple Itô integrals and coefficient functions

(A) Multi-index

Let $\alpha:=(j_1,\cdots,j_l)$ be multi-indices with $j_i\in\{0,1,\cdots,m\}$, $i=1,\cdots,l$. Set $l(\alpha)=l$ be the length of α, and $n(\alpha)$ be the number of the components of α which are equal to zero. Denote by v the multi-index of length zero, i.e., $l(v)=0$.

Let $\mathscr{M}$ be the set of all multi-indices, i.e.,

$$\mathscr{M}=\Big\{(j_1,j_2,\cdots,j_l):j_i\in\{0,1,\cdots,m\},\ i\in\{1,2,\cdots,l\},\ l=1,2,3,\cdots\Big\}\cup\{v\}.$$

Given $\alpha\in\mathscr{M}$ with $l(\alpha)\ge1$, $-\alpha$ and $\alpha-$ are two multi-indices obtained by deleting the first and the last component of α, respectively.

(B) Multiple Itô integrals

Now we introduce the multiple Itô integral operator I_α on adapted right continuous stochastic processes $\{f=f(t),t\ge0\}$ with left limits, which is defined recursively as

$$I_\alpha[f(\cdot)]_{\rho,\tau}:=\begin{cases}f(\tau), & l=0,\\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s}ds, & l\ge1 \text{ and } j_l=0,\\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s}dW_s^{j_l}, & l\ge1 \text{ and } j_l\ge1,\end{cases}\tag{2.14}$$

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where ρ and τ are two stopping times satisfying $0 \le \rho(w) \le \tau(w) \le T$, a.s. and we assume that all the integrals exist. We give the following examples to illustrate the terminology

$$I_v[f(\cdot)]_{0,t} = f(t), \qquad I_{(0)}[f(\cdot)]_{\rho,\tau} = \int_\rho^\tau f(s)ds, \tag{2.15a}$$

$$I_{(1)}[f(\cdot)]_{\rho,\tau} = \int_\rho^\tau f(s)dW_s^1, \quad I_{(0,1)}[f(\cdot)]_{0,t} = \int_0^t \int_0^{s_2} f(s)ds_1 dW_{s_2}^1. \tag{2.15b}$$

(C) Itô coefficient functions

For each index $\alpha = (j_1, \cdots, j_l)$ and function $f \in C^r(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d)$ with $r = l(\alpha) + n(\alpha)$, we define recursively coefficient function f_α^β by

$$f_\alpha^\beta = \begin{cases} f^\beta, & l = 0, \\ L^{j_1} f_{-\alpha}^\beta, & l \ge 1. \end{cases} \tag{2.16}$$

Let $m = d = 1$. If $f(t, x', x) = x$, then we have

$$f_{(0)}^\beta = b^\beta, \quad f_{(1)}^\beta = \sigma^\beta, \quad f_{(1,1)}^\beta = L^1\sigma^\beta = \sigma^\beta \frac{\partial \sigma^\beta}{\partial x},$$
$$f_{(0,1)}^\beta = L^0\sigma^\beta = \frac{\partial \sigma^\beta}{\partial t} + \frac{\partial \sigma^\beta}{\partial x} b^\beta + \frac{1}{2}\frac{\partial^2 \sigma^\beta}{\partial x^2}(\sigma^\beta)^2.$$

In order to see more clearly the notation f_α^β, we take $f(t, x', x) = \sin(x'x)$ and $f^X(t, X_t) = \mathbb{E}[\sin(X_t x)]\Big|_{x=X_t}$ as an example, where the X_t is the solution of MSDE in Example 2.1. By Theorem 2.3, we have

$$\begin{aligned}
f_{(1)}^X &= L^1 f^X = \frac{\partial f^X}{\partial x}\sigma^X = \mathbb{E}[X_t \cos(X_t x)]\Big|_{x=X_t} \mathbb{E}[\sin(X_t x)]\Big|_{x=X_t}, \\
f_{(0)}^X &= L^0 f^X = \mathbb{E}\big[x\cos(X_t x)\big(\mathbb{E}[\cos(X_t x)]\big|_{x=X_t}\big) \\
&\quad - \frac{1}{2}x^2 \sin(X_t x)\Big(\mathbb{E}[\sin(X_t x)]\big|_{x=X_t}\Big)^2\Big]\Big|_{x=X_t} \\
&\quad + \mathbb{E}[X_t \cos(X_t x)]\Big|_{x=X_t} \mathbb{E}[\cos(X_t x)]\Big|_{x=X_t} \\
&\quad - \frac{1}{2}\mathbb{E}[(X_t)^2 \sin(X_t x)]\Big|_{x=X_t} \Big(\mathbb{E}[\sin(X_t x)]\Big|_{x=X_t}\Big)^2, \\
f_{(1,1)}^X &= L^1(L^1 f^X) = \Big(\big(\mathbb{E}[X_t \cos(X_t x)]\big)^2 \\
&\quad - \mathbb{E}[(X_t)^2 \sin(X_t x)]\mathbb{E}[\sin(X_t x)]\Big)\Big|_{x=X_t} \mathbb{E}[\sin(X_t x)]\Big|_{x=X_t}.
\end{aligned}$$

(D) Hierarchical and remainder sets

We call a subset $\mathscr{A} \subset \mathscr{M}$ a hierarchical set if it satisfies

$$\mathscr{A} \ne \emptyset, \quad \sup_{\alpha \in \mathscr{A}} l(\alpha) < \infty, \quad \text{and} \quad -\alpha \in \mathscr{A} \ \text{ for each } \alpha \in \mathscr{A} \backslash \{v\};$$

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and its remainder set $\mathscr{B}(\mathscr{A})$ is defined by

$$\mathscr{B}(\mathscr{A}) = \{\alpha \in \mathscr{M} \setminus \mathscr{A} : -\alpha \in \mathscr{A}\}.$$

For example, let

$$\mathscr{A}_0 = \{v, (0), (1)\}, \quad \mathscr{A}_1 = \{v, (0), (1), (1,1)\}.$$

Note that $\mathscr{A}_0$ and $\mathscr{A}_1$ are hierarchical sets, and we have their remainder sets

$$\mathscr{B}(\mathscr{A}_0) = \{(0,0), (1,0), (0,1), (1,1)\},$$
$$\mathscr{B}(\mathscr{A}_1) = \{(0,0), (1,0), (0,1), (0,1,1), (1,1,1)\}.$$

2.3.2. Mean-field Itô-Taylor expansions

In this subsection, we will develop the mean-field Itô-Taylor expansion of $f = f^{\beta}(t, X_t) = \mathbb{E}[f(t, \beta_t, x]\big|_{x=X_t}$ with X_t being the solution of MSDE

$$X_t = X_{t_0} + \int_{t_0}^{t} b^{\beta}(s, X_s)ds + \int_{t_0}^{t} \sigma^{\beta}(s, X_s)dW_s, \tag{2.17}$$

where $t \in [t_0, T]$ and β_t defined by (2.7) satisfies $\mathbb{E}\big[\int_0^T |\beta_t|^2\big]dt < \infty$.

Motivated by the method used to develop the Itô-Taylor expansion for SDEs [11], we deduce the Itô-Taylor expansion for MSDEs by iterated application of the mean-field Itô formula (2.9). We assume that the coefficients b and σ are sufficiently smooth satisfying the Assumptions (A1) and (A2). Then for a $C^{1,2,2}$ function f mapping from $[0,\infty) \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, by the mean-field Itô formula, we have

$$f^{\beta}(t, X_t) = f^{\beta}(t_0, X_{t_0}) + \int_{t_0}^{t} L^0 f^{\beta}(s, X_s)ds + \int_{t_0}^{t} \overrightarrow{L}^1 f^{\beta}(s, X_s)dW_s \tag{2.18}$$

for $t \in [t_0, T]$. If $f(t, x', x) = x$, we have $L^0 f^{\beta} = b^{\beta}$ and $\overrightarrow{L}^1 f^{\beta} = \sigma^{\beta}$. In this case, (2.18) reduces to the original Itô equation for X_t, i.e.,

$$X_t = X_{t_0} + \int_{t_0}^{t} b^{\beta}(s, X_s)ds + \int_{t_0}^{t} \sigma^{\beta}(s, X_s)dW_s. \tag{2.19}$$

By applying Itô's formula (2.9) to the functions b^{β} and σ^{β} in (2.19), we obtain

$$\begin{aligned} X_t = {} & X_{t_0} + \int_{t_0}^{t} \left(b^{\beta}(t_0, X_{t_0}) + \int_{t_0}^{s} L^0 b^{\beta}(z, X_z)dz + \int_{t_0}^{s} \overrightarrow{L}^1 b^{\beta}(z, X_z)dW_z \right) ds \\ & + \int_{t_0}^{t} \left(\sigma^{\beta}(t_0, X_{t_0}) + \int_{t_0}^{s} L^0 \sigma^{\beta}(z, X_z)dz + \int_{t_0}^{s} \overrightarrow{L}^1 \sigma^{\beta}(z, X_z)dW_z \right) dW_s \\ = {} & X_{t_0} + b^{\beta}(t_0, X_{t_0}) \int_{t_0}^{t} ds + \sigma^{\beta}(t_0, X_{t_0}) \int_{t_0}^{t} dW_s + R \end{aligned} \tag{2.20}$$

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with the remainder R being

$$R=\int_{t_0}^{t}\int_{t_0}^{s}L^0b^\beta(z,X_z)dzds+\int_{t_0}^{t}\int_{t_0}^{s}\overrightarrow{L}^1b^\beta(z,X_z)dW_zds$$
$$+\int_{t_0}^{t}\int_{t_0}^{s}L^0\sigma^\beta(z,X_z)dzdW_s+\int_{t_0}^{t}\int_{t_0}^{s}\overrightarrow{L}^1\sigma^\beta(z,X_z)dW_zdW_s.$$

The (2.20) is the simplest nontrivial Itô-Taylor expansion for MSDEs. Further, by applying the Itô formula (2.9) to $\overrightarrow{L}^1\sigma^\beta$ appearing in (2.20), we deduce

$$X_t=X_{t_0}+b^\beta(t_0,X_{t_0})\int_{t_0}^{t}ds+\sigma^\beta(t_0,X_{t_0})\int_{t_0}^{t}dW_s$$
$$+\overrightarrow{L}^1\sigma^\beta(t_0,X_{t_0})\int_{t_0}^{t}\int_{t_0}^{s}dW_zdW_s+\bar{R} \tag{2.21}$$

with the remainder $\bar{R}$ being

$$\bar{R}=\int_{t_0}^{t}\int_{t_0}^{s}L^0b^\beta(z,X_z)dzds+\int_{t_0}^{t}\int_{t_0}^{s}\overrightarrow{L}^1b^\beta(z,X_z)dW_zds$$
$$+\int_{t_0}^{t}\int_{t_0}^{s}L^0\sigma^\beta(z,X_z)dzdW_s+\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z}L^0\overrightarrow{L}^1\sigma^\beta(u,X_u)dudW_zdW_s$$
$$+\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z}\overrightarrow{L}^1\overrightarrow{L}^1\sigma^\beta(u,X_u)dW_udW_zdW_s.$$

Continuing the above procedure, we can obtain our high-order mean-field Itô-Taylor expansion, which is stated in the following theorem.

Theorem 2.4. *Let ρ and τ be two stopping times with $t_0\le\rho(w)\le\tau(w)\le T$, and let $\mathscr{A}\subset\mathscr{M}$ be a hierarchical set, and $f:\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$. Then the mean-field Itô-Taylor expansion*

$$f^\beta(\tau,X_\tau)=\sum_{\alpha\in\mathscr{A}}I_\alpha\left[f_\alpha^\beta(\rho,X_\rho)\right]_{\rho,\tau}+\sum_{\alpha\in\mathscr{B}(\mathscr{A})}I_\alpha\left[f_\alpha^\beta(\cdot,X_\cdot)\right]_{\rho,\tau} \tag{2.22}$$

holds, provided all of the derivatives of f^β, b^β and σ^β and all of the multiple Ito integrals appearing in (2.22) *exist.*

By using the mean-field Itô formula (2.9) in Theorem 2.3, the proof of the above theorem is similar to that of the Itô-Taylor expansion for the standard SDEs given in [11], so we omit it here. Notice that the mean-field Itô-Taylor expansion can be regarded as a generalization of the Itô-Taylor expansion for SDEs.

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3. Itô-Taylor schemes for MSDEs

In this section, based on the mean-field Itô formula (2.9) and the mean-field Itô-Taylor expansion (2.22) for MSDEs, we will propose our numerical schemes for solving MSDEs.

For the time interval $[0,T]$, we introduce a uniform time partition

$$0 = t_0 < t_1 < \cdots < t_{N_T} = T$$

with time step $h = t_{k+1} - t_k$. Denote by ΔW_k the increment $W_{t_{k+1}} - W_{t_k}$ of Brownian motion W_t. We also use X_k to denote the approximation value of the solution X_t of (2.1) at time $t = t_k$, $k = 0, 1, \cdots, N_T$.

In the sequel, for any function $f : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we denote by $f^{X_k}(t_k, X_k)$ the expectation $\mathbb{E}[f(t_k, X_k, x)]\big|_{x=X_k}$.

3.1. Strong Itô-Taylor scheme

For $\gamma = 0.5, 1.0, 1.5, \cdots$, we define the hierarchical set $\mathscr{A}_\gamma$ by

$$\mathscr{A}_\gamma = \left\{ \alpha \in \mathscr{M} : l(\alpha) + n(\alpha) \le 2\gamma \ \text{ or } \ l(\alpha) = n(\alpha) = \gamma + \frac{1}{2} \right\} \tag{3.1}$$

and denote its remainder set by $\mathscr{B}(\mathscr{A}_\gamma)$. Take $f(t, x', x) = x$, then by Theorem 2.4, we have the Itô-Taylor expansion

$$X_{t_{k+1}} = \sum_{\alpha \in \mathscr{A}_\gamma} I_\alpha \left[f_\alpha^X(t_k, X_{t_k}) \right]_{t_k, t_{k+1}} + R_k^\gamma, \tag{3.2}$$

where the remiainder term $R_k^\gamma = \sum_{\alpha \in \mathscr{B}(\mathscr{A}_\gamma)} I_\alpha \left[f_\alpha^X(\cdot, X_\cdot) \right]_{t_k, t_{k+1}}$.

Now letting $\beta_t = X_t$ in (2.16) and removing the remainder term R_k^γ in (3.2), we propose the following general strong order γ Itô-Taylor scheme for solving the MSDE (2.1).

Scheme 3.1 (Strong order γ Itô-Taylor scheme)**.**

$$X_{k+1} = \sum_{\alpha \in \mathscr{A}_\gamma} I_\alpha \left[f_\alpha^{X_k}(t_k, X_k) \right]_{t_k, t_{k+1}}. \tag{3.3}$$

With specific values of γ, we write Scheme 3.1 in explicit forms as follows.

3.1.1. The Euler scheme

Take $\gamma = 0.5$. By definitions of $\mathscr{A}_\gamma$ and $\mathscr{B}(\mathscr{A}_\gamma)$, we have

$$\mathscr{A}_{0.5} = \{v, (0), (1)\} \ \text{ and } \ \mathscr{B}(\mathscr{A}_{0.5}) = \{(0,0), (1,0), (0,1), (1,1)\}.$$

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Utilizing the Eq. (3.2), we deduce

$$X_{t_{k+1}} = X_{t_k} + b^X(t_k, X_{t_k}) \int_{t_k}^{t_{k+1}} ds + \sigma^X(t_k, X_{t_k}) \int_{t_k}^{t_{k+1}} dW_s + R_1, \tag{3.4}$$

where

$$\begin{aligned} R_1 = &\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} L^0 b^X(z, X_z) dz ds + \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} L^1 b^X(z, X_z) dW_z ds \\ &+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} L^0 \sigma^X(z, X_z) dz dW_s + \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} L^1 \sigma^X(z, X_z) dW_z dW_s. \end{aligned}$$

Removing the remainder term R_1, we get the strong order 0.5 Itô-Taylor scheme

$$X_{k+1} = X_k + b^{X_k}(t_k, X_k)\Delta t_k + \sigma^{X_k}(t_k, X_k)\Delta W_k, \tag{3.5}$$

which, in fact, is the so-called Euler scheme.

3.1.2. The Milstein scheme

Taking $\gamma = 1$, we have

$$\mathscr{A}_1 = \{v, (0), (1), (1,1)\} \text{ and } \mathscr{B}(\mathscr{A}_1) = \{(0,0), (1,0), (0,1), (0,1,1), (1,1,1)\}.$$

By (3.2), we have

$$\begin{aligned} X_{t_{k+1}} = X_{t_k} &+ b^X(t_k, X_{t_k}) \int_{t_k}^{t_{k+1}} ds + \sigma^X(t_k, X_{t_k}) \int_{t_k}^{t_{k+1}} dW_s \\ &+ L^1\sigma^X(t_k, X_{t_k}) \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dW_z dW_s + R_2, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} R_2 = &\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} L^0 b^X(z, X_z) dz ds + \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} L^1 b^X(z, X_z) dW_z ds \\ &+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} L^0 \sigma^X(z, X_z) dz dW_s + \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \int_{t_k}^{z} L^0 L^1 \sigma^X(u, X_u) du dW_z dW_s \\ &+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \int_{t_k}^{z} L^1 L^1 \sigma^X(u, X_u) dW_u dW_z dW_s. \end{aligned}$$

Withdrawing the truncation term R_2, we obtain the strong order 1.0 Itô-Taylor scheme

$$\begin{aligned} X_{k+1} = &X_k + b^{X_k}(t_k, X_k)\Delta t_k + \sigma^{X_k}(t_k, X_k)\Delta W_k \\ &+ \frac{1}{2} L^1\sigma^{X_k}(t_k, X_k) \left((\Delta W_k)^2 - \Delta t_k \right), \end{aligned} \tag{3.7}$$

which is called the Milstein scheme.

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3.1.3. The strong order 1.5 Itô-Taylor scheme

Taking $\gamma = 1.5$, we have

$$\begin{aligned}\mathscr{A}_\gamma &= \{v,(0),(1),(1,1),(0,0),(1,0),(0,1),(1,1,1)\},\\ \mathscr{B}(\mathscr{A}_\gamma) &= \{(0,1,1),(0,0,0),(1,0,0),(0,1,0),(1,1,0),\\ &\qquad (0,0,1),(1,0,1),(0,1,1,1),(1,1,1,1)\}.\end{aligned}$$

Similarly, we obtain the strong order 1.5 Itô-Taylor scheme

$$\begin{aligned}X_{k+1} = X_k &+ b^{X_k}(t_k,X_k)\Delta t_k + \sigma^{X_k}(t_k,X_k)\Delta W_k\\ &+ \frac{1}{2}L^1\sigma^{X_k}(t_k,X_k)\big((\Delta W_k)^2 - \Delta t_k\big)\\ &+ L^1 b^{X_k}(t_k,X_k)\Delta Z_k + \frac{1}{2}L^0 b^{X_k}(t_k,X_k)(\Delta t_k)^2\\ &+ L^0\sigma^{X_k}(t_k,X_k)(\Delta W_k\Delta t_k - \Delta Z_k)\\ &+ \frac{1}{2}L^1L^1\sigma^{X_k}(t_k,X_k)\big(\frac{1}{3}(\Delta W_k)^2 - \Delta t_k\big)\Delta W_k.\end{aligned} \tag{3.8}$$

Here ΔZ_k is a random variable defined by the double integrals

$$\Delta Z_k = \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} dW_z ds = \Delta W_k\Delta t_k - \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} dz dW_s.$$

3.2. Weak Itô-Taylor schemes

For $\eta = 1.0, 2.0, 3.0, \cdots$, we define the hierarchical set

$$\Gamma_\eta = \{\alpha \in \mathscr{M} : l(\alpha) \le \eta\} \tag{3.9}$$

and denote its remainder set by $\mathscr{B}(\Gamma_\eta)$. Take $f(t,x',x) = x$, then by Theorem 2.4, we have the Itô-Taylor expansion

$$X_{t_{k+1}} = \sum_{\alpha\in\Gamma_\eta} I_\alpha[f_\alpha^X(t_k,X_{t_k})]_{t_k,t_{k+1}} + R_k^\eta, \tag{3.10}$$

where $R_k^\eta = \sum_{\alpha\in\Gamma_\eta} I_\alpha[f_\alpha^X(\cdot,X_\cdot)]_{t_k,t_{k+1}}$. Removing the remainder term R_k^η from (3.10), we propose the following weak order η Itô-Taylor scheme for solving the MSDE (2.1).

Scheme 3.2 (Weak order η Itô-Taylor scheme)**.**

$$X_{k+1} = \sum_{\alpha\in\Gamma_\eta} I_\alpha[f_\alpha^{X_k}(t_k,X_k)]_{t_k,t_{k+1}}. \tag{3.11}$$

In the following subsections, we write Scheme 3.2 in explicit form with specific η.

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3.2.1. The Euler scheme

Taking $\eta = 1.0$ in (3.9) leads to

$$\Gamma_{1.0} = \{v,(0),(1)\} \text{ and } \mathscr{B}(\Gamma_{1.0}) = \{(0,1),(0,0),(1,0),(1,1)\}.$$

By (3.10), we deduce

$$X_{t_{k+1}} = X_{t_k} + b^X(t_k,X_{t_k})\int_{t_k}^{t_{k+1}} ds + \sigma^X(t_k,X_{t_k})\int_{t_k}^{t_{k+1}} dW_s + R_1, \tag{3.12}$$

where

$$\begin{aligned} R_1 = &\int_{t_k}^{t_{k+1}}\int_{t_k}^{s} L^0 b^X(z,X_z)dzds + \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} L^1 b^X(z,X_z)dW_z ds \\ &+ \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} L^0\sigma^X(z,X_z)dzdW_s + \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} L^1\sigma^X(z,X_z)dW_z dW_s. \end{aligned}$$

Removing the remainder term R_1, we get the Euler scheme (3.5)

$$X_{k+1} = X_k + b^{X_k}(t_k,X_k)\Delta t_k + \sigma^{X_k}(t_k,X_k)\Delta W_k,$$

which is also the weak order 1.0 Itô-Taylor scheme.

3.2.2. The weak order 2.0 Itô-Taylor scheme

Taking $\eta = 2.0$ in (3.9) gives

$$\begin{aligned} \Gamma_{2.0} &= \{v,(0),(1),(1,1),(0,0),(1,0),(0,1)\}, \\ \mathscr{B}(\Gamma_{2.0}) &= \{(0,1,1),(0,0,0),(0,1,0),(0,0,1), \\ &\qquad (1,1,1),(1,0,0),(1,1,0),(1,0,1)\}. \end{aligned}$$

Then the explicit form of Scheme 3.2 is

$$\begin{aligned} X_{k+1} = X_k &+ b^{X_k}(t_k,X_k)\Delta t_k + \sigma^{X_k}(t_k,X_k)\Delta W_k \\ &+ \frac{1}{2}L^1\sigma^{X_k}(t_k,X_k)\big((\Delta W_k)^2 - \Delta t_k\big) \\ &+ L^1 b^{X_k}(t_k,X_k)\Delta Z_k + \frac{1}{2}L^0 b^{X_k}(t_k,X_k)(\Delta t_k)^2 \\ &+ L^0\sigma^{X_k}(t_k,X_k)(\Delta W_k\Delta t_k - \Delta Z_k), \end{aligned} \tag{3.13}$$

which is the weak order 2.0 Itô-Taylor scheme.

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Remark 3.1. To show the differences between numerical schemes for solving standard SDEs and those for MSDEs, we take the weak order 2.0 scheme as an example. Now we consider the corresponding SDE of Example 5.4

$$dX_s = \cos(X_s)ds + \sin(X_s)dW_s, \quad s \in [0, T]. \tag{3.14}$$

Then the weak order 2.0 scheme for solving SDE (3.14) in [11] is

$$\begin{aligned} X_{k+1} = X_k &+ \cos(X_k)\Delta t_k + \sin(X_k)\Delta W_k \\ &+ \frac{1}{2}\cos(X_k)\sin(X_k)\big((\Delta W_k)^2 - \Delta t_k\big) - \sin(X_k)\sin(X_k)\Delta Z_k \\ &- \frac{1}{2}\Big(\sin(X_k)\cos(X_k) + \frac{1}{2}\cos(X_k)\big(\sin(X_k)\big)^2\Big)(\Delta t_k)^2 \\ &+ \Big(\cos(X_k)\cos(X_k) - \frac{1}{2}\sin(X_k)\big(\sin(X_k)\big)^2\Big)(\Delta W_k\Delta t_k - \Delta Z_k). \end{aligned} \tag{3.15}$$

The weak order 2.0 scheme for the MSDE (5.4) is

$$\begin{aligned} X_{k+1} = X_k &+ b^{X_k}(t_k, X_k)\Delta t_k + \sigma^{X_k}(t_k, X_k)\Delta W_k \\ &+ \frac{1}{2}L^1\sigma^{X_k}(t_k, X_k)\big((\Delta W_k)^2 - \Delta t_k\big) + L^1 b^{X_k}(t_k, X_k)\Delta Z_k \\ &+ \frac{1}{2}L^0 b^{X_k}(t_k, X_k)(\Delta t_k)^2 + L^0\sigma^{X_k}(t_k, X_k)(\Delta W_k\Delta t_k - \Delta Z_k) \\ = X_k &+ \big(\mathbb{E}\big[\cos(X_k x)\big]\Big|_{x=X_k}\big)\Delta t_k + \big(\mathbb{E}\big[\sin(X_k x)\big]\Big|_{x=X_k}\big)\Delta W_k \\ &+ \frac{1}{2}\Big(\mathbb{E}\big[X_k\cos(X_k x)\big]\mathbb{E}\big[\sin(X_k x)\big]\Big)\Big|_{x=X_k}\big((\Delta W_k)^2 - \Delta t_k\big) \\ &- \Big(\mathbb{E}\big[X_k\sin(X_k x)\big]\mathbb{E}\big[\sin(X_k x)\big]\Big)\Big|_{x=X_k}\Delta Z_k \\ &+ \frac{1}{2}\Big(\mathbb{E}\Big[-x\sin(X_k x)\big(\mathbb{E}\big[\cos(X_k x)\big]\Big|_{x=X_k}\big) \\ &- \frac{1}{2}x^2\cos(X_k x)\big(\mathbb{E}\big[\sin(X_k x)\big]\Big|_{x=X_k}\big)^2\Big]\Big|_{x=X_k} \\ &+ \Big(\mathbb{E}\big[-X_k\sin(X_k x)\big]\mathbb{E}\big[\cos(X_k x)\big]\Big)\Big|_{x=X_k} \\ &+ \frac{1}{2}\Big(\mathbb{E}\big[-(X_k)^2\cos(X_k x)\big]\big(\mathbb{E}\big[\sin(X_k x)\big]\big)^2\Big)\Big|_{x=X_k}\Big)(\Delta t_k)^2 \\ &+ \Big(\mathbb{E}\Big[x\cos(X_k x)\big(\mathbb{E}\big[\cos(X_k x)\big]\Big|_{x=X_k}\big) \\ &- \frac{1}{2}x^2\sin(X_k x)\big(\mathbb{E}\big[\sin(X_k x)\big]\Big|_{x=X_k}\big)^2\Big]\Big|_{x=X_k} \\ &+ \Big(\mathbb{E}\big[X_k\cos(X_k x)\big]\mathbb{E}\big[\cos(X_k x)\big]\Big)\Big|_{x=X_k} \\ &- \frac{1}{2}\Big(\mathbb{E}\big[(X_k)^2\sin(X_k x)\big]\big(\mathbb{E}\big[\sin(X_k x)\big]\big)^2\Big)\Big|_{x=X_k}\Big)(\Delta W_k\Delta t_k - \Delta Z_k). \end{aligned} \tag{3.16}$$

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Compared with the numerical schemes for the standard SDEs, schemes for MSDEs would be much more complicated. The computations of the expectations of unknowns in the coefficients of MSDEs are also very time-consuming. And in order to achieve the needed accuracy, the number M of paths *used in the Monte-Carlo method to compute the expectations should be big enough.*

4. Error Analysis

In this section, we will give error estimate for the strong order γ Itô-Taylor Scheme 3.1. We follow the idea used by Milstein et al. in [20] to rigorously obtain the global convergence rate of Scheme 3.1 based on the relationship between the local and global convergence rates.

4.1. The general error estimate theorem

Let $\{X_{t,X}(s)\}_{t\le s\le T}$ be the solution of MSDE (2.1) starting from the point (t,X). By the Eqs. (2.1) and (2.6), it also holds that

$$X_{t,X}(t+\theta)=X+\int_t^{t+\theta} b^{X_{t,X}}\left(s,X_{t,X}(s)\right)ds+\int_t^{t+\theta}\sigma^{X_{t,X}}\left(s,X_{t,X}(s)\right)dW_s \tag{4.1}$$

for $0\le\theta\le T-t$. Let $\bar{X}_{t,X}(t+h)$ be the one-step approximation of $X_{t,X}(t+h)$. We have the following theorem.

Theorem 4.1. *Let $X_{t,X}(t+h)$ be defined as* (4.1). *If $\bar{X}_{t,X}(t+h)$ satisfies*

$$\left|\mathbb{E}\left[X_{t,X}(t+h)-\bar{X}_{t,X}(t+h)|\mathscr{F}_t\right]\right|\le C^*\left(1+\mathbb{E}\left[|X|^2\right]+|X|^2\right)^{\frac{1}{2}}h^{p_1}, \tag{4.2a}$$

$$\left(\mathbb{E}\left[|X_{t,X}(t+h)-\bar{X}_{t,X}(t+h)|^2\middle|\mathscr{F}_t\right]\right)^{\frac{1}{2}}\le C^*\left(1+\mathbb{E}\left[|X|^2\right]+|X|^2\right)^{\frac{1}{2}}h^{p_2} \tag{4.2b}$$

for $t\in[t_0,T-h]$ and $x\in\mathbb{R}^d$, where p_1 and p_2 are parameters satisfying $p_2\ge\frac{1}{2}$, $p_1\ge p_2+\frac{1}{2}$, and C^* a positive constant independent of h, $X_{t,X}(t+h)$ and $\bar{X}_{t,X}(t+h)$. Then for $k=1,\cdots,N_T$, it holds that

$$\left(\mathbb{E}[|X_{t_0,X_0}(t_k)-\bar{X}_{t_0,X_0}(t_k)|^2]\right)^{\frac{1}{2}}\le C\left(1+\mathbb{E}[|X_0|^2]\right)^{\frac{1}{2}}h^{p_2-\frac{1}{2}}, \tag{4.3}$$

where $X_{t_0,X_0}(t_0)=X_0$ and $\bar{X}_{t_0,X_0}(t_k)$ is the solution of the one-step scheme

$$\bar{X}_{t_0,X_0}(t_k)=\bar{X}_{t_{k-1},\bar{X}_{t_0,X_0}(t_{k-1})}(t_k), \tag{4.4}$$

and C a constant independent of h, $X_{t,X}(t+h)$ and $\bar{X}_{t,X}(t+h)$.

Theorem 4.1 tells us how to estimate the strong orders of one-step schemes for MSDEs. The condition (4.2a) is the weak local error estimate of (4.4) with order p_1. The condition (4.2b) is the strong local error estimate of (4.4) with order p_2. The estimate (4.3) means that the global strong order of (4.4) is $p_2-\frac{1}{2}$.

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4.2. Lemmas

In this subsection, we will introduce some lemmas needed in the proof of Theorem 4.1.

Denote $X_{t_0,X_0}(t_k)$ by $X(t_k)$ and $\bar{X}_{t_0,X_0}(t_k)$ by $\bar{X}_k$. Then the one-step scheme (4.4) becomes

$$\bar{X}_{k+1} = \bar{X}_{t_k,\bar{X}_k}(t_{k+1}) = \bar{X}_{t_0,X_0}(t_{k+1}). \tag{4.5}$$

Lemma 4.1. *Let $X_{t,X}(s)$ and $X_{t,Y}(s)$ be the solutions of the MSDE* (2.1) *with initial conditions $X_{t,X}(t) = X$ and $X_{t,Y}(t) = Y$, respectively. Let $Z = X_{t,X}(t+h) - X_{t,Y}(t+h) - (X-Y)$, then under the Assumptions* (A1) *and* (A2)*, we have*

$$\mathbb{E}\big[|Z|^2\big] \le C_L \mathbb{E}\big[|X-Y|^2\big]h, \tag{4.6a}$$

$$\mathbb{E}\big[|X_{t,X}(t+h) - X_{t,Y}(t+h)|^2\big] \le (1 + C_L h)\mathbb{E}\big[|X-Y|^2\big], \tag{4.6b}$$

where C_L is a positive constant depending on the Lipschitz constant L.

Proof. For any $0 \le \theta \le h$, from (4.1) and the standard Itô formula and Itô isometry, it follows that

$$\begin{aligned}
&\mathbb{E}\big[|X_{t,X}(t+\theta) - X_{t,Y}(t+\theta)|^2\big] \\
&= \mathbb{E}\big[|X-Y|^2\big] + 2\mathbb{E}\bigg[\int_t^{t+\theta} \Big(X_{t,X}(s) - X_{t,Y}(s)\Big)\Big(b^{X_{t,X}}\big(s, X_{t,X}(s)\big) \\
&\quad - b^{X_{t,Y}}\big(s, X_{t,Y}(s)\big)\Big)ds\bigg] + \mathbb{E}\bigg[\int_t^{t+\theta} \Big|\sigma^{X_{t,X}}\big(s, X_{t,X}(s)\big) - \sigma^{X_{t,Y}}\big(s, X_{t,Y}(s)\big)\Big|^2 ds\bigg].
\end{aligned}$$

Then under Assumption (A2), we deduce

$$\begin{aligned}
&\mathbb{E}\big[|X_{t,X}(t+\theta) - X_{t,Y}(t+\theta)|^2\big] \\
&\le \mathbb{E}\big[|X-Y|^2\big] + 2L\mathbb{E}\bigg[\int_t^{t+\theta} \Big|X_{t,X}(s) - X_{t,Y}(s)\Big| \\
&\quad \cdot \Big(\mathbb{E}\big[\big|X_{t,X}(s) - X_{t,Y}(s)\big|\big] + \big|X_{t,X}(s) - X_{t,Y}(s)\big|\Big)ds\bigg] \\
&\quad + 2L\mathbb{E}\bigg[\int_t^{t+\theta} \Big(\mathbb{E}\big[\big|X_{t,X}(s) - X_{t,Y}(s)\big|^2\big] + \big|X_{t,X}(s) - X_{t,Y}(s)\big|^2\Big)ds\bigg] \\
&\le \mathbb{E}\big[|X-Y|^2\big] + 8L\mathbb{E}\bigg[\int_t^{t+\theta} \mathbb{E}\big[|X_{t,X}(s) - X_{t,Y}(s)|^2\big]ds\bigg].
\end{aligned} \tag{4.7}$$

Now by using the Gronwall lemma, we obtain

$$\mathbb{E}\big[|X_{t,X}(t+\theta) - X_{t,Y}(t+\theta)|^2\big] \le e^{8Lh}\mathbb{E}\big[|X-Y|^2\big], \quad 0 \le \theta \le h, \tag{4.8}$$

which implies the inequality (4.6b) for small h.

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By the definition of Z, we have

$$Z = \int_t^{t+h} \Big(b^{X_{t,X}}\big(s, X_{t,X}(s)\big) - b^{X_{t,Y}}\big(s, X_{t,Y}(s)\big) \Big) ds + \int_t^{t+h} \Big(\sigma^{X_{t,X}}\big(s, X_{t,X}(s)\big) - \sigma^{X_{t,Y}}\big(s, X_{t,Y}(s)\big) \Big) dW_s. \tag{4.9}$$

Taking square on the both sides of (4.9), and then taking the expectation $\mathbb{E}[\cdot]$, we obtain

$$\mathbb{E}[|Z|^2] \le 2\mathbb{E}\left[\Big| \int_t^{t+h} \Big(b^{X_{t,X}}\big(s, X_{t,X}(s)\big) - b^{X_{t,Y}}\big(s, X_{t,Y}(s)\big) \Big) ds \Big|^2 \right] + 2\mathbb{E}\left[\int_t^{t+h} \Big| \sigma^{X_{t,X}}\big(s, X_{t,X}(s)\big) - \sigma^{X_{t,Y}}\big(s, X_{t,Y}(s)\big) \Big|^2 ds \right]. \tag{4.10}$$

Then by the Assumption (A2) and estimate (4.8), we deduce

$$\begin{aligned}
\mathbb{E}[|Z|^2] \le & 2Lh\mathbb{E}\left[\int_t^{t+h} \Big(\mathbb{E}\big[|X_{t,X}(s) - X_{t,Y}(s)|^2\big] + |X_{t,X}(s) - X_{t,Y}(s)|^2 \Big) ds \right] \\
& + 2L\mathbb{E}\left[\int_t^{t+h} \Big(\mathbb{E}\big[|X_{t,X}(s) - X_{t,Y}(s)|^2\big] + |X_{t,X}(s) - X_{t,Y}(s)|^2 \Big) ds \right] \\
\le & 4L(1+h)\left[\int_t^{t+h} \mathbb{E}\big[|X_{t,X}(s) - X_{t,Y}(s)|^2\big] ds \right] \\
\le & 4L(1+h)\left[\int_t^{t+h} \mathbb{E}\big[|X - Y|^2\big] \times e^{8Lh} ds \right] \le 4L(1+h)e^{8Lh}\mathbb{E}\big[|X - Y|^2\big]h,
\end{aligned}$$

which proves the estimate (4.6a). The proof is completed. □

Lemma 4.2. *Under Assumptions* (A1) *and* (A2)*, it holds that*

$$\mathbb{E}\Big[\Big| \mathbb{E}\big[X_{t_k, \bar{X}_k}(t_{k+1}) - \bar{X}_k | \mathscr{F}_{t_k} \big] \Big|^2 \Big] \le C_K \big(1 + \mathbb{E}\big[|\bar{X}_k|^2\big] \big) h^2 \tag{4.11}$$

for $k = 0, 1, \cdots, N_T - 1$*, where the positive constant* C_K *depends on the linear growth constant* K*.*

Proof. By (4.1), we easily get

$$\mathbb{E}\big[X_{t_k, \bar{X}_k}(t_{k+1}) - \bar{X}_k | \mathscr{F}_{t_k} \big] = \mathbb{E}\Big[\int_{t_k}^{t_{k+1}} b^{X_{t_k, \bar{X}_k}}\big(s, X_{t_k, \bar{X}_k}(s)\big) ds \Big| \mathscr{F}_{t_k} \Big].$$

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Then

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\mathbb{E}\big[X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_k|\mathscr{F}_{t_k}\big]\Big|^2\Big]\\
\le&\mathbb{E}\Big[\mathbb{E}\Big[\Big|\int_{t_k}^{t_{k+1}} b^{X_{t_k,\bar{X}_k}}\big(s,X_{t_k,\bar{X}_k}(s)\big)ds\Big|^2\Big|\mathscr{F}_{t_k}\Big]\Big]\\
\le&h\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[\mathbb{E}\big[|b(s,X_{t_k,\bar{X}_k}(s),x)|^2\big]\Big|_{x=X_{t_k,\bar{X}_k}(s)}\Big]ds.
\end{aligned}$$

By the linear growth Assumption (A1), we have

$$\big|b(s,X_{t_k,\bar{X}_k}(s),x)\big|^2\le K\Big(1+\big|X_{t_k,\bar{X}_k}(s)\big|^2+\big|x\big|^2\Big).$$

Then by Theorem 2.1, we deduce

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}\big[|b(s,X_{t_k,\bar{X}_k}(s),x)|^2\big]\Big|_{x=X_{t_k,\bar{X}_k}(s)}\Big]\\
\le&K\mathbb{E}\Big(1+\mathbb{E}\big[|X_{t_k,\bar{X}_k}(s)|^2\big]+|X_{t_k,\bar{X}_k}(s)|^2\Big)\\
\le&K\Big(1+2\mathbb{E}\big[|X_{t_k,\bar{X}_k}(s)|^2\big]\Big)\le K\Big(1+2C_2\mathbb{E}\big[|\bar{X}_k|^2\big]\Big).
\end{aligned}$$

Thus, we have the conclusion

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\mathbb{E}\big[X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_k|\mathscr{F}_{t_k}\Big|^2\Big]\\
\le&h\int_{t_k}^{t_{k+1}}K\big(1+2C_2\mathbb{E}\big[|\bar{X}_k|^2\big]\big)ds\le K\big(1+2C_2\mathbb{E}\big[|\bar{X}_k|^2\big]\big)h^2,
\end{aligned}$$

which completes the proof. □

Lemma 4.3 (Discrete Gronwall lemma)**.** *Suppose that for arbitrary N and $k=0,\cdots,N$ we have*

$$u_{k+1}\le(1+Ah)u_k+Bh^p, \tag{4.12}$$

where $h=T/N$, $A>0$, $B>0$, $p\ge1$, $u_k\ge0$, $k=0,\cdots,N$. Then

$$u_k\le e^{AT}u_0+\frac{B}{A}(e^{AT}-1)h^{p-1}. \tag{4.13}$$

To simplify presentations, in the sequel, we take C as a generic constant, which does not depend on the time step h, the solutions of MSDEs and the Itô-Taylor schemes for solving MSDEs, and may be different from line to line.

Lemma 4.4. *Suppose that $\{\bar{X}_k\}_{k=0}^{N_T}$ satisfy the conditions* (4.2a) *and* (4.2b) *in Theorem 4.1. Then, for all $k=0,1,\cdots,N_T$, it holds that*

$$\mathbb{E}\big[|\bar{X}_k|^2\big]\le C\big(1+\mathbb{E}\big[|X_0|^2\big]\big). \tag{4.14}$$

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Proof. Suppose $\mathbb{E}\big[|\bar{X}_k|^2\big] < \infty$. Then by Theorem 2.1 and the condition (4.2a) in Theorem 4.1, we get

$$\mathbb{E}\big[|X_{t_k,\bar{X}_k}(t_{k+1})|^2\big] \le C\big(1+\mathbb{E}\big[|\bar{X}_k|^2\big]\big) < \infty, \tag{4.15a}$$

$$\mathbb{E}\big[|X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big|^2\big] \le C\big(1+\mathbb{E}\big[|\bar{X}_k|^2\big]\big)h^{2p_2} < \infty, \tag{4.15b}$$

which implies

$$\mathbb{E}\big[|\bar{X}_{k+1}|^2\big] < \infty.$$

Since $\mathbb{E}\big[|\bar{X}_0|^2\big] < \infty$, by induction we have

$$\mathbb{E}\big[|\bar{X}_k|^2\big] < \infty, \quad k = 0, 1, \cdots, N_T.$$

Consider the equality

$$\begin{aligned}
\mathbb{E}\big[|\bar{X}_{k+1}|^2\big] =& \mathbb{E}\big[|\bar{X}_k + \bar{X}_{t_k,\bar{X}_k}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1}) + X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k|^2\big] \\
=& \mathbb{E}\big[|\bar{X}_k|^2 + \mathbb{E}\big[|\bar{X}_{t_k,\bar{X}_k}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})|^2\big] \\
&+ \mathbb{E}\big[|X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k|^2\big]\big] + 2\mathbb{E}\big[\bar{X}_k\big(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k\big)\big] \\
&+ 2\mathbb{E}\big[\bar{X}_k\big(\bar{X}_{t_k,\bar{X}_k}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\big)\big] \\
&+ 2\mathbb{E}\big[\big(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k\big)\big(\bar{X}_{t_k,\bar{X}_k}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\big)\big].
\end{aligned}$$

By Theorem 2.1 and the condition (4.2a) in Theorem 4.1, we have

$$\mathbb{E}\big[|X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k|^2\big] \le C\big(1+\mathbb{E}\big[|\bar{X}_k|^2\big]\big)h, \tag{4.16a}$$

$$\mathbb{E}\big[|\bar{X}_{t_k,\bar{X}_k}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})|^2\big] \le C\big(1+\mathbb{E}\big[|\bar{X}_k|^2\big]\big)h^{2p_2}. \tag{4.16b}$$

Then we get

$$\begin{aligned}
&2\mathbb{E}\big[(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k)\big(\bar{X}_{t_k,\bar{X}_k}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\big)\big] \\
\le& 2\Big(\mathbb{E}\big[|X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k|^2\big]\Big)^{1/2}\Big(\mathbb{E}\big[|\bar{X}_{t_k,\bar{X}_k}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})|^2\big]\Big)^{1/2} \\
\le& C\big(1+\mathbb{E}\big[|\bar{X}_k|^2\big]\big)h^{p_2+1/2}.
\end{aligned} \tag{4.17}$$

And by Lemma 4.2 and the condition (4.2a) in Theorem 4.1, we deduce

$$\begin{aligned}
&2\mathbb{E}\Big[\bar{X}_k(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k)\Big] = 2\mathbb{E}\Big[\bar{X}_k\mathbb{E}\big[X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k|\mathscr{F}_{t_k}\big]\Big] \\
\le& 2\Big(\mathbb{E}\big[|\bar{X}_k|^2\big]\Big)^{1/2}\Big(\mathbb{E}\Big[\Big|\mathbb{E}\big[X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_k|\mathscr{F}_{t_k}\big]\Big|^2\Big]\Big)^{1/2} \\
\le& C\big(1+\mathbb{E}\big[|\bar{X}_k|^2\big]\big)h,
\end{aligned} \tag{4.18}$$

and

$$2\mathbb{E}\big[\bar{X}_k\big(\bar{X}_{t_k,\bar{X}_k}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\big)\big] \le C\big(1+\mathbb{E}\big[|\bar{X}_k|^2\big]\big)h^{p_1}. \tag{4.19}$$

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From (4.16)-(4.19) and the conditions $p_1 \geq 1$ and $p_2 \geq 1/2$, we have

$$\mathbb{E}\big[|\bar{X}_{k+1}|^2\big] \leq \mathbb{E}\big[|\bar{X}_k|^2\big] + C\big(1+\mathbb{E}\big[|\bar{X}_k|^2\big]\big)h = (1+Ch)\mathbb{E}\big[|\bar{X}_k|^2\big] + Ch, \tag{4.20}$$

which, by the Gronwall lemma in Lemma 4.3, leads to

$$\mathbb{E}\big[|\bar{X}_k|^2\big] \leq e^{CT}\mathbb{E}\big[|\bar{X}_0|^2\big] + (e^{CT}-1) \leq C(1+\mathbb{E}\big[|\bar{X}_0|^2\big]). \tag{4.21}$$

This completes the proof. □

4.3. The proof of Theorem 4.1

Now we turn to prove Theorem 4.1 by using the lemmas in the above subsection.

Proof. By definition, $\bar{X}_{k+1} = \bar{X}_{t_k,\bar{X}_k}(t_{k+1})$. Then

$$\begin{aligned} &X(t_{k+1}) - \bar{X}_{k+1} = X_{t_k,X(t_k)}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}) \\ =&\big(X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\big) + \big(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big). \end{aligned} \tag{4.22}$$

For the first term on the right hand side of (4.22), we have

$$X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1}) = X(t_k) - \bar{X}_k + Z,$$

where $Z = X_{t,X(t_k)}(t+h) - X_{t,\bar{X}_k}(t+h) - (X(t_k) - \bar{X}_k)$.

Define $e_k := \big(\mathbb{E}\big[|X(t_k) - \bar{X}_k|^2\big]\big)^{\frac{1}{2}}$ $(k = 0, 1, \cdots, N_T)$. Then taking mean square on both sides of (4.22), we obtain

$$\begin{aligned} e_{k+1}^2 =&\mathbb{E}\Big[\big|X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\big|^2\Big] + \mathbb{E}\Big[\big|X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big|^2\Big] \\ &+ 2\mathbb{E}\Big[\big(X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\big)\big(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big)\Big] \\ =&\mathbb{E}\Big[\big|X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\big|^2\Big] + \mathbb{E}\Big[\big|X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big|^2\Big] \\ &+ 2\mathbb{E}\Big[\big(X(t_k) - \bar{X}_k\big)\big(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big)\Big] \\ &+ 2\mathbb{E}\Big[Z\big(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big)\Big]. \end{aligned} \tag{4.23}$$

By Lemma 4.1, we have

$$\begin{aligned} &\mathbb{E}\Big[\big|X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\big|^2\Big] \\ \leq&(1+Ch)\mathbb{E}\Big[\big|X(t_k) - \bar{X}_k\big|^2\Big] = (1+Ch)e_k^2. \end{aligned} \tag{4.24}$$

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Using the condition (4.2a) in Theorem 4.1 and Lemma 4.4, we get

$$\begin{aligned}
&\mathbb{E}\Big[\big|X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big|^2\Big]\\
&=\mathbb{E}\Big[\mathbb{E}\big[\big|X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big|^2\big|\mathscr{F}_{t_k}\big]\Big]\\
&\le C\mathbb{E}\big[1+\mathbb{E}[|\bar{X}_k|^2]+|\bar{X}_k|^2\big]h^{2p_2}\\
&\le C\big(1+\mathbb{E}\big[|\bar{X}_k|^2\big]\big)h^{2p_2}\le C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)h^{2p_2}.
\end{aligned}\tag{4.25}$$

Then, by the condition (4.2a) in Theorem 4.1, Lemma 4.4 and Schwartz inequality, we deduce

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\big(X(t_k)-\bar{X}_k\big)\big(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big)\Big]\Big|\\
&=\Big|\mathbb{E}\Big[\mathbb{E}\big[\big(X(t_k)-\bar{X}_k\big)\big(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big)\big|\mathscr{F}_{t_k}\big]\Big]\Big|\\
&\le\big(\mathbb{E}\big[|X(t_k)-\bar{X}_k|^2\big]\big)^{1/2}C\big(\mathbb{E}\big[1+\mathbb{E}[|\bar{X}_k|^2]+|\bar{X}_k|^2\big]\big)^{1/2}h^{p_1}\\
&\le C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)^{1/2}h^{p_1}e_k.
\end{aligned}\tag{4.26}$$

Combining the condition (4.2b), Lemma 4.1 and Lemma 4.4 leads to

$$\begin{aligned}
&\Big|\mathbb{E}\big[Z\big(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\big)\big]\Big|\\
&\le C\big(\mathbb{E}\big[|X(t_k)-\bar{X}_k|^2\big]\big)^{1/2}\big(\mathbb{E}\big[1+\mathbb{E}[|\bar{X}_k|^2]+|\bar{X}_k|^2\big]\big)^{1/2}h^{p_2+1/2}\\
&\le C\big(\mathbb{E}\big[|X(t_k)-\bar{X}_k|^2\big]\big)^{1/2}\big(1+\mathbb{E}\big[|X_0|^2\big]\big)^{1/2}h^{p_2+1/2}\\
&\le C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)^{1/2}h^{p_2+1/2}e_k.
\end{aligned}\tag{4.27}$$

From (4.24)–(4.27), and the inequalities $p_1\ge p_2+1/2$ and $2ab\le a^2+b^2$ for any $a,b\in\mathbb{R}$, it follows that

$$\begin{aligned}
e_{k+1}^2&\le e_k^2(1+Ch)+C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)^{1/2}h^{p_2+1/2}e_k+C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)h^{2p_2}\\
&\le e_k^2(1+Ch)+C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)h^{2p_2}+Che_k^2+C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)h^{2p_2}\\
&\le e_k^2(1+Ch)+C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)h^{2p_2}.
\end{aligned}\tag{4.28}$$

Now by Lemma 4.3, noting that $e_0=0$, we deduce

$$\begin{aligned}
e_k^2&=\mathbb{E}\big[|X(t_k)-\bar{X}_k|^2\big]\\
&\le C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)\big(e^{CT}-1\big)h^{2p_2-1}\le C\big(1+\mathbb{E}\big[|X_0|^2\big]\big)h^{2p_2-1}.
\end{aligned}$$

Hence the conclusion (4.3) in Theorem 4.1 is proved. □

By applying Theorem 4.1, we will give an error estimate of the strong order γ Itô-Taylor scheme in the next subsection.

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4.4. Error estimate of the strong order γ Itô-Taylor scheme

For convenience, let $W_t^0 = t$. Notice that $f^{X_{t,X}}(s, X_{t,X}(s)) = \mathbb{E}\big[f(s, X_{t,X}(s), x)\big]\big|_{x=X_{t,X}(s)}$ and $X_{t,X}(t) = X$ for $s \in [t, T]$, hence $f^{X_{t,X}}(t, X_{t,X}(t)) = f^{X_{t,X}}(t, X)$.

Let $\alpha = (i_1, i_2, \cdots, i_k)$ with $i_j \in \{0, 1, \cdots, m\}$, $j = 1, 2, \cdots, k$. We have the following Lemmas 4.5 and 4.6.

Lemma 4.5. *Let β and f^β be defined by (2.7) and (2.8), respectively. Suppose f_α^β and $I_\alpha[f^\beta(\cdot)]_{t,t+h}$ exist and $|f_\alpha^\beta(t,x)| \le C(1 + \mathbb{E}[|\beta_t|^2] + |x|^2)^{1/2}$. Then we have*

$$\mathbb{E}\Big[\big(I_\alpha\big[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\big]_{t,t+h}\big)^2\Big|\mathbb{F}_t\Big] \le C\big(1 + \mathbb{E}[|X|^2] + |X|^2\big)h^{l(\alpha)+n(\alpha)}. \tag{4.29}$$

Proof. By the definition of $I_\alpha\big[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\big]_{t,t+h}$, we have

$$\begin{aligned} &I_\alpha\Big[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\Big]_{t,t+h} \\ &= \int_t^{t+h}\int_t^s \cdots \int_t^{s_{k-2}} f_\alpha^{X_{t,X}}(s_{k-1}, X_{t,X}(s_{k-1}))\, dW_{s_{k-1}}^{i_1} \cdots dW_{s_1}^{i_{k-1}} dW_s^{i_k}. \end{aligned}$$

If $\alpha = v$, i.e., $l(\alpha) + n(\alpha) = 0$, we get

$$\begin{aligned} &\mathbb{E}\left[\left(I_\alpha\left[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\right]_{t,t+h}\right)^2\Big|\mathbb{F}_t\right] = \mathbb{E}\left[\left|f_\alpha^{X_{t,X}}(t,X)\right|^2\Big|\mathbb{F}_t\right] \\ \le& C\mathbb{E}\left[1 + \mathbb{E}[|X|^2] + |X|^2\Big|\mathbb{F}_t\right] = C\left(1 + \mathbb{E}[|X|^2] + |X|^2\right), \end{aligned} \tag{4.30}$$

which implies the estimate (4.29) with $p(v) = l(\alpha) + n(\alpha) = 0$. Now we consider $l(\alpha) > 0$. If $i_k \ne 0$, then

$$\begin{aligned} &\mathbb{E}\Big[\big(I_\alpha\big[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\big]_{t,t+h}\big)^2\Big|\mathbb{F}_t\Big] \\ =& \int_t^{t+h} \mathbb{E}\Big[\big(I_{\alpha-}\big[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\big]_{t,s}\big)^2\Big|\mathbb{F}_t\Big]\, ds. \end{aligned} \tag{4.31}$$

And if $i_k = 0$, i.e., $dW_s^{i_k} = ds$, then

$$\begin{aligned} &\mathbb{E}\Big[\big(I_\alpha\big[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\big]_{t,t+h}\big)^2\Big|\mathbb{F}_t\Big] \\ =& \mathbb{E}\left[\Big|\int_t^{t+h} I_{\alpha-}\big[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\big]_{t,s} ds\Big|^2\Big|\mathbb{F}_t\right] \\ \le& h\int_t^{t+h} \mathbb{E}\Big[\big(I_{\alpha-}\big[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\big]_{t,s}\big)^2\Big|\mathbb{F}_t\Big]\, ds. \end{aligned} \tag{4.32}$$

The estimates (4.31) and (4.32) imply the recurrence relation

$$p(\alpha) = p(\alpha-) + (2 - i'_{l(\alpha)}) = \sum_{j=1}^{k}(2 - i'_j) = l(\alpha) + n(\alpha)$$

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with i'_j defined by

$$i'_j = \begin{cases} 0, & i_j = 0, \\ 1, & i_j \neq 0. \end{cases}$$

The proof is completed. □

Lemma 4.6. *Under the conditions of Lemma 4.5, we have*

$$\left|\mathbb{E}\left[I_\alpha[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))]_{t,t+h}\middle|\mathbb{F}_t\right]\right| \begin{cases} =0, & \text{if } l(\alpha) \neq n(\alpha), \\ \leq C\left(1+\mathbb{E}\left[|X|^2\right]+|X|^2\right)^{1/2} h^{n(\alpha)}, & \text{if } l(\alpha) = n(\alpha). \end{cases}$$

Proof. The proof is similar to that of Lemma 4.5, so we omit it here. □

Theorem 4.2. *Let $X(t)$ and $\bar{X}_k$ be the solutions of the MSDE* (2.1) *and the strong order γ Itô-Taylor scheme* (3.3) *with $X(t_0) = X_0$, respectively. Suppose $f^{X_{t,X}}(s, X_{t,X}(s)) = X_{t,X}(s)$ has the Itô-Taylor expansion* (2.22) *with $\mathscr{A} = \mathscr{A}_\gamma$ and*

$$\left|f_\alpha^\beta(t,x)\right| \leq C\left(1+\mathbb{E}\left[|\beta_t|^2\right]+|x|^2\right)^{1/2},$$

for $\alpha \in \mathscr{A}_\gamma \cup \mathscr{B}(\mathscr{A}_\gamma)$, $(t,x) \in [0,T] \times \mathbb{R}^d$, and β defined by (2.7). *Then it holds that*

$$\max_{k \in \{1,2,\cdots,N_T\}} \mathbb{E}\left[|X_{t_k} - \bar{X}_k|^2\right] \leq C\left(1+\mathbb{E}\left[|X_0|^2\right]\right)(\Delta t)^{2\gamma}.$$

Proof. We just give the proof for the case $d = 1$. The proof of the general case is similar. Let β be the Itô process $\{X_{t,X}(s) : t \leq s \leq T\}$. Then by Theorem 2.4, we have

$$X_{t,X}(t+h) = X + \sum_{\alpha \in .\mathscr{A}_\gamma \setminus v} I_\alpha\left[f_\alpha^{X_{t,X}}(t,X)\right]_{t,t+h} + R^\gamma, \tag{4.33}$$

where

$$R^\gamma = \sum_{\alpha \in \mathscr{B}(.\mathscr{A}_\gamma)} I_\alpha\left[f_\alpha^{X_{t,X}}(\cdot, X_{t,X}(\cdot))\right]_{t,t+h}.$$

The strong order γ Itô-Taylor scheme (3.3) also has the form

$$\bar{X}_{t,X}(t+h) = X + \sum_{\alpha \in .\mathscr{A}_\gamma \setminus v} I_\alpha\left[f_\alpha^{X_{t,X}}(t,X)\right]_{t,t+h}. \tag{4.34}$$

Subtracting (4.33) from (4.34) leads to

$$R^\gamma = X_{t,X}(t+h) - \bar{X}_{t,X}(t+h). \tag{4.35}$$

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By Lemma 4.5, we deduce

$$\begin{aligned}\mathbb{E}\Big[|R^{\gamma}|^2\Big|\mathbb{F}_t\Big] &= \mathbb{E}\Big[\Big|\sum_{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)} I_\alpha\big[f_\alpha^{X_{t,X}}(\cdot,X_{t,X}(\cdot))\big]_{t,t+h}\Big|^2\Big|\mathbb{F}_t\Big]\\ &\le C\sum_{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)}\mathbb{E}\Big[\Big|I_\alpha\big[f_\alpha^{X_{t,X}}(\cdot,X_{t,X}(\cdot))\big]_{t,t+h}\Big|^2\Big|\mathbb{F}_t\Big]\\ &\le C\sum_{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)}\big(1+\mathbb{E}\big[|X|^2\big]+|X|^2\big)h^{l(\alpha)+n(\alpha)}\\ &\le C\big(1+\mathbb{E}\big[|X|^2\big]+|X|^2\big)h^{2p_2},\end{aligned}\tag{4.36}$$

where $2p_2=\min\limits_{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)}\{l(\alpha)+n(\alpha)\}$. Since $\mathscr{A}_\gamma=\{\alpha\in\mathscr{M}:l(\alpha)+n(\alpha)\le 2\gamma \text{ or } l(\alpha)=n(\alpha)=\gamma+\frac{1}{2}\}$, we have

$$p_2=\frac{1}{2}\min_{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)}\{l(\alpha)+n(\alpha)\}=\gamma+\frac{1}{2}.$$

Now we prove that $p_1\ge p_2+\frac{1}{2}$. Using Lemma 4.6, we get

$$\begin{aligned}\Big|\mathbb{E}\big[R^{\gamma}\big|\mathbb{F}_t\big]\Big| &= \Big|\mathbb{E}\Big[\sum_{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)} I_\alpha\big[f_\alpha^{X_{t,X}}(\cdot,X_{t,X}(\cdot))\big]_{t,t+h}\Big|\mathbb{F}_t\Big]\Big|\\ &\le \sum_{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)}\Big|\mathbb{E}\Big[I_\alpha\big[f_\alpha^{X_{t,X}}(\cdot,X_{t,X}(\cdot))\big]_{t,t+h}\Big|\mathbb{F}_t\Big]\Big|\\ &\le C\sum_{\substack{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)\\ l(\alpha)=n(\alpha)}}\big(1+\mathbb{E}\big[|X|^2\big]+|X|^2\big)^{1/2}h^{n(\alpha)}\\ &\le C\big(1+\mathbb{E}\big[|X|^2\big]+|X|^2\big)^{1/2}h^{p_1},\end{aligned}$$

where $p_1=\min\limits_{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)}\{n(\alpha):n(\alpha)=l(\alpha)\}$. Simple calculation yields

$$p_1=\min_{\alpha\in\mathscr{B}(\mathscr{A}_\gamma)}\{n(\alpha):n(\alpha)=l(\alpha)\}=\begin{cases}\gamma+1,\ \gamma=1,2,\cdots\\ \gamma+\frac{3}{2},\ \gamma=1.5,2.5,\cdots,\end{cases}$$

which implies $p_1\ge p_2+\frac{1}{2}$. Thus by Theorem 4.1, we complete the proof. □

5. Numerical Experiments

In this section, we use four numerical examples to demonstrate the effectiveness and accuracy of our schemes. For each example, we test the following schemes: the Euler scheme (3.5), the Milstein scheme (3.7), the strong order 1.5 scheme (3.8) and the weak order 2.0 scheme (3.13).

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Table 5.1: Strong errors and convergence rates for Example 5.1.

N_T	2	4	8	16	32	
	$\mathbb{E}[\lvert X(T)-X_{N_T}\rvert]$					CR
Euler	0.9383	0.5645	0.3119	0.1625	0.0810	0.89
Milstein	0.9383	0.5645	0.3119	0.1625	0.0810	0.89
Strong-1.5	0.2422	0.0737	0.0203	0.0053	0.0013	1.88
Weak-2.0	0.2422	0.0737	0.0203	0.0053	0.0013	1.88

Table 5.2: Weak errors and convergence rates for Example 5.1.

N_T	2	4	8	16	32	
	$\lvert\mathbb{E}[X(T)-X_{N_T}]\rvert$					CR
Euler	0.2910	0.1955	0.1121	0.0590	0.0296	0.83
Milstein	0.2910	0.1955	0.1121	0.0590	0.0296	0.83
Strong-1.5	0.1475	0.0437	0.0117	0.0028	0.0007	1.94
Weak-2.0	0.1475	0.0437	0.0117	0.0028	0.0007	1.94

The time interval $[0,T]$ is uniformly partitioned into N_T parts with time step $h=\frac{T}{N_T}$. In the experiments, we are to test the strong errors $\mathbb{E}\big[\lvert X(T)-X_{N_T}\rvert\big]$, the weak errors $\big\lvert\mathbb{E}[X(T)-X_{N_T}]\big\rvert$, and convergence rates of these schemes, where $X(T)$ and X_{N_T} are the terminal solutions of MSDEs and the schemes, respectively. The Monte Carlo method is used to approximate the expectation $\mathbb{E}[\cdot]$ appeared in the coefficients with sample times M. The "exact" solution of the MSDEs is identified with the numerical one using a small step-size $h_{\text{exact}}=2^{-9}$ for the first three examples and $h_{\text{exact}}=2^{-8}$ for the last example. The convergence rates with respect to time step h are obtained by least square fitting.

In our computations, we take $T=1$, $X_0=1$ and $M=20000$ in each example. In all the tables, the CR represents convergence rate, and Euler, Milstein, Strong-1.5 and Weak-2.0 represent the Euler scheme, the Milstein scheme, the strong order 1.5 scheme, and the weak order 2.0 scheme, respectively.

Example 5.1. Consider the following nonlinear additive mean-field SDE

$$dX_s=\big(\mathbb{E}[\cos(X_s)]+X_s\big)ds+e^s dW_s,\ \ s\in[0,T]. \tag{5.1}$$

The errors in strong sense and weak sense, and the corresponding convergence rates are listed in Tables 5.1-5.2 for Example 5.1.

Example 5.2. Choose the nonlinear mean-field SDE as

$$dX_s=\big(\mathbb{E}[\sin(X_s)]\cos(X_s)+\mathbb{E}[\cos(X_s)]\sin(X_s)\big)ds+\big(\mathbb{E}[X_s]\big)dW_s,\ \ s\in[0,T]. \tag{5.2}$$

Note that the above equation is also an additive MSDE.

We list the strong and weak errors and convergence rates of the schemes in Table 5.3 and Table 5.4, respectively.

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Table 5.3: Strong errors and convergence rates for Example 5.2.

N_T	2	4	8	16	32	
	$\mathbb{E}[\lvert X(T)-X_{N_T}\rvert]$					CR
Euler	0.2221	0.0974	0.0453	0.0219	0.0104	1.10
Milstein	0.2221	0.0974	0.0453	0.0219	0.0104	1.10
Strong-1.5	0.0990	0.0276	0.0084	0.0028	0.0016	1.61
Weak-2.0	0.0990	0.0276	0.0084	0.0028	0.0016	1.61

Table 5.4: Weak errors and convergence rates for Example 5.2.

N_T	2	4	8	16	32	
	$\lvert\mathbb{E}[X(T)-X_{N_T}]\rvert$					CR
Euler	0.1499	0.0667	0.0306	0.0145	0.0068	1.11
Milstein	0.1499	0.0667	0.0306	0.0145	0.0068	1.11
Strong-1.5	0.0488	0.0112	0.0027	0.0007	0.0002	2.04
Weak-2.0	0.0488	0.0112	0.0027	0.0007	0.0002	2.04

Table 5.5: Strong errors and convergence rates for Example 5.3.

N_T	2	4	8	16	32	
	$\mathbb{E}[\lvert X(T)-X_{N_T}\rvert]$					CR
Euler	0.2317	0.1106	0.0643	0.0406	0.0268	0.77
Milstein	0.2822	0.1107	0.0489	0.0233	0.0112	1.16
Strong-1.5	0.1804	0.0575	0.0193	0.0066	0.0023	1.57
Weak-2.0	0.1981	0.0645	0.0233	0.0098	0.0045	1.37

Example 5.3. The aim of this example is to test the performance of our schemes for solving non-additive nonlinear MSDEs. Consider the following MSDE

$$dX_s = \big(\mathbb{E}[\sin(X_s)] + \cos(X_s)\big)ds + \big(\mathbb{E}[\cos(X_s)] + \sin(X_s)\big)dW_s,\ \ s \in [0,T]. \tag{5.3}$$

The strong and weak errors and convergence rates of the schemes are shown in Table 5.5 and Table 5.6, respectively.

Example 5.4. This example considers a more complex non-additive nonlinear mean-field SDE than the one in Example 5.3, which is

$$dX_s = \big(\mathbb{E}[\cos(X_s x)]\big|_{x=X_s}\big)ds + \big(\mathbb{E}[\sin(X_s x)]\big|_{x=X_s}\big)dW_s,\ \ s \in [0,T]. \tag{5.4}$$

The numerical results are given in Table 5.7 and Table 5.8.

All the errors and convergence rates listed in Tables 5.1–5.8 show that the Itô-Taylor schemes are effective and accurate for solving MSDEs. And combining the numerical results with the numerical theory for standard SDEs, we draw some conclusions as follows.

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Table 5.6: Weak errors and convergence rates for Example 5.3.

N_T	2	4	8	16	32	
	$\lvert\mathbb{E}[X(T)-X_{N_T}]\rvert$					CR
Euler	0.0738	0.0344	0.0182	0.0089	0.0039	1.04
Milstein	0.1330	0.0418	0.0193	0.0092	0.0044	1.20
Strong-1.5	0.0546	0.0074	0.0011	0.0003	0.00005	2.48
Weak-2.0	0.0587	0.0064	0.0009	0.0003	0.0001	2.24

Table 5.7: Strong errors and convergence rates for Example 5.4.

N_T	2	4	8	16	32	
	$\mathbb{E}[\lvert X(T)-X_{N_T}\rvert]$					CR
Euler	0.1354	0.0755	0.0503	0.0340	0.0227	0.63
Milstein	0.1542	0.0694	0.0328	0.0158	0.0075	1.08
Strong-1.5	0.1105	0.0395	0.0139	0.0050	0.0017	1.49
Weak-2.0	0.1127	0.0443	0.0181	0.0083	0.0039	1.21

Table 5.8: Weak errors and convergence rates for Example 5.4.

N_T	2	4	8	16	32	
	$\lvert\mathbb{E}[X(T)-X_{N_T}]\rvert$					CR
Euler	0.0667	0.0252	0.0121	0.0055	0.0025	1.16
Milstein	0.1068	0.0366	0.0144	0.0063	0.0028	1.30
Strong-1.5	0.0309	0.0020	0.0005	0.0001	0.00003	2.44
Weak-2.0	0.0305	0.0012	0.0004	0.0001	0.00007	2.04

1. Additive noises
   The fact that the MSDEs in Examples 5.1 and 5.2 are additive implies that the Milstein scheme and the Euler scheme are the same, so are the strong order 1.5 scheme and the weak order 2.0 scheme. Our numerical results also show the implications, and the convergence rates listed in Tables 5.1–5.4 are consistent with our theoretical conclusions.

2. Non-additive noises

   1) Strong convergence.
      From Tables 5.5 and 5.7, we see that the Euler scheme, the Milstein scheme and the strong order 1.5 scheme for solving MSDEs with non-additive noises have orders 0.5, 1.0 and 1.5 of strong convergence, respectively. These numerical results are in a good agreement with those for standard SDEs theory. However, the strong convergence rate of the weak order 2.0 scheme is less than 1.5.

   2) Weak convergence.
      The numerical results displayed in Tables 5.6 and 5.8 tell us that the Euler scheme and the Milstein scheme have the weak convergence rate of order 1.0, while both

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the strong order 1.5 scheme and the weak order 2.0 scheme can achieve an order 2.0 of weak convergence.

6. Conclusions

In this paper, we developed the mean-field Itô formula and the mean-field Itô-Taylor expansion, and based on them we proposed the strong order γ and weak order η Itô-Taylor schemes for solving MSDEs. By choosing $\gamma = 0.5, 1.0, 1.5$ and $\eta = 2.0$, the proposed schemes become the Euler, the Milstein, the strong order 1.5, and the weak order 2.0 schemes, respectively. Under some regularity conditions, we theoretically proved the convergence rate γ of our strong order γ Itô-Taylor scheme. Our numerical experiment results showed that the schemes are stable, effective and accurate for solving MSDEs, and that the convergence rates are consistent with our theoretical conclusions.

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